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(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–6, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, May 13.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 13.
MICHIGAN, Ann Arbor, March 18.
MINNESOTA.
MISSOURI, Springfield, April 28.
NEBRASKA.
OHIO, Columbus, April 6.
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 6.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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## THE OCTOBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The eleventh regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the Aluminum Research Laboratories of the Aluminum Company of America at New Kensington, Pennsylvania, on Saturday, October 22, 1938. Professor H. L. Black, chairman of the Section, presided both at the morning and at the afternoon session.

Eighty-six representatives from twelve colleges, seven research laboratories, and four high schools attended the meeting, including the following twenty-nine members of the Association: C. S. Atchison, O. F. H. Bert, H. L. Black, A. M. Bryson, Helen Calkins, W. E. Cleland, Elizabeth B. Cowley, L. L. Dines, H. L. Dorwart, Beatrice L. Hagen, H. C. Hicks, A. V. Karpov, H. R. Leifer, M. L. Manning, David Moskovitz, Frederick Mosteller, L. T. Moston, C. T. Oergel, E. G. Olds, N. C. Riggs, J. B. Rosenbach, E. A. Saibel, S. R. Smith, J. C. Stayer, R. G. Sturm, J. S. Taylor, R. W. Thomas, W. J. Wagner, E. D. Wells.

At the annual business meeting the following officers of the Section were elected: Chairman, H. L. Black, Westminster College; Secretary-Treasurer, David Moskovitz, Carnegie Institute of Technology; Member of Executive Committee, L. T. Moston, Waynesburg College. Professor H. C. Hicks, Carnegie Institute of Technology, continues in office for the second year of his term as additional member of the executive committee.

The morning session was preceded by a very interesting and instructive tour of the Aluminum Research Laboratories, furnishing a splendid background for the first three papers on the program which constituted a symposium on the applications of mathematics to the work of the Laboratories. Following a welcoming address by Dr. F. C. Frary, Director of Research of the Aluminum Research Laboratories, the following five papers were read:

1. "The analysis and interpretation of engineering data" by Dr. R. G. Sturm, Aluminum Research Laboratories.
2. "Mathematics in structural analysis" by Dr. M. Holt, Aluminum Research Laboratories, introduced by Dr. Sturm.
3. "Applications of statistical methods" by Dr. R. B. Mears, Aluminum Research Laboratories, introduced by Dr. Sturm.
4. "A schematic method for obtaining invariants of certain algebraic forms" by P. N. Carpenter, Grove City College, introduced by the Secretary.
5. "The Sylvester determinant" by Dr. H. L. Krall, Pennsylvania State College, introduced by Professor Owens.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Sturm emphasized the utilitarian concept of mathematics, illustrating the importance of and difficulties involved in the reduction of particular engineering problems to problems in mathematics. Problems leading to differ-

ential equations, numerical integration, and curve fitting were discussed from the point of view of the application of mathematics to practical problems. Particular consideration was given to the usefulness of such examples to teachers of mathematics.

2. Dr. Holt presented a variety of problems encountered in structural engineering research and explained methods of analysis which gave theoretical results that agree satisfactorily with experimental results. The distribution of load among the rivets in a riveted joint was studied by an algebraic analysis. The angle of twist of structural frames was determined by geometry. A series solution was applied in the study of the torsion of structural shapes. True deflections were obtained from measured components by graphics. Partial differential equations were used in problems of elastic instability.

3. Dr. Mears explained the concept of "corrosion probability" and discussed the application of this concept to steel and aluminum. The use of the Poisson expression in the calculation of frequency of occurrence of sites of attack was outlined. Mention was made of the application of tests to determine the significance of differences of means and some typical examples were presented. The use of correlation coefficients was described in making estimates of the effect of any of the constituents of a given metal or alloy on either its resistance to corrosion or its corrosion probability. It was concluded that statistical methods are of considerable aid in treating data which are not susceptible to analysis by the other commonly employed methods.

4. Mr. Carpenter presented an application of a schematic method for exhausting all the possibilities of associating indices on tensor forms developed by C. M. Cramlet (Bull. Amer. Math. Soc., May–June, 1928, pp. 334–335) and thereby obtaining complete systems of invariants for general algebraic forms. Complete systems were found for the $n$-ary quadratic and the binary cubic forms; and certain invariants were completely evaluated for the ternary cubic and the binary quartic. The following general theorems were established: (1) The $n$-ary $p$-ic form, when $p$ is even, has an unique non-vanishing invariant of weight $p$ and degree $n$; (2) The $n$-ary $p$-ic form, when $n$ is even and $p$ is odd, has a non-vanishing invariant of weight $np$ and degree $n^2$.

5. Dr. Krall presented an elementary proof of the converse of the Sylvester theorem on elimination; namely, that if the Sylvester Determinant is zero, then the two algebraic equations have a common root. The interest in the paper consisted both in the elementary nature of the analysis involved and the ingenuity of the methods employed.

J. S. TAYLOR, *Secretary*

# SOME RECENT ADVANCES IN ALGEBRA

SAUNDERS MACLANE, University of Chicago

The rapid exploitation of the new techniques of abstract algebra and the introduction of many new types of algebraic systems, though perhaps superficially confusing, has actually centered about several well defined lines of investigation: function-fields, linear algebras, $p$-adic fields, Lie algebras, matrices. The present direction and the close interrelation of these fields of investigation was clearly indicated in a conference on algebra recently held at the University of Chicago. We here attempt to summarize some of the ideas and inter-connections brought out at this conference, not in the fashion of a handbook or monograph, but rather as a survey for the generally interested mathematical public. After the necessary background has been filled in, we state the sorts of problems considered and the types of answers obtained. For detailed statements of results we refer to the skeleton bibliography at the end. In the last paragraph (§12) we attempt to state a general definition of algebra and a summary of its fundamental problems. To the authors whose ideas are here assembled we apologize for the omission, inevitably attendant upon such a summary, of the many difficulties and subtleties of their work.

**1. Algebraic geometry, power series, and valuations.** The relation of algebra to algebraic geometry was a lively topic of debate, stimulated by a paper of Lefschetz on the use of formal power series in algebraic geometry [**21**].* The origin of this algebraic-geometric connection might be described thus: the geometry of an algebraic curve can be reduced to the algebra of a certain corresponding field;† specifically, if $k$ denotes the field of all complex numbers, and if a curve is defined in the plane by an irreducible polynomial equation $f(x, y)=0$, then the corresponding field $K$ is the totality $k(x, y)$ of all rational functions $z=g(x, y)/h(x, y)$ with complex coefficients of the two quantities $x$ and $y$. It should be emphasized that $x$ and $y$ do not figure in this field as variables taking on values in the sense of analysis, but merely as quantities on which rational operations of addition, multiplication, *etc.*, can be performed, subject always to the proviso that the rational combination $f(x, y)$ is to be 0. The use of this "function field" $k(x, y)$ corresponding to the curve has one considerable advantage: a birational transformation of the curve will leave the corresponding field invariant.

Near a point $x=a$, $y=b$ of the curve $f(x, y)=0$, the variable $y$ can be expanded in one or more series of the form

$$(1) \qquad y = b + b_1t + b_2t^2 + \cdots, \qquad t = (x-a)^{1/n},$$

when $t$ is a suitably chosen integral or fractional power of $x-a$. This series (1),

* Numbers in brackets refer to references at the end of the paper.

† For the definitions of a few fundamental algebraic terms (group, field, and the like) one may refer to the glossary in Albert [**1**].

the Puiseux expansion, is usually treated as a convergent series defining an element or "cycle" of the algebraic function $y$ of $x$, but Lefschetz emphasizes the purely formal character of this expansion. The central fact is that the Puiseux series (1), if substituted in $f(x, y)$, yields an identity $f(x, y) \equiv 0$ in $t$. Conversely, any series (1) which formally satisfies the equation $f=0$ is the Puiseux series corresponding to some branch of the algebraic curve, or, alternatively, determines a corresponding point $P$ on the Riemann surface of the algebraic function $y(x)$. Any rational function $z=R(x, y)$ in the field $K=k(x, y)$ becomes a power series in $t$ after substitution of the series for $y$ and that for $x$, $x=a+t^n$;

$$z = t^\nu(a_0 + a_1 t + a_2 t^2 + \cdots), \tag{2}$$

where the integer $\nu$ may be positive, negative, or zero, and the $a_i$ are in the coefficient field $k$ of all complex numbers. The set of all such possible power series forms a field $k\{t\}$, because such series can be added, divided, and multiplied by the usual formal procedures. The point $P$ determined by (1) on the Riemann surface can be said to yield a one-to-one map (2) of the functions $z$ of the field $K$ onto a subset of the power series field $k\{t\}$, and this map is an *isomorphism* because any rational relation which holds between several functions of the field must hold between their corresponding power series.

This formal treatment by power series will apply now to algebraic curves even when the field $k$ of constants is not the classical complex number field, but any field $k$ which is algebraically closed, in the sense that every polynomial equation over $k$ has a root in $k$; that is, for any field $k$ in which the fundamental theorem of algebra holds. Lefschetz pointed out that on this basis most of the classical treatment of algebraic functions of one and two variables, as presented, for instance, by Picard or Weierstrass, can be developed as pure algebra. The abelian integrals and their classification can be managed algebraically in terms of abelian differentials $g(x, y)dx$. Moreover, the genus of an algebraic curve, an important invariant often defined topologically as the number of holes in the Riemann surface (a pretzel), can be defined algebraically, in terms of these integrals, as the maximum number of linearly independent differentials of the "first kind." One or two theorems of an essentially topological character cannot be generalized, but Lefschetz conjectured that the Riemann-Roch theorem could be treated algebraically not only by the usual arithmetic proof [**31**], but even by one of the classical geometric proofs [**21**]. This formal series treatment is of utility in other parts of analysis, notably in the treatment of algebraic functions of Dirichlet series [**26**].

The importance of the application of algebra to geometry was emphasized by Zariski's new proof that the singularities of an algebraic surface can be eliminated by suitable birational transformations. Previously the only sound proof of this important theorem had been one due to Walker [**32**], using analytic functions. If the algebraic surface $S$ is given in three-space by a single homogeneous algebraic equation $f(x_0, x_1, x_2, x_3)=0$, then the singular points are those points of the surface at which certain partial derivatives simultaneously vanish.

The theorem requires that the surface $S$ be represented in $n$-space with coördinates $y_0, y_1, \cdots, y_n$ by a non-singular surface $S'$ which is a birational transform of the original surface $S$ in the sense that the $y$'s are rational functions of the $x$'s and conversely. One difficulty in eliminating singular points arises because a transformation carefully constructed to eliminate one singular point may explode some other singular point into a whole singular curve. Zariski treats this difficulty by repeatedly using an "integral closure" process which gets rid of singular curves. In terms of the non-homogeneous coördinates $\xi_1 = x_1/x_0$, $\xi_2 = x_2/x_0$, $\xi_3 = x_3/x_0$, a rational function $\eta = \eta(\xi_1, \xi_2, \xi_3)$ is called *integral* if it satisfies a polynomial equation,

$$\eta^m + a_1(\xi_1, \xi_2, \xi_3)\eta^{m-1} + \cdots + a_m(\xi_1, \xi_2, \xi_3) = 0,$$

with first coefficient 1 and the other coefficients polynomials in the $\xi_i$. The non-homogeneous integral closure process is a birational transformation replacing the coördinates $\xi_1, \xi_2, \xi_3$ by integral functions $\eta_1, \eta_2, \cdots, \eta_n$ chosen so that every integral function of the $\xi_i$ is a polynomial in these new coördinates $\eta_j$. After this process has removed singular curves, the nature of one of the remaining isolated singular points $P$ depends on the sorts of curves (or "branches") passing through $P$ on the surface. Such a branch can be represented algebraically by a certain corresponding "valuation." The reduction of the singularity is effected by applying to suitable branches at $P$ a "uniformization lemma," which expresses the coördinates as holomorphic functions (integral power series) of two parameters $u$ and $v$—where these parameters can be chosen as rational functions of the original coördinates (for such valuations, *cf.* Zariski [36]).

The study of valuations occurs not only in algebraic geometry but also in algebraic number theory and other arithmetical questions. If the power series expansion (2) of an algebraic function $z$ at a point $P$ begins with a non-vanishing term $a_0t^\nu$, then the order or *value* $V(z)$ at $P$ may be defined to be the exponent $\nu$ of that term.

$$V(z) = \nu \quad \text{if} \quad z = a_0t^\nu + a_1t^{\nu+1} + \cdots, \qquad (a_0 \neq 0). \tag{3}$$

When $z$ has a zero at $t=0$, $V(z)$ is the order of this zero; when $z$ has a pole at $t=0$, $V(z)$ is the negative of the order of the pole. This valuation for a sum or product of two functions $z$ and $w$ can be shown to have the properties

$$V(zw) = V(z) + V(w), \qquad V(z + w) \geqq \min (V(z), V(w)). \tag{4}$$

Any real-valued function $V(z)$ defined in a field $K$ and having these two properties is known as a *valuation* of that field. Any such $V$ can also be converted into an "absolute value" $\|z\| = e^{-V(z)}$, with corresponding properties

$$\|zw\| = \|z\| \cdot \|w\|, \qquad \|z + w\| \leqq \max (\|z\|, \|w\|).$$

The second of these properties is even stronger than the usual triangle axiom $|z+w| \leqq |z| + |w|$ for complex numbers. Thus $\|z\|$ behaves like the absolute value of $z$, and limits with respect to it can be defined in the usual way. Especially important are the fields $K'$ which are *complete* with respect to an absolute

value, in the sense that every sequence $a_1, a_2, \cdots$ which is a Cauchy sequence has a limit $a$ in the field $K'$. For instance, the field $K'=k\{t\}$ of all formal power series (2) with $\|z\|=\exp(-V(z))$ given by (3) is a typical complete field.

Arithmetically any prime number $p$ determines a valuation $V_p$ of the integers, if $V_p(n)$ is defined as the exponent of the highest power of $p$ dividing $n$,

$$V_p(p^\nu b) = \nu, \qquad n = p^\nu b, \qquad b \text{ prime to } p. \tag{5}$$

The field $R$ of rational numbers $n/m$ with this "$p$-adic" valuation $V_p(n/m) = V_p(n) - V_p(m)$ can be embedded in a larger *$p$-adic number field* $R_p$ complete with respect to $V_p$. The structure of this field $R_p$ is determined by the behavior of the residues (mod $p$), which themselves form a field, the Galois field containing $p$ elements $0, 1, 2, \cdots, p-1$. Similarly the residues (mod $t$) of the power series (3) form a field, the field $k$ of coefficients. Such *residue-class fields* occur for other fields $K$ with the valuations $V$. Any complete field $K$ whose valuation function $V$ takes on only integral values is essentially determined by its residue-class field.

Just as in the Galois theory, the structure of such a field and the form of its subfields depends upon the possible "symmetries" of the field. A symmetry of a field $F$ is technically known as an automorphism: a map of the field upon itself which preserves rational relations. In other words, an *automorphism* $S$ of $F$ is a one-to-one correspondence $x \longleftrightarrow x^S$ of the field $F$ to itself such that sums and products correspond to sums and products:

$$(x + y)^S = x^S + y^S; \qquad (xy)^S = x^S y^S. \tag{6}$$

The successive application of two automorphisms $S$ and $T$ yields a new automorphism $x \longleftrightarrow (x^S)^T$ called the product $ST$. Under this product the automorphisms of $F$ form a group. For certain complete fields like the $p$-adic fields this group $G$ has been investigated by MacLane, who finds certain subgroups of $G$ analogous to Hilbert's "inertial" and "ramification" groups for prime ideals (*cf.* also MacLane [**24**]).

**2. The lattice representation of the structure of groups.** Recent algebraic investigations have shown that the structure of a group depends vitally upon the number and arrangement of its subgroups. An instance in point is the Jordan-Hölder theorem, which asserts that certain chains of relatively normal subgroups of a group must always have the same length. Two subgroups $H$ and $K$ of a given group can be combined in two ways, to yield the *intersection* $H \cap K$ of the two subgroups and the *union* $H \cup K$, which is the smallest subgroup containing both $H$ and $K$. Relative to these two operations the subgroups are said to form a *lattice* or *structure*. A lattice can also be defined abstractly in terms of the associative and other laws satisfied by intersection and union [**25, 6**].

Since the lattice of subgroups represents many of the properties of a group, one comes inevitably to the question: when is the nature of the group $G$ completely determined by its lattice of subgroups? Since $G$ is an abstract group, its

subgroup lattice $L$ will be exactly like the subgroup lattice of any group $G'$ *isomorphic* to $G$. To say that $G$ is isomorphic to $G'$ means that there is a one-to-one correspondence $T$: $A \longleftrightarrow A^T$ between the elements $A$ of the group $G$ and the elements $A' = A^T$ of the group $G'$ such that products correspond to products:

$$\text{(7)} \qquad \text{If} \quad A \longleftrightarrow A^T, \qquad B \longleftrightarrow B^T, \quad \text{then} \quad AB \longleftrightarrow A^T B^T.$$

Under this isomorphism $T$ each subgroup $H$ of $G$ goes into a subset $H^\tau$ of $G'$ composed of all images $A^T$ of elements $A$ of $H$. This correspondence $H \longleftrightarrow H^\tau$ is a one-to-one correspondence between the subgroups of $G$ and those of $G'$ such that unions and intersections are preserved:

$$\text{(8)} \qquad (H \cap K)^\tau = H^\tau \cap K^\tau, \qquad (H \cup K)^\tau = H^\tau \cup K^\tau.$$

Conversely, a one-to-one correspondence $H \longleftrightarrow H^\tau$ between the subgroups $H$ of $G$ and the subgroups $H'$ of another group $G'$ is called a *subgroup-isomorphism* of $G$ to $G'$ if property (8) holds. Hence the natural question: Are two subgroup-isomorphic groups $G$ and $G'$ necessarily (elementwise) isomorphic? An affirmative answer would mean that the structure of a group is actually determined by the lattice of its subgroups. The answer is not always affirmative, but Baer has an answer in the case of abelian groups. If the group is one of certain listed types, which are all "small" groups in the sense that they have relatively few subgroups, the answer is no; for other abelian groups, the answer is yes: subgroup-isomorphic groups are in fact isomorphic, and in many cases the only subgroup isomorphisms are those generated in the above fashion by ordinary isomorphisms.

**3. Generalized quaternions.** Systems of elements subject to the rational operations of addition and multiplication but not satisfying the commutative law for multiplication were first discovered in the guise of quaternions during the last century. A real quaternion algebra $Q$ is the set of all linear combinations $Y$ of four basal elements $1, i, j, ij$,

$$\text{(9)} \qquad Y = y_1 + y_2 i + y_3 j + y_4 ij,$$

where the components $y_1, \cdots, y_4$ are real numbers. The rational operations on such elements are *Scalar multiplication*: $Y$ is multiplied by a real number $b$ by multiplying each component $y_i$ by that number $b$; *Addition*: two elements $Y$ and $Z$ are added by adding corresponding components; *Multiplication*: the product of two elements $Y$ and $Z$ is found by using the usual distributive and associative laws and the following table for the products of the four basal elements:

$$\text{(10)} \qquad i^2 = -1, \qquad j^2 = -1, \qquad ij = -ji.$$

The system $Q$ is called a linear associative algebra because it is a set of elements closed with respect to these three operations, and because these operations sat-

isfy the usual algebraic laws (excluding the law $YZ=ZY$). This particular system $Q$ is an algebra *over* the field of real numbers because the components are elements from that field.

Wedderburn first introduced the consideration of such linear algebras over fields other than the real number field. Over the field of rational numbers, for instance, one has quaternion algebras $Q(\alpha, \beta)$ consisting of all elements $Y$ of (9) with rational components and a multiplication table

$$i^2 = \alpha, \qquad j^2 = \beta, \qquad ij = -ji, \tag{11}$$

where $\alpha$ and $\beta$ are fixed rational integers. Aside from the fact that the same algebra might be represented with different basal units $i'$ and $j'$ using different constants $\alpha'$ and $\beta'$, there are infinitely many essentially different quaternion algebras $Q(\alpha,\beta)$, two algebras being the same only if a certain "fundamental number" determined by $\alpha$ and $\beta$ is the same.* The importance and variety of the algebras possible over the field of rational numbers and other fields was recognized by Dickson, whose researches have paved the way for the far reaching theory of algebras over many types of fields.

The structure of an algebra depends upon the form of its subfields and subalgebras. The quaternion algebra $Q(\alpha, \beta)$ contains the set $R(\sqrt{\alpha})$ of all elements $x=y_1+y_2i$. This set is a field; it is obtained from the field $R$ of rational numbers by adjoining $i$ with $i^2=\alpha$, for $R(\sqrt{\alpha})$ consists of all rational functions of $i$ with coefficients in $R$. Each element $x=y_1+y_2i$ of the field $R(\sqrt{\alpha})$ has a conjugate $\bar{x}=y_1-y_2i$, and the correspondence $x\longleftrightarrow\bar{x}$ is a one-to-one correspondence of the field $R(\sqrt{\alpha})$ to itself. Since $\overline{x_1x_2}=\bar{x}_1\bar{x}_2$, $\overline{x_1+x_2}=\bar{x}_1+\bar{x}_2$, this correspondence preserves sums and products and hence is an automorphism of the field in the sense of (6). In terms of this subfield $R(\sqrt{\alpha})$ the general quaternion (9) can be written as

$$Y = (y_1 + y_2i) + j(y_3 - y_4i) = x_1 + jx_2 \tag{12}$$

where $x_1=y_1+y_2i$ and $x_2=y_3-y_4i$ are elements in the field $R(\sqrt{\alpha})$. The multiplication table (11) implies that

$$x_1j = (y_1 + y_2i)j = y_1j - y_2ji = j(y_1 - y_2i) = j\bar{x}_1,$$

and hence we can write a new table in terms of conjugates as

$$j^2 = \beta, \qquad xj = j\bar{x}, \quad \text{for} \quad x \quad \text{in} \quad R(\sqrt{\alpha}). \tag{13}$$

This automorphism formulation of the quaternions has latent potentialities for generalization.

**4. Arithmetics of quaternions.** Arithmetic can be considered in an algebra if one can select in the algebra a suitable set $J$ of integral numbers which, like the ordinary integers, form a *ring*: that is, a subset $J$ of the algebra closed under

* In terms of this fundamental number a certain simplified canonical form of the table (11) is possible, as shown in Albert [2] and Latimer [19].

addition, subtraction, and multiplication. In the quadratic algebraic number field $K=R(\sqrt{\alpha})$ every element $x=y_1+y_2\sqrt{\alpha}$ satisfies a quadratic equation

$$[t-(y_1+y_2\sqrt{\alpha})][t-(y_1-y_2\sqrt{\alpha})]=t^2-2y_1t+(y_1^2-y_2^2\alpha)=0$$

with rational coefficients. The number $x$ is called an *integer* if this equation has integral coefficients.* The set of all integers then forms a ring. For quaternions one could attempt the same definition, for the quaternion $Y$ of (12) satisfies a rational equation whose roots are $Y=x_1+jx_2$ and its conjugate $\overline{Y}=\bar{x}_1-jx_2$,

$$(t-Y)(t-\overline{Y})=t^2-2y_1t+N(Y)=0. \tag{14}$$

Here the constant term $N(Y)=Y\overline{Y}$ is the so-called *norm* of $Y$,

$$N(Y)=Y\overline{Y}=(x_1+jx_2)(\bar{x}_1-jx_2)=x_1\bar{x}_1-\beta x_2x_2. \tag{15}$$

If the previous definition of an integer is now applied, so that $Y$ is called integral if this equation (14) has rational integers as coefficients, then the set of integers unfortunately may no longer be a ring because there can be two integers whose sum is not integral.

As a substitute one may consider the set $\mathfrak{g}$ of all quaternions $Y=x_1+jx_2$ for which the numbers $x_1$ and $x_2$ of the field $R(\sqrt{\alpha})$ are integers. This set is too much dependent upon the particular choice of $j$, but has most of the requisite properties:

(C): $\mathfrak{g}$ is a ring;

$(R')$: every element of $\mathfrak{g}$ satisfies a polynomial equation (for instance the equation (14)) which has a first coefficient 1 and the remaining coefficients rational integers;

$(U')$: $\mathfrak{g}$ contains all the rational integers of $R$ and contains just as many linearly independent elements over $R$ as does the whole quaternion algebra $Q(\alpha,\beta)$.

This last property is immediate, for $\mathfrak{g}$ contains the four basal elements $1, i, j$, and $ij$ which are linearly independent by the construction of the algebra. A subset $\mathfrak{g}$ of an algebra having these three properties $(C)$, $(R')$, and $(U')$ is called an *order* of the algebra. Dickson recognized that a set of integers in more general algebras is more suitably defined as a *maximal order*† $\mathfrak{g}$; that is, an order which is contained in no larger order of the algebra. Every order is then contained in at least one maximal order; in particular, the ring $\mathfrak{g}$ defined above for the quaternions is not usually itself a maximal order, but can be extended so as to become one.

The arithmetic properties of a maximal order $\mathfrak{O}$ again depend upon a suitable type of subsystem of the order: the *ideals* of the order. A subset $\mathfrak{a}$ of $\mathfrak{O}$ is a *left ideal* of $\mathfrak{O}$ if the difference of two elements of $\mathfrak{a}$ is again an element of $\mathfrak{a}$, and if the product $ba$ is in $\mathfrak{a}$ for any $b$ in $\mathfrak{O}$ and $a$ in $\mathfrak{a}$. One also requires that an ideal

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* Zariski's treatment of the singularities of surfaces involves essentially the same notion of an integer (§1).

† This terminology is not that used by Dickson himself. See Dickson [9].

contain at least one rational integer. In particular for any element $a_0$ in $\mathfrak{O}$ the set $(a_0)$ of all elements $ba_0$ for $b$ in $\mathfrak{O}$ is an ideal. This ideal is the *principal ideal* generated by $a_0$. Similarly, left and right ideals can be considered in any ring.

**5. Quadratic forms and quaternions.** Quaternion algebras reflect many properties of quadratic and hermitian forms, because the norm of a quaternion is itself such a form. The norm $N(Y)$ as calculated in (15) is a simple hermitian form in the variables $x_1$ and $x_2$. Directly in terms of the original components $y_1, \cdots, y_4$ of (9) the norm $N(\overline{Y}) = Y\overline{Y}$ with $\overline{Y} = y_1 - y_2 i - y_3 j - y_4 ij$, becomes the quadratic form

$$N(Y) = y_1^2 - \alpha y_2^2 - \beta y_3^2 + \alpha\beta y_4^2.$$

This hermitian form interpretation has been investigated in terms of ideals by Latimer. The order $\mathfrak{g}$ of integers used above has a *basis* $1, j$, in the sense that $\mathfrak{g}$ consists of all linear combinations $x_1 + jx_2$ of 1 and $j$ with coefficients $x_1$ and $x_2$ integers of the quadratic field $R(\sqrt{\alpha})$. Similarly, any left ideal $\mathfrak{a}$ of this ring has a basis $\omega_1$ and $\omega_2$, so that any element of $\mathfrak{a}$ can be expressed as $x\omega_1 + y\omega_2$ for $x$ and $y$ again integers of $R(\sqrt{\alpha})$. The norm of this element $x\omega_1 + y\omega_2$ is, except for a constant factor, a form

$$ax\bar{x} + b\bar{x}y + \bar{b}x\bar{y} + cy\bar{y}. \tag{16}$$

This form is hermitian because it is equal to its own conjugate—the coefficients $a$ and $c$ are rational integers and the cross product terms have conjugate coefficients $b$ and $\bar{b}$. The determinant $b\bar{b} - ac$ of this form turns out to be the constant $\beta$ used in the multiplication tables (13) and (11).

An ideal $\mathfrak{a}$ with a given basis $\omega_1, \omega_2$, then corresponds to a hermitian form of determinant $\beta$ over the field $R(\sqrt{\alpha})$, and conversely. If the basis of the ideal is changed, one gets a new form equivalent to the first in the sense that it can be obtained from the first by a linear homogeneous transformation of determinant 1. The possible classes of equivalent forms can further be put into correspondence with certain* classes of ideals—where two ideals belong to the same class when they differ by principal ideal factors (with positive norms). This means that facts about the classes of forms can be translated into facts about classes of ideals. In particular, Latimer has found certain quaternion algebras with just one class of ideals. This statement means that every (regular) ideal is a principal ideal and hence that the ideal structure of the ring is as simple as possible [**18, 20**].

The quaternion algebras can also be used as a starting point for the treatment of properties of quadratic forms in many variables, of the type

$$f(x_1, x_2, \cdots, x_n) = \sum a_{ij}x_i x_j \qquad (i, j = 1, \cdots, n),$$

where the coefficients $a_{ij}$ are in the field $R$ of rational numbers and can be chosen so that $a_{ij} = a_{ji}$. Many questions of the representation of numbers by such forms

* Because the ring $\mathfrak{O}$ is not a maximal order, it is necessary here to consider not all ideals, but only certain "regular" ideals.

lead to the "representation of zero" by numbers in any field $F$ containing the rational field $R$. The form $f$ is said to *properly represent* 0 in the field $F$ if there are numbers $x_1, x_2, \cdots, x_n$ not all zero in the field $F$ such that $f(x_1, \cdots, x_n)=0$. Here $F$ might be the field $R_0$ of all real numbers, which is complete with respect to the ordinary absolute value, or a $p$-adic number field $R_p$ which contains $R$ and is complete with respect to the $p$-adic valuations of (5), §1. If $f$ represents 0 in the original field $R$, then it certainly represents 0 in any one of these larger fields $R_0$ and $R_p$. A theorem due to Hasse now states a converse: If the form $f(x_1, \cdots, x_n)$ properly represents 0 in the real number field $R_0$ and in every $p$-adic field $R_p$, then it also properly represents 0 in the field of rationals [**34, 11**]. This type of theorem, which starts from the behavior "locally" for each prime $p$ and derives the behavior in the original field $R$, has attracted much current interest (see also the class field theory discussed in §11). Artin has now a new and elegant proof for this theorem of Hasse, using an induction on the number of variables combined with a treatment of the four variable case in terms of the norm forms of quaternion algebras. His proof also applies when $R$ is replaced by an algebraic number field.

**6. The structure of algebras.** We turn now to the general definition of a linear associative algebra over an arbitrary field $F$. An algebra $A$ consists of all linear combinations

$$a = \alpha_1 u_1 + \cdots + \alpha_n u_n = \sum \alpha_i u_i \tag{17}$$

of $n$ basal elements $u_1, \cdots, u_n$ with coefficients $\alpha_i$ in the field $F$. The sum of two such elements is given by

$$(\sum \alpha_i u_i) + (\sum \beta_i u_i) = \sum (\alpha_i + \beta_i) u_i \qquad (i = 1, \cdots, n); \tag{18}$$

and the scalar product of an element $a$ by an element $\beta$ of the field $F$ is determined by

$$\beta(\sum \alpha_i u_i) = \sum (\beta\alpha_i) u_i, \qquad (i = 1, \cdots, n). \tag{19}$$

The product of two elements $a$ and $b$ in the algebra is defined by the formula

$$(\sum \alpha_i u_i)(\sum \beta_j u_j) = \sum (\alpha_i \beta_j) u_i u_j, \qquad (i, j = 1, \cdots, n) \tag{20}$$

which is completed by a table giving the product $u_i u_j$ of any two basal elements as some particular element of the algebra. This multiplication table must be such that the product $a \cdot b$ satisfies the associative law. The number $n$ of basal elements of the algebra $A$ is its *order*; it can be characterized by the statement that the $n$ basal elements are linearly independent, while any $n+1$ elements of $A$ are necessarily linearly dependent.

We shall assume that the algebra $A$ has a unit element 1 with the property that $1 \cdot a = a \cdot 1 = a$ for every $a$ of the algebra—for if there were no such element we could construct it and add it to the algebra. The algebra then contains the elements $1 \cdot \alpha$ which behave like the corresponding $\alpha$'s of the field $F$ and thus

can be identified with the elements of $F$, so that the algebra then contains its coefficient field $F$.

Certain types of algebras are important. A *division algebra* is an algebra in which every $a \neq 0$ has an inverse $a^{-1}$ such that $a^{-1}a = aa^{-1} = 1$. An algebra is *simple* if it contains no proper subset which is a two-sided ideal (both a left and a right ideal, as at the end of §4). Such algebras exist: any division algebra is necessarily simple in this sense. A simple algebra is *normal* if it is as uncommutative as possible: that is, if $b$ is an element of the algebra such that $bx = xb$ is true for every $x$, then $b$ is necessarily an element $b = \beta$ of the coefficient field $F$.

A major problem in the study of linear algebras is the construction of all algebras out of fields or out of simpler algebras in systematic fashion. The object of such investigations is then a more concrete representation for algebras than that given by the general definition. Matrices yield algebras; for the set of all $m \times m$ square matrices with elements in a field $F$ is an algebra $M_m$ (a so-called *total matrix algebra*). The $m^2$ matrices with 1 in one position and 0 elsewhere can be considered as its $n = m^2$ basal elements $u_1, \cdots, u_n$. This matrix construction can be generalized if we consider the set of all $m \times m$ matrices with the usual matrix multiplication, but with elements themselves taken from a given normal division algebra $D$ over the field $F$. The algebras which can be obtained in this fashion are all the normal simple algebras; they are the important type because all other algebras can be decomposed into such normal simple ones by certain structure theorems of Wedderburn [**33**, ch. 10].

New algebras can be formed by combinations of old ones. If $A$ and $B$ are two algebras over a field $F$ one can construct a new algebra, the direct product $A \times B$, which contains both the given algebras $A$ and $B$. Specifically, $A \times B$ consists of all sums

$$c = b_1 u_1 + \cdots + b_n u_n \tag{21}$$

formed from the basal elements $u_i$ of $A$ in the same fashion as in the element (17) of $A$, except that the coefficients $b_i$ are now elements of $B$. Multiplication is determined by the multiplication in $A$ and $B$ and by the rule that $u_i b = b u_i$. To be specific, any $2 \times 2$ matrix with elements $b_{ij}$ in a division algebra $B$ can be written in the form (21) as

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + b_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + b_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence the normal simple algebra of all such matrices is simply the direct product $M_2 \times D$ of the division algebra $D$ by the $2 \times 2$ total matrix algebra!

**7. Cyclic algebras.** Since the matrix algebras are readily written down, the construction of normal simple algebras becomes essentially the construction of division algebras. Here explicit constructions are possible after the manner of quaternion algebras. The quaternion algebras have a multiplication table (13) depending on a certain quadratic subfield $R(\sqrt{\alpha})$ and on an automorphism

$x \longleftrightarrow \bar{x}$ of this quadratic subfield. There are similar algebraic extensions $K=F(\theta)$ of other fields $F$, obtained by adjoining to $F$ a root $\theta$ of an irreducible polynomial equation $f(x)=0$ whose coefficients are in $F$. The degree $m$ of this irreducible equation is called the *degree* of the field $K$ over $F$. The nature of this extension depends on the structure of the group $G$ of those automorphisms $x \longleftrightarrow x^S$ of $K$ (*cf.* the definition of (6)) which leaves every element of $F$ fixed. The simplest case arises when there is one such automorphism $S$ with $S^m=I$, $I$ the identity automorphism, so that the order $m$ of $S$ in the group $G$ is exactly the degree of the field. Every automorphism of $G$ is then necessarily one of the powers of $S$, so that the automorphism group $G$ is the cyclic group $G=[I, S, S^2, \cdots, S^{m-1}]$. A field $K$ with such a group over $F$ is a *cyclic* field and is often denoted by $Z$. Such fields certainly exist—for let $F$ be the field $R(\omega)$ obtained by adjoining to the rationals a $p$-th root $\omega$ of unity, and let $Z=F(\theta)$ be generated by the adjunction to $F$ of a $p$-th root $\theta=\sqrt[p]{\alpha}$ not already in $F$. Then $\omega\theta$ is also a $p$-th root of $\alpha$, and the correspondence $S$ defined by

$$\theta^S = \omega\theta, \qquad [b(\theta)]^S = b(\omega\theta),$$

for any polynomial $b(\theta)$ in the field $Z=F(\theta)$, carries a root $\theta$ of $t^p-\alpha=0$ into another root $\omega\theta$ of the same equation and therefore can be shown to be an automorphism. Its powers are

$$S^i\colon \quad \theta^{S^i} = \omega^i\theta, \qquad S^p = I,$$

so that the group and field are actually cyclic.

From any such cyclic field $Z=F(\theta)$ with its automorphism $S$ of order $m$, one can construct corresponding "cyclic" algebras over $F$. The algebra is to consist of all elements

$$a = z_0 + z_1y + z_2y^2 + \cdots + z_{m-1}y^{m-1}, \tag{22}$$

where the coefficients $z_i$ are now elements of the cyclic field $Z$ and the multiplication rules are given in terms of a fixed element $\gamma$ of $F$ by the table

$$y^m = \gamma, \qquad zy = yz^S, \quad \text{for} \quad z \quad \text{in} \quad Z. \tag{23}$$

The symbol $(Z, S, \gamma)$ designates the algebra so obtained. The quaternions of (12) and (13) are the special case when $Z$ is the quadratic field $R(\sqrt{\alpha})$, $m=2$, $y=j$, and $\gamma=\beta$. Such a cyclic algebra need not always be a division algebra, but the important fact is that any normal division algebra over a field $F$ of algebraic numbers is necessarily a cyclic algebra [**8**, ch. 7].

The structure of such cyclic algebras is now a matter of central interest. The direct product of two cyclic algebras of the same degree over a field $F$ can be itself represented as another cyclic algebra multiplied directly by a suitable total matrix algebra. A cyclic algebra can even be combined with itself in this fashion to form a new cyclic algebra! The properties of this "direct power" of an algebra are fundamental. Albert [**3**] has shown that these properties may be obtained elegantly by systematically using another simple type of algebra, called the

*cyclicsemi-fields*. Such a semi-field is a commutative algebra consisting of all $t \times t$ diagonal matrices whose elements are taken from a given cyclic field $Z$. Such a diagonal matrix, which has zeros everywhere but on the main diagonal, may be represented by simply writing down the diagonal elements $\{z_1, \cdots, z_t\}$. The automorphism $S$ of the field then generates an automorphism of these matrices which may be obtained by permutating the diagonal elements cyclicly and then applying $S$ to the last element

$$\{z_1, \cdots, z_t\} \longleftrightarrow \{z_2, \cdots, z_t, z_1^S\}.$$

Relative to this correspondence the semi-fields behave very much like ordinary cyclic fields.

An outstanding problem concerns the nature of normal division algebras and normal simple algebras over general fields $F$—where there might be division algebras which are not cyclic algebras. The cyclic algebras $A = (Z, S, \gamma)$ (*cf.* (23)) are characterized by the fact that they contain a cyclic subfield $Z$ which is maximal (contained in no larger subfield). Any normal division algebra over $F$ still contains a maximal subfield $K$—but this subfield may have too few automorphisms to be normal* or may not have a cyclic automorphism group. The multiplication table in the algebra may still be described by a more complicated multiplication table depending on certain automorphisms after the fashion of (23). The structure of such algebras has been analyzed by Brauer [7] who makes considerable use of two tools: Firstly, any algebra $A$ over $F$ can be extended to a larger algebra $A_K$ over a given field $K$ containing $F$—one need only consider the element (17) of the algebra with the same basal units, but now with coefficients $\alpha_i$ taken from the larger field $K$. Secondly, any algebra $A$ can be considered as an algebra of certain matrices with components in $F$, for any element $a$ of the algebra has certain products with the basal elements,

$$au_i = \alpha_{i1}u_1 + \cdots + \alpha_{in}u_n. \tag{24}$$

The correspondence between the element $a$ and the matrix $a^* = (\alpha_{ij})$ "represents" $A$ as a matrix algebra. By simplifying these matrices one obtains reductions of this representation to certain irreducible representations over $F$ or $K$; their construction by Brauer yields information about the algebra.

**8. Arithmetic of cyclic algebras.** The arithmetic study of a general algebra can be reduced to the study of the normal simple algebras. Here again the basic necessity is the definition of a set of integers—and such sets are defined as maximal orders in exactly the fashion discussed above (§4) for the quaternion algebras. But the definition of such sets and the description of their properties is not the only problem. How can these maximal orders be explicitly found in the case of particular algebras, say for cyclic algebras? How many such orders are there? Some answers have been found for the special case when the cyclic algebra is a quaternion algebra, and Hull has extended the method to any cyclic algebra

* A field $K$ over $F$ is *normal* if it is obtained by adjoining to $F$ *all* roots of an equation; a normal field of degree $n$ over $F$ always has $n$ automorphisms over $F$.

generated over the field of rational numbers by a cyclic field $Z$ of odd prime degree. Such a cyclic algebra might be represented in several different ways by the generation $(Z, S, \gamma)$ of (22) and (23), but there is a certain canonical generation which is found by considering the algebra not over the rational number field, but over the larger $p$-adic number fields constructed in §1, from the valuation (5) belonging to a rational prime $p$. In this generation one might naturally define an integer to be an element (22) of the algebra in which the coefficients $z_i$ are integers of the algebraic number field $Z$. This set of integers is an order, but not a maximal order, and Hull [**12**] finds that it can be embedded in exactly $n$ distinct maximal orders which can be explicitly represented in terms of the solutions of certain congruences. All other maximal orders can be obtained from any one of these. This method of finding formulas for maximal orders is a fruitful one, for it has been extended by Perlis [**27**] to the case of cyclic algebras over the rationals where the degree of the cyclic field is not a prime, but a power of a prime.

In any maximal order $\mathfrak{D}$ the study of arithmetic questions of divisibility leads inevitably to the study of the divisors of 1. Such numbers, called *units* of $\mathfrak{D}$, are simply the elements $a \neq 0$ in $\mathfrak{D}$ which have reciprocals $1/a$ also in the order $\mathfrak{D}$. These units form a group under multiplication. Recently [**10**] advances have been made in studying the totality of units and the structure of the group which they form; the typical theorem gives an expression of the units in terms of a finite set of "fundamental" units. Hull has successfully applied these methods to the orders of quaternion algebras, where the group $G$ can be represented geometrically as a certain Fuchsian group.

The ideal theory in a maximal order of a normal simple algebra (we recall that such algebras include the cyclic algebras) has been developed extensively. MacDuffee [**23**] has shown that many of the requisite computations with ideals can be carried out effectively by the use of matrices. This use of matrices depends directly upon the representation of an algebra in terms of matrices, discussed above in (24). Specifically, an ideal $\mathfrak{a}$ in a maximal order $\mathfrak{D}$ of such an algebra over the rational number field always has a basis $\omega_1, \cdots, \omega_n$ such that the ideal consists of all linear combinations

$$b = \beta_1\omega_1 + \cdots + \beta_n\omega_n, \tag{25}$$

for rational integral coefficients $\beta_i$. The elements of this basis of the ideal can be represented in terms of the basal elements of the algebra in the form

$$\omega_i = g_{i1}u_1 + g_{i2}u_2 + \cdots + g_{in}u_n, \qquad i = 1, \cdots, n. \tag{26}$$

The matrix of these coefficients $(g_{ij})$ is said to correspond to the given ideal. MacDuffee has then determined when two matrices correspond to different bases of the same ideal and when a matrix corresponds to an ideal. Greatest common divisor computations on the ideals can then be done elegantly in terms of computations of the greatest common left divisors of the corresponding matrices. Furthermore, the question of the equivalence of two ideals (in the sense

that ideals are equivalent when they differ by factors which are principal ideals, *cf.* §4), which was discussed above in connection with hermitian forms, can be treated in terms of matrices. Thus the calculation of the number of classes of equivalent ideals becomes tangible.

**9. Canonical forms for matrices.** A matrix $A=(a_{ij})$ can always be considered as a linear transformation $y_i=\sum a_{ij}x_j$ $(i=1, \cdots, n)$ which carries a vector $(x_1, \cdots, x_n)$ with $n$ components into another vector $(y_1, \cdots, y_n)$. A change of the coördinate system to which these vectors are referred changes the matrix of the transformation to a new matrix.

$$B = TAT^{-1} \qquad (T \text{ a non-singular matrix}). \tag{27}$$

Such a $B$ is called *similar* to $A$. Elementary divisors are ordinarily used to reduce $A$ to a canonical form under such similarity transformations. Ingraham has considered this problem in the more general case when the matrix $A$ is one whose elements are taken not in a field but in a division algebra. By using systematically the properties of the space which $A$ transforms he has been able to carry through the study of similarity in this case [**14, 15**]. Because the elements of the matrix are taken from an algebra without a commutative law, the polynomials in a matrix, which are important for the theory, must be redefined in a suitable manner. More generally, the equation (27) can be written in the form $BT=TA$, and this equation can be studied even when $T$ is singular, to determine the maximal possible rank for a matrix $T$ satisfying the equation. Analogous methods work for other matrix equations [**13**].

When the elements of a matrix $A$ are complex numbers, the matrix $A$ is called *hermitian* if $A^*=A$, *unitary* if $AA^*=I$, where $A^*=(\bar{a}_{ji})$ is the conjugate transpose of the matrix $A$. The reductions of such matrices to canonical form by suitable similarity transformations are important classical problems. These problems can now be treated, as in Williamson [**35**], as a special type of a general reduction of certain normal matrices. A matrix is called *normal* if $AA^*=A^*A$. Both unitary and hermitian matrices have this property. It can be shown that a matrix $A$ is normal in this sense if and only if $A$ can be expressed as a polynomial $f(A^*)$ in its conjugate transpose matrix $A^*$. This leads to a more general definition: A matrix $B$ is *normal* relative to a non-singular hermitian matrix $H$ if $BH=Hf(B^*)$ for some polynomial $f$. This includes the previous case with $H=I$. The canonical forms for such relatively normal matrices have been successfully treated by Williamson, using similarity transformations like (27) subject to the natural side condition that the given hermitian matrix $H$ be unchanged by $T$ in the sense that $THT^*=H$.

**10. Lie algebras.** In the study of the so-called commutators of a group of continuous transformations certain non-associative algebras arise. These algebras differ from the linear associative algebras considered above essentially in the replacement of the associative law of multiplication by two other laws:†

† These laws are not unfamiliar. They are satisfied by the vector product of two vectors. For complete definition of terms, see Jacobson [**16, 17**].

$$(28) \qquad [x, y] = -[y, x], \qquad [x[y, z]] + [y[z, x]] + [z[x, y]] = 0.$$

Here, in accord with usage, the product of two elements in a Lie algebra is denoted by $[x, y]$. Such a Lie algebra may readily be obtained from an associative algebra $A$ by the simple device of defining the multiplication of the Lie algebra in terms of the given associative multiplication by the equation

$$[x, y] = xy - yx,$$

so that $[x, y]$ is the "commutator" of $x$ and $y$. The fact that the so-defined commutator actually satisfies the conditions (28) can be readily verified. By using a suitable automorphism $S$ of the given associative algebra $A$ it is possible to define still other Lie algebras and even to describe their automorphism groups. By systematic use of such construction Jacobson has reduced the problem of determining all "simple" Lie algebras over ordinary fields to questions in underlying associative algebras. These questions can be solved explicitly when the algebra is one over the field of real numbers, but here, as in Cartan's classical theory of Lie groups, certain troublesome exceptional algebras can arise.

**11. Local class field theory.** An extraordinary combination of the study of fields with valuations, algebraic extensions of fields, automorphism groups of fields and normal simple algebras over fields has arisen in the class field theory. This theory aims at an explicit description of the fields $K$ which are abelian extensions of an algebraic number field $k$. An *abelian* extension of $k$ is a normal extension of $k$ (*cf.* footnote in §7) such that the group of all automorphisms (6) of $K$ which leave $k$ fixed is *abelian*. The class field theory describes these extensions in terms of automorphism groups by a device which represents the automorphism group in terms of an isomorphic group derived from the multiplicative group in the field $k$ [**4**]. Furthermore, the character of these abelian extensions proves to be intimately connected to the variety and the properties of the normal simple algebras and cyclic algebras possible over $k$. For many purposes this study of fields and algebras over the rational number field $k$ can be simplified by treating each prime number in $k$ separately; one then considers the extensions not of $k$ but of the $p$-adic field $k_p$ which is complete with respect to the $p$-adic valuation (5) defined by this prime $p$. Schilling has shown in several papers [**28, 29, 30**] that most of the theorems of local class field theory can also be obtained over fields $k$ complete with respect to more general valuations. The $p$-adic valuations (5) are *discrete* in the sense that the value $V(a)$ of every element $a$ of the field is an integer, but the abelian extensions turn out to have much the same structure if one considers valuations in which $V(a)$ runs over all rational numbers or even includes some irrationalities.

**12. What is algebra?** To summarize this survey we shall essay here a description of the general tendency of these investigations. Algebra concerns itself with the postulational description of certain systems of elements in which some or all of the four rational operations are possible: fields, linear algebras, Lie algebras, groups. The abstract or postulational development of these systems must

then be supplemented by an investigation of their "structure." Under "structure" we include:

(a) the number and interrelations of the subsystems of a given system, either subsystems just like the whole system (lattice of subgroups), or subsystems with especially characteristic properties (sets of integers, maximal orders, ideals, subfields of an algebra, *etc.*);

(b) the group of automorphisms of a system, and connections between the subgroups of this group and the subsystems of the given system (Galois theory, class field theory);

(c) the construction of all systems of specific types out of simpler systems of the same or other types (the construction of cyclic algebras and matrix algebras, the reduction of a given surface to a birationally equivalent surface without singularities, construction of Lie algebras);

(d) alternatively, the description of given systems as subsystems of larger systems (complete fields, power series fields);

(e) criteria or invariants to determine when two explicitly but differently constructed systems are abstractly the same or *isomorphic* (the canonical generation of a cyclic algebra; the genus as an invariant defined by the differentials of a function-field).

With this explanation we venture the characterization:

*Algebra tends to the study of the explicit structure of postulationally defined systems closed with respect to one or more rational operations.* This summary does not account well for the use of topological operations in algebra, of which the valuations form but one example. Furthermore, the reduction of matrices to canonical forms is only indirectly an instance of the criteria intended in (e). As with many hyper-generalizations, our statements fit the facts only when the facts are first slightly distorted!

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# A LEMMA ON SQUARES

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Some time ago the author was led to consider a problem which may be stated roughly as follows: *Given a fixed unit square, how many non-overlapping unit squares may touch the perimeter but no other point of the fixed square?*

For convenience let us speak of an *admissible set* as any set of non-overlapping unit squares touching the fixed square as above. The problem appears to be an extremely simple one and a little reflection shows that an admissible set of eight squares may be selected, but certainly, no such set of more than thirteen squares exists. The layman, thinking of a checkerboard, will admit as obvious the statement that *no admissible set can have more than eight squares.* This assertion, however, was found by no means easy to prove in spite of the elementary discussion offered below.*

Before going any further we shall state the lemma in a more precise form.

LEMMA: *Let there be given in the xy-plane any finite system* $s_0, s_1, \cdots, s_n$ *of closed squares with side length one, such that*

1) *No two squares of the set* $s_0, s_1, \cdots, s_n$ *have interior points in common.*

2) *Each square of the set* $s_1, \cdots, s_n$ *has at least one boundary point in common with* $s_0$. *Then* $n \leqq 8$.

*Proof.* Join $c_0$, the center of $s_0$, by a ray to $c_j$, the center of $s_j$, $(j=1, \cdots, n)$.

The center $c_0$ cannot be on the same ray with two or more of the centers $c_j$ $(j=1, \cdots, n)$ since conditions 1) and 2) imply that†

$$1 \leqq d(c_0, c_j) \leqq \sqrt{2}, \qquad 1 \leqq d(c_i, c_j), \tag{1}$$

where $i \neq j$; $i=1, \cdots, n$; $j=1, \cdots, n$. Hence we may assume that the squares are numbered in a counterclockwise direction, so that $c_j$ and $c_{j+1}$ are on adjacent rays.

Consider the rays to $c_j$ and $c_{j+1}$, and let $d(c_0, c_j)=x$; $d(c_0, c_{j+1})=y$; $d(c_j, c_{j+1})=z$; $\angle c_jc_0c_{j+1}=\phi$. Now from (1) it follows that

$$1 \leqq x, y \leqq \sqrt{2}, \qquad 1 \leqq z. \tag{2}$$

We shall prove the lemma by showing that $\cos\phi < \cos 2\pi/9$; *i.e.*, $\phi > 2\pi/9$, so that there cannot be nine rays. We have

$$\cos\phi = \frac{x^2+y^2-z^2}{2xy} \leqq \frac{x^2+y^2-1}{2xy} \equiv f(x, y).$$

Let us examine $f(x, y)$ for $x$ and $y$ in the range given by (2). For fixed $y$, we have

* An alternate and somewhat lengthy argument was presented to the Mathematics Club at Ohio State University. The present proof is based on a subsequent remark to the author.

† By $d(a, b)$ we shall understand the distance between the points $a$ and $b$.

$$f_x(x, y) = \frac{x^2 - y^2 + 1}{2x^2 y} \geqq \frac{x^2 - 1}{2x^2 y} \geqq 0.$$

Hence for fixed $y$, the function $f(x, y)$ takes on its maximum on the boundary, at $x = \sqrt{2}$. Similarly, for fixed $x$, the function $f(x, y)$ takes on its maximum on the boundary, at $y = \sqrt{2}$. Therefore

$$\cos \phi \leqq \max [f(x, y)] = 3/4 < \cos \frac{2\pi}{9},$$

whence $\phi > 2\pi/9$. Thus the proof is complete.

Before taking up any applications of the lemma it should be noted that though this proves eight to be the maximum number of squares in any admissible set it does not say anything about the *position* of the squares of the set. One position in which the admissible set has eight squares is suggested by a checkerboard. It is natural to ask: *Does the admissible set attain its maximum of eight squares in any other position?* No definite information appears to be available.

The lemma has an application in the theory of functions of sets; indeed, the question answered by the lemma was aroused by a paper of Banach on this subject.*

For the sake of simplicity we shall consider $F(s)$ a function of squares defined for all closed squares $s$ comprised in a fixed closed square $S_0$. We have then the following two definitions of a function of *bounded variation*:

1. $F(s)$ is of *bounded variation* if there exists a positive constant $M$ such that for all sets of squares $s_1, \cdots, s_m$ satisfying the condition that (a) no two of the squares $s_1, \cdots, s_m$ have points in common, it is true that (b) $\sum_{j=1}^{m} |F(s_j)| \leqq M$.

2. The second definition differs from the first only in that (b) is required to be a consequence of the following condition (a′) no two of the squares $s_1, \cdots, s_m$ have *interior* points in common.

Banach does not give an explicit proof of the equivalence of these two definitions, but it is easily seen that an explicit proof would depend on the above lemma in the following modified form.

*If $s_0, s_1, \cdots, s_n$ is any system of closed squares in the xy-plane such that*
1) *no two of the squares $s_0, s_1, \cdots, s_n$ have interior points in common,*
2) *each $s_1, \cdots, s_n$ has at least one boundary point in common with $s_0$,*
3) *the area of $s_0 \leqq$ the area of $s_j$, $(j = 1, \cdots, n)$, then $n \leqq 8$.*

As a matter of fact, an immediate consequence of the above is the following:

LEMMA. *Given any finite system of closed squares such that no two have common interior points, the squares of the system can be distributed among at most nine groups, $G_1, \cdots, G_9$, such that in any one of the groups no two squares have points in common.*

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* Sur une classe de fonctions d'ensemble, Fundamenta Mathematicae, vol. 6, 1924, pp. 170–188.

*Proof.* The statement is trivial if there is but one square. Assume the statement to be true if there are $n$ squares. Consider any system of $(n+1)$ squares. Let $s_0$ be a square which is not larger than any of the others. By the preceding lemma at most eight squares of the system may have a boundary point in common with the boundary of $s_0$. Omitting $s_0$ we have a system of $n$ squares which may be distributed among at most nine groups having the property mentioned above. At least one of these groups has no representative touching $s_0$. Let $s_0$ be a member of that group and our induction is complete.

The equivalence of the two definitions for $F(s)$ of bounded variation is now an immediate consequence of the above lemma. In fact, suppose $F(s)$ is of bounded variation by the first definition. Let $s_1, \cdots, s_m$ be any system of closed squares such that no two of the squares have common interior points. The system can be divided into at most nine groups such that no two of the squares in any one group have points in common with each other. Hence

$$\sum_{s \in G_j} | F(s) | \leqq M, \qquad (j = 1, \cdots, 9).$$

Therefore

$$\sum_{k=1}^{m} | F(s_k) | = \sum_{j=1}^{9} \sum_{s \in G_j} | F(s) | \leqq 9M.$$

And now to finish the proof of the equivalence we merely have to point out that the second definition obviously implies the first.

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## ON SOME SIMPLE TYPES OF SEMI-RINGS*

H. S. VANDIVER, University of Texas

In a previous paper† the writer described briefly certain types of finite algebras, closely related to ordinary arithmetic, in which the cancellation law of addition does not always hold. In spite of their elementary character, these systems appear to be new. Dr. R. Brauer informed the writer that Dedekind had considered, in some of his lectures, a set of elements

$$a_1, a_2, \cdots$$

which contained repetitions of the type described in connection with the set (1) of the paper just referred to, or the set (C) of the present paper, but that he did not set up any algebra based on this idea.

In the present paper we shall develop notions of the first paper further. The treatment is largely independent of that previously given since we shall adopt at the outset the point of view which is mentioned but not developed in §5 of the first paper. In effect, here we derive the theory from elementary arithmetic,

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* Presented to the Texas Section of the Mathematical Association of America at its meeting in Houston, April, 1937.

† Bulletin of the American Mathematical Society, vol. 40, 1934, pp. 914–20.

whereas previously the theory was developed independently of elementary arithmetic.

Consider the natural numbers and introduce a relation between them called $(i, j)$ equivalence, where $j \geqq i$, symbolized by $\cong$ and defined by the following (each letter denoting a natural number):

I. If $a < i$ then $a \cong b$ if and only if $a = b$, and if $a \geqq i$ then $a \cong b$ if and only if $a \equiv b \pmod{m}$, where $m = j - i + 1$.

In view of the above there are exactly $j$ natural numbers which are not $(i, j)$ equivalent, namely $1, 2, \cdots, j$. Also, in view of the congruence properties of natural numbers and the above definition, if $A$, $B$, and $D$ denote combinations involving natural numbers, symbols denoting them, addition $(+)$ also signs, multiplication $(\cdot)$ signs, (understood) and parentheses of the types employed in elementary algebra then we have:

II. $$A \cong A.$$

III. $$\text{If } A \cong B \quad \text{then} \quad B \cong A.$$

IV. $$\text{If } A \cong B \quad \text{and} \quad A \cong D \quad \text{then} \quad B \cong D.$$

V. The substitution axiom, which includes the theorem:

$$\text{If } A \cong B \text{ then } A + D \cong B + D.$$

Also, if $a$, $b$, and $d$ denote natural numbers we have:

VI. $$a + (b + d) \cong (a + b) + d.$$

VII. $$a + b \cong b + a.$$

VIII. $$a(bd) \cong (ab)d.$$

IX. $$ab \cong ba.$$

X. $$a(b + d) \cong ab + ad.$$

Now let us consider a set of elements

(C) $$C_1, C_2, \cdots, C_i, \cdots, C_j, \cdots$$

$i$ and $j$ denoting natural numbers, which may be put into one-to-one correspondence with the natural numbers, namely, $a$ corresponds to $C_a$. Then we may state a set of postulates numbered 2, 3, 4, and 5, obtained in order from II, III, IV, and V by replacing natural numbers by $C$'s in the combinations and the symbol of equivalence $(\cong)$ by the symbol of equality $(=)$. Similarly we may set up a system of postulates numbered 6, 7, 8, 9, and 10 for the $C$'s by replacing the natural numbers mentioned in VI–X inclusive by the corresponding $C$'s and $(\cong)$ by $(=)$. If this is done we have a correspondence for the two systems which is an extension of the ordinary idea of isomorphism. In view of this correspondence *the set $C$, containing $j$ distinct (unequal) elements shall be said to exist and to*

*have the properties given in* I–X. The cancellation law of addition does not hold for the elements (C) since if $i=4$, $j=7$ we have

$$C_3 + C_1 = C_7 + C_1,$$

but

$$C_3 \neq C_7.$$

In the previous paper I pointed out that this algebra is a semi-ring, the latter being a system of elements forming a semi-group under addition, a semi-group under multiplication, and the right and left distributive laws hold. Further a semi-group is a system closed under an associative operation for which the equivalence postulates and the substitution postulate hold. Now semi-rings appear often in elementary number theory, for example, the system of natural numbers under ordinary addition and multiplication. But in the study of such a system it is usually found convenient to consider it as being imbedded in a ring or field, as the ring of all rational integers in elementary number theory, and the field of all complex numbers as in analytic number theory. Some of the principal landmarks of arithmetic have obviously been associated with adjunctions of this kind. *But the set* (C) *does not have such properties in general*, in fact we have the

THEOREM. *A semi-ring exists which cannot be imbedded in any ring whatever.*

Suppose in (C) that $i=2$ and $j=5$; then $C_6=C_2$, and

$$C_1 + C_1 = C_5 + C_1, \tag{1}$$

but

$$C_5 \neq C_1.$$

We shall show that this semi-ring $S$ has the property described in the Theorem.

Now a ring may be defined as a semi-ring whose additive semi-group is a commutative group. If $S$ is contained in a ring $R$, it follows from the group property under addition that a zero element $A_0$ exists in $R$ and if $A_i$ is any element of $R$ then there is an element $A_i'$ in $R$ such that

$$A_i + A_i' = A_0;$$

and since $S$ is in $R$ then for any $C_k$ in $S$ there is an element $C_k'$ in $R$ such that

$$C_k + C_k' = A_0.$$

Applying this to the relation (2) we have for $C_k=C_1$,

$$\begin{aligned} C_1 + C_1 + C_1' &= C_5 + C_1 + C_1', \\ C_1 + (C_1 + C_1') &= C_5 + (C_1 + C_1'), \\ C_1 + A_0 &= C_5 + A_0, \\ C_1 &= C_5, \end{aligned}$$

which is a contradiction. Hence our Theorem.

Much work has been done in the investigation of various types of rings but the above shows that *the ring is not the fundamental system for associative algebra of double composition.*

In the previous paper I pointed out that the elements $C_i, \cdots, C_j$ in (C) form a ring for $i=1$. Mr. F. C. Biesele proved* that these elements formed a cyclic group for $i$ arbitrary so that they form a ring with $i$ arbitrary which is isomorphic with the ring of residue classes modulo $(j-i+1)$.

There is a simple physical example† of the algebra formed by (C). Suppose an object travels in a straight line through $i$ equal units of distance and then travels in a circular path whose circumference is equal to $(j-i+1)$ of these same units of distance. Designate the operation of bringing the object through $a$ units of distance in this way by $C_a$. If by another operation $C_b$ we bring the object from the same starting point to the same position as found for $C_a$ we write

$$C_a = C_b.$$

Also set

$$C_a + C_b = C_{a+b}, \qquad C_aC_b = C_{ab}.$$

It is easy to see from these definitions that the commutative and associative laws of addition and multiplication hold, and also the distributive law, but for $j=6$, $i=4$, with $C_7=C_4$ we have $C_1+C_6=C_1+C_3$, but $C_6\neq C_3$.

The above example suggests the question as to what type of algebra we obtain if the object travels for a time in a circular path and then after a certain number of complete revolutions goes off in a straight line. For example, let the circumference of the circular path be 5 units and suppose the object makes one complete revolution around the path and then goes off from its starting point in a straight line divided into units equal to those employed on the circumference of the circular path. Then using the notation and definitions employed before we may introduce a symbol $C_0$ meaning the operation of bringing the object through zero units and hence

$$C_0 = C_5;$$

but

$$C_0 + C_1 \neq C_5 + C_1,$$

so that the *substitution axiom as employed in ordinary algebra, similar to postulate 5 above, does not hold*; for although

$$C_5 + C_1 = C_5 + C_1, \tag{2}$$

and

$$C_0 = C_5,$$

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* In his M. A. thesis at the University of Texas, 1933.

† Due to Professor W. L. G. Williams.

we cannot substitute $C_0$ for $C_5$ in the left hand member of (2). It is easy to see that the cancellation law of addition holds, however, in this system. By combining the characteristics of the two systems, we obtain an algebra in which neither the cancellation law of addition nor the substitution axiom holds.

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## PLANETARY ORBITS IN GENERAL RELATIVITY

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In the following paper, the differential equation of a planetary orbit arising in the relativity theory of gravitation is integrated in terms of elliptic functions, and some of the physical implications of the results are indicated. The paper is an extension of an article by W. B. Morton.* The material has been presented from a somewhat different viewpoint, however, and the analysis extended to include all values of the constants, $m$ and $h$, which provide a real orbit. Also considered are the effects of imposing the additional condition of real initial velocity, and finally, from astronomical data, the range in which the results are practically significant.

In the relativity theory of gravitation the differential equation of the orbit is†

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2m}{h^2}u + \frac{c^2-1}{h^2}, \tag{1}$$

and the additional integrals are

$$r^2\frac{d\phi}{ds} = h, \tag{2}$$

$$\frac{dt}{ds} = \frac{c}{1-2mu}, \tag{3}$$

where $r(=1/u)$ and $\phi$ are the polar coördinates in the orbital plane with origin at the primary, and $c$, $h$, and $m$ are constants, the last being the mass of the primary in gravitational units. Writing $v_\phi = r(d\phi/dt)$, where $v_\phi$ is the azimuthal component of the velocity, and combining (2) and (3), we obtain

$$\frac{c}{u(1-2mu)}v_\phi = h. \tag{4}$$

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* W. B. Morton, The forms of planetary orbits in the relativity theory, London Philosophical Magazine, vol. 42, 1921, pp. 511–522. My attention was called to an article by Y. Hagihara in the Japanese Journal of Astronomy and Geophysics, vol. 8, pp. 67–176 (abstract in the Fortschritte der Mathematik, vol. 57, part 8, 1938, p. 1175). The original article was not available to me, but I gather from the abstract that the author covered much the same ground as is covered in the first part of this paper, using, however, the Weierstrass elliptic functions rather than the Jacobian functions.

† Eddington, The Mathematical Theory of Relativity, p. 86.

It is evident that the necessary condition for a real orbit ($u$ and $\phi$ real) is that the right hand side of (1) be real and positive. If the cubic on the right in (1) has three real roots, say $a$, $b$, $c$ ($a \geqq b \geqq c$), then (a) if $m > 0$, $u$ may only vary in such a way that $u \geqq a$ or $b \geqq u \geqq c$ and (b) if $m < 0$, then $u \leqq c$ or $b \leqq u \leqq a$. If the cubic has but one real root, say $u = a$, then $m > 0$ implies $u \geqq a$ and $m < 0$ implies $u \leqq a$. Since the roots of the cubic are also zero points of $(dr/d\phi)$, they are also the apses of the orbit, so that the type of orbit depends on the position of the roots. If initially $u \geqq a$ ($m > 0$) then $u$ is bounded by $u = a$ and $u = \infty$ ($r = 0$) and the orbit is "captured." If initially $b \geqq u \geqq c$ ($m > 0$) or $a \geqq u \geqq b$ ($m < 0$), $u$ is confined between the two roots and if they have the same sign, the orbit is found to be quasi-elliptic (elliptic with the relativity advance of perihelion). If the roots differ in sign, or if one is zero, there will be one or two points for which $u = 0$ ($r = \infty$) and the orbit is quasi-parabolic or hyperbolic.

Since the cubic in the right hand member of (1) has at least one real zero, we may let such a zero be $u = 1/a$. This gives for the constant of integration, $c$, the relation

$$c^2 = \left(1 + \frac{h^2}{a^2}\right)\left(1 - \frac{2m}{a}\right).$$

If $m$ and $r$ are multiplied by a constant while $v_\phi$ remains unchanged, it follows from (4) that $h$ is multiplied by the same constant, and hence equation (1) remains invariant in form, so that the shape of the orbit is unchanged. It is convenient, then, to take

$$r' = \frac{r}{a}, \qquad m' = \frac{m}{a}, \qquad h' = \frac{h}{a},$$

where $a$ = value of $r$ at an apse; thus $u' = 1$ at an apse. Dropping primes hereafter, we get

$$c^2 = (1 + h^2)(1 - 2m), \tag{5}$$

and (1) becomes

$$\left(\frac{du}{d\phi}\right)^2 = 2m(u - 1)\left\{u^2 - \left(\frac{1}{2m} - 1\right)u + \frac{1}{h^2} - \left(\frac{1}{2m} - 1\right)\right\}. \tag{6}$$

The problem to be considered is: Let a body be projected from the apse $u = 1$, what will be the shape of the orbit under the permissible values of $m$ and $h$?

From the condition that the right side of (6) be real, we see that it is sufficient to consider all possible combinations of $m$ and $h$ with $m$ real and $h$ either real or a pure imaginary. Let $\alpha$, $\beta (\alpha \geqq \beta)$ be the two zeros (hereafter referred to as roots) of the quadratic factor on the right in (6); then

$$\alpha, \beta = \frac{1}{2}\left(\frac{1}{2m} - 1\right) \pm \frac{1}{2}\left\{\left(\frac{1}{2m} + 1\right)^2 - 4\left(1 + \frac{1}{h^2}\right)\right\}^{1/2}. \tag{7}$$

We first consider the case of $h$ and $m$ both real. If we let $x = 1/m$, $y = 1/h$, then relations between the roots will appear as curves in the $xy$ plane. We thus plot (Fig. 1) the crucial curves:

$$C_1\colon\ \alpha=\beta,\qquad C_2\colon\ \alpha \text{ or } \beta=1,\qquad C_3\colon\ \beta=0;$$

which from (7) are:

$$C_1\colon\ y^2=(x^2+4x-12)/16;$$
$$C_2\colon\ y^2=x-3;$$
$$C_3\colon\ y^2=(x-2)/2.$$

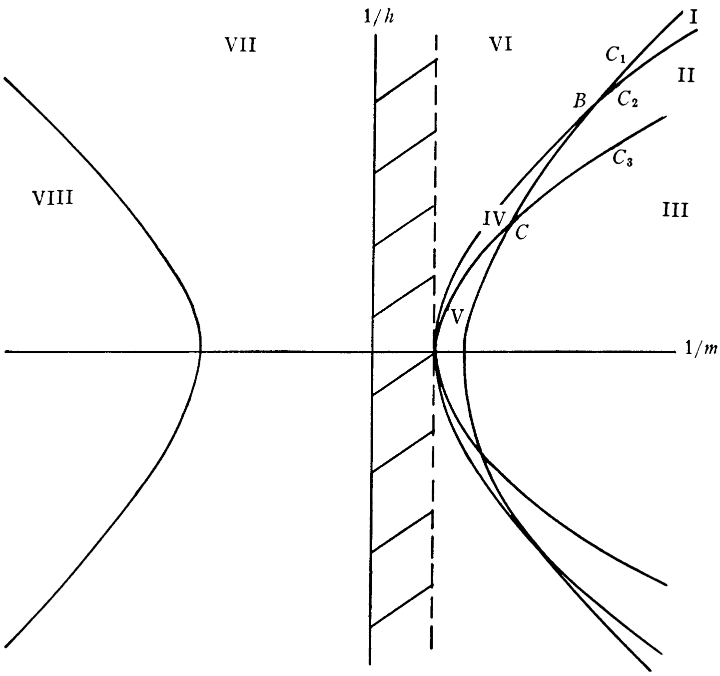


FIG. 1

From (7) we see that in the regions designated, the roots satisfy the following conditions:

| | | | |
|---|---|---|---|
| I. | $\alpha>\beta>1$ | VI. | $\alpha, \beta$ imag., $m>0$ |
| II. | $\alpha>1>\beta,\quad \beta>0$ | VII. | $\alpha, \beta$ imag., $m<0$ |
| III. | $\alpha>1>\beta,\quad \beta<0$ | VIII. | $1>0>\alpha>\beta,\quad m<0$ |
| IV & V. | $1>\alpha>\beta$ | | |

Thus in accordance with the discussion above, in regions I and II the orbit is quasi-elliptic, $u=1$ being the aphelion in I, and the perihelion in II. In region III, the two least roots differ in sign, so the orbit is quasi-hyperbolic. In regions IV and V, $u=1$ is the largest root and the orbit is captured. Since in regions VI and VII, $u=1$ is the only real root, the orbit is captured in VI and hyperbolic in VII. In region VIII, $m<0$ and the two largest roots differ in sign, so the orbit is also hyperbolic.

To obtain the orbital equation for region I, we transform to a new dependent variable by the substitution

$$u = 1 + (\beta - 1)\sin^2\psi, \tag{8}$$

where initially ($u=1$, $\phi=0$) $\psi=0$. Then substituting in (6) and integrating both sides we have

$$\int_0^\psi \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \tfrac{1}{2}p\phi, \qquad p^2 = 2m(\alpha - 1), \qquad k^2 = \frac{\beta - 1}{\alpha - 1},$$

where, from the conditions on the roots in region I, $p^2>0$ and $0<k^2<1$. Inverting the integral and substituting in (8), we find

$$u = 1 + (\beta - 1)\, sn^2\, \tfrac{1}{2}p\phi. \tag{9}$$

If $4K$ and $4iK'$ are the real and imaginary periods of the elliptic function, in view of the properties of the $sn$-function, we have from (9)

$$u = \beta, \quad \text{for} \quad \tfrac{1}{2}p\phi = K,$$

and

$$u = \alpha, \quad \text{for} \quad \tfrac{1}{2}p\phi = K + iK'.$$

Hence the equation may be referred to the middle root, $u=\beta$ or the largest root $u=\alpha$, as initial point, by the transformations, respectively,

$$\phi' = \phi - \frac{2K}{p}, \tag{10}$$

$$\phi' = \phi - \frac{2}{p}(K + iK'). \tag{11}$$

In regions II and III, $u=1$ is the middle root; hence to obtain the orbital equation it is only necessary to rearrange the roots (interchange 1 and $\beta$) and transform by (10). Thus we obtain

$$u = \beta + (1 - \beta)\frac{cn^2\,\frac{1}{2}p\phi}{dn^2\,\frac{1}{2}p\phi}, \qquad p^2 = 2m(\alpha - \beta), \qquad k^2 = \frac{1-\beta}{\alpha - \beta}. \tag{12}$$

Similarly for regions IV, V, and VIII, $u=1$ is the largest root, so that upon rearranging roots and transforming by (11) we have

$$u = \alpha + \frac{1-\alpha}{cn^2\,\frac{1}{2}p\phi}, \qquad p^2 = 2m(1 - \beta), \qquad k^2 = \frac{\alpha - \beta}{1 - \beta}. \tag{13}$$

In VI and VII, there being but one real root, we must proceed directly from the differential equation. Transforming by

$$u = 1 + H\tan^2\psi, \quad \text{where} \quad H = \sqrt{\frac{1}{h^2} - \frac{1}{m} + 3},$$

and proceeding as in the case of region I, we obtain

$$u = 1 + H \frac{sn^2 \tfrac{1}{2}p\phi \, dn^2 \tfrac{1}{2}p\phi}{cn^2 \tfrac{1}{2}p\phi}, \qquad p^2 = 2mH, \qquad k^2 = \frac{H + \dfrac{1}{4m} - \dfrac{3}{2}}{2H}.$$

On the left hand side of the plane, we have $m<0$; to preserve the reality of the elliptic function argument, we set $p=ip'$ and transform. Thus, for instance, in region VIII we obtain from (13),

$$u = \alpha + (1-\alpha)\, cn^2 \tfrac{1}{2}p'\phi, \qquad p'^2 = 2m(\beta - 1), \qquad k'^2 = \frac{1-\alpha}{1-\beta}.$$

The orbits for the limiting curves are obtained by specializing the equations obtained for the various regions. Thus we find that on $C_1$ above $B$, the orbit is asymptotic circular with a limiting circle $u=\beta$; below $B$, the orbit is captured. On $C_2$ the orbit is circular, with equation $u=1$. On $C_3$ above $C$, the orbit is quasi-parabolic, below $C$, captured. In the article of W. B. Morton, already referred to, the orbital equations for these various regions and special cases are set down and a number of orbits drawn.

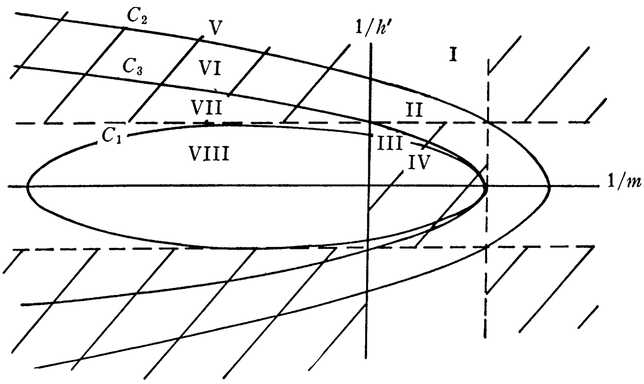


FIG. 2

We may apply an entirely similar analysis to the case in which $h$ is a pure imaginary. Let $h=ih'$, and plot the crucial curves in the plane $x=1/m$, $y=1/h'$ (Fig. 2) as follows:

$$C_1\colon\ \alpha = \beta; \qquad C_2\colon\ \alpha = 1; \qquad C_3\colon\ \alpha = 0.$$

which from (7) are:

$$\begin{aligned} C_1\colon\ y^2 &= -(x^2 + 4x - 12)/16; \\ C_2\colon\ y^2 &= -(x-3); \\ C_3\colon\ y^2 &= -(x-2)/2. \end{aligned}$$

Since substituting $-h'^2$ for $h^2$ in (6) does not change the form of the equation, the same orbital equations are obtained as in the real case. Making this substitution, we obtain, in a manner analogous to that of the real case, the following results:

| | | |
|---|---|---|
| I. | $\alpha > 1 > \beta, \quad \beta < 0$ | $(m > 0)$ hyperbolic; |
| II, III. | $1 > \alpha > \beta$ | $(m > 0)$ captured; |
| IV. | $\alpha, \beta$ imag. | $(m > 0)$ captured; |
| V. | $\alpha > 1 > \beta$ | $(m < 0)$ quasi-elliptic; |
| VI. | $1 > \alpha > \beta, \quad \alpha > 0$ | $(m < 0)$ quasi-elliptic; |
| VII. | $1 > \alpha > \beta, \quad \alpha < 0$ | $(m < 0)$ hyperbolic; |
| VIII. | $\alpha, \beta$ imag. | $(m < 0)$ hyperbolic, |

and the special cases

$C_1$ left side, hyperbolic; right side, captured;

$C_2$ circular;

$C_3$ left side, quasi-parabolic; right side, captured.

We now consider some of the physical implications of the above results. Combining (4) and (5), we have that initially $(u=1)$,

$$v_0 = \sqrt{\frac{1-2m}{1+1/h^2}}, \tag{14}$$

where the initial velocity, $v_0 = v_\phi$ for the apse. For the path of a ray of light, we have the relation $ds=0$ which from (2) implies that $h=\infty$. Hence from (14), the velocity of light is, initially

$$v_0 = \sqrt{1-2m}.$$

Thus as $h$ ranges through all real values, $v_0$ goes through all velocities less than the velocity of light. If we impose the added condition that the initial velocity be real, from (14) for real values of $h$ we have

$$m \leqq \tfrac{1}{2}.$$

This eliminates the shaded region in Figure 1. In equation (14) it is evident that imaginary values of $h$ make $v_0$ either imaginary or greater than the velocity of light. Then from (14) the condition that $v_0$ be real limits us to the regions for which

$$h'^2 \geqq 1 \quad \text{and} \quad m \leqq \tfrac{1}{2}, \tag{15}$$

or

$$h'^2 \leqq 1 \quad \text{and} \quad m \geqq \tfrac{1}{2}, \tag{16}$$

which eliminates the shaded region in Figure 2 and leaves only orbits which are hyperbolic (in I, VII, VIII), circular (on $C_2$), or captured (in region II). In the region corresponding to (15), as $h'$ ranges from 1 to $\infty$, $v_0$ ranges from $\infty$ down to the velocity of light. In the region of (16), it is interesting to note that although the velocity of light is imaginary, the initial velocity and the orbit are real.

Certain other physical facts eliminate other regions. Negative values of $m$, which would correspond to a center of repulsion, are not found in nature.

Neither do we find velocities greater than the velocity of light, or regions in which the velocity of light is imaginary. Astronomically, the most extreme case known is that of Sirius's companion. At the surface of this star, $m$ (which, as we have seen, is the ratio of the mass of the primary in gravitational units to the radial distance) is calculated to be about $3\times 10^{-5}$. In every other known case the value of $m$ is much smaller, so that practically we need consider only the region to the far right in Figure 1.

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## MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This Department affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

### THE CONTENT OF A COURSE IN ALGEBRA FOR PROSPECTIVE SECONDARY SCHOOL TEACHERS OF MATHEMATICS*

W. H. ERSKINE, Bethany College

First let us bring to a focus the phase of the topic to which we shall direct our attention. We are not referring here to a methods course for high school teachers nor to a course in advanced high school algebra, but rather to one which is usually given to juniors and seniors by most colleges and which is, in a sense, the terminal undergraduate course in algebra for prospective secondary school teachers. To some extent the content of the course is uniform and consists of the theory of equations, solution of equations with numerical coefficients and an introduction to the theory of determinants. A study of course descriptions in college bulletins will indicate that from 3 to 4 semester hours are given over to this part of the course. The content of the remainder of the course, to which 1 to 2 semester hours are devoted, depends largely upon the college and professor, and often varies from year to year. It is to the content of this remaining part of the course that I wish to direct your attention.

No doubt, many of you will have had experience in teaching such a course and will have suggestions to make concerning its content. Indeed one of the purposes for which we are gathered here is to take advantage of this opportunity to exchange ideas. In conversations with teachers of mathematics I have noted that considerable interest has been shown in the material and methods which have made up the course I am now conducting at Bethany College. And so to start the discussion it was thought advisable that I sketch briefly the course which I have devised. The plan is not entirely original but is related to that of Klein as indicated in his *Elementary Mathematics from an Advanced Standpoint.*

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* Read before the Allegheny Mountain Section of the Mathematical Association of America, May 1938.

Some of you may have already tried a similar scheme and we would be very much interested in your deductions.

However, before we go into the discussion of the content, I would like to make some general observations. These observations are not meant as criticisms of existing mathematical curricula and teaching methods but represent certain deficiencies which might be corrected in a measure by this course.

1. Most courses in mathematics have as their aim the consideration of some particular topic in more or less complete form. Upon completion of the course the student is expected to know the processes and manipulations taught in the course. The relation of the course to other branches of mathematics takes a position of secondary importance. Usually it is only a small fraction of the class who become really aware of these overtones. The majority of the students cannot see the extent and unity of mathematics—they cannot see the woods for the trees.

In determining the content of the course which we are discussing, I have been influenced by the desire that the students learn to see the woods.

2. For reasons known to all of you the procedure in college courses in mathematics has been to study intensively one textbook or set of notes to the almost complete neglect of other texts. Students rarely become familiar with books on mathematics and rarely know of the leading mathematicians of the present day. This I feel sure is one of the causes for the fact that the high school teacher's knowledge of mathematics and hence all of his teaching is confined to the textbook alone. Indeed is it not true that both the teacher and students think of the year's work in terms of the mastery of a text and the solution of a certain percentage of its exercises? To continue the previous analogy could it not be said that the students are led into the woods along a single path and soon acquire the belief that there is but one path through the woods?

A terminal course in algebra for teachers should acquaint the members of the class with other texts and authors and make use of a variety of methods. Klein recommends the visual methods of geometry even in demonstrating or proving algebraic relations.

3. The layman's misconception of mathematics is indicated by his surprise upon hearing that there are unsolved problems in mathematics and learning of the amount of original work that is continually being published in mathematical journals. Most people have the impression that calculus is the *Ultima Thule* of research in mathematics and that geometry, algebra, and arithmetic are closed chapters in the development of knowledge, and indeed have been closed since Euclid wrote the *Elements*.

A course in Algebra for teachers should have as one of its aims the acquainting of the student with the direction of present day investigations in algebra. They should be introduced to topics in which they could do reading and research, and thus keep alive their interest in the subject which is going to feed and clothe them. The course should point out to them other beautiful woods and valleys through which the paths of mathematics may lead them, and cer-

tainly a few of the mountain peaks which have been as insurmountable as Everest.

4. In the usual course in mathematics the pressing necessity for acquiring special techniques leaves little time available for acquainting the students with situations where the methods of the course are not applicable. For complete understanding it is important to know the limitations of the methods and techniques of the course. I wonder how many high school teachers have considered the fundamental laws of algebra in this respect. How many have experienced a number system in which the commutative law does not hold, or where unique factorization is not possible, or where the whole is not greater than its parts? I believe that it was Gauss who said that one does not understand an operation until he knows also its inverse.

Somewhere in the preparation of high school teachers there should be an opportunity for work with numbers where the laws of ordinary algebra do not hold. With your permission I should like to say it this way. . . . Sometimes the prospective teacher should be led into woods where the ordinary means of locomotion are impossible, where he cannot ride but must walk, or crawl, or swim.

5. Because of the insistent demands for prerequisite mathematics for the physical sciences, the courses in pure mathematics are limited in number. In the larger schools the requirements for self sufficiency of each course prohibit the teacher from dwelling long on the relations of the course to others in the curriculum. Indeed it is only in courses in content and teaching methods of secondary school mathematics that the college curriculum makes a definite effort to contribute to the prospective teacher's preparation. In fact the prospective teacher is given the impression that advanced mathematics has nothing to do with the subject he is preparing to teach. Is this the reason why some teachers' colleges do not give any mathematics above the calculus? After completing the course in calculus at Bethany a former student of mine transferred to another college, and when she inquired about further courses leading to a major she was told she already knew too much mathematics for a high school teacher. That mathematical research is extending roots deeper and deeper into the foundations of the subject as well as growing new branches is not the common understanding among the high school teachers of mathematics.

A course in algebra for teachers should give a treatment of certain elementary or fundamental matters from the point of view of higher mathematics. To use our analogy one last time, the prospective teacher should be taught to look at individual trees from the standpoint of a forester who knows every tree in the woods.

It has been with these five observations in mind that I have set up the plans for the course in algebra which we are here to discuss.

Whenever I have suggested to other teachers a course in the fundamental concepts of mathematics, I have been told of the difficulty of giving visible body

to a course which concerns matters so essentially abstract. I have been informed also of the difficulty of obtaining unity to such a course.

As you see from the list of topics below, I have followed the advice of Professor G. H. Hardy when he said at the conclusion of a lecture before the American Mathematical Society, "The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general, and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same amount of 'calculus for engineers.'" Following the suggestion of the early chapters of Felix Klein's book on *Elementary Mathematics from an Advanced Standpoint*, I have sought to give unity to the course by centering my discussion of the fundamental concepts of mathematics about the "Integers."

The methods I have used distinguish the course from the usual one in the theory of numbers. For instance, I have used the visual methods of geometry when convenient. One hour and a half a week is spent in lecture and class discussion and one hour in exercises on the fundamental concepts. The class is given a mimeographed list of definitions, theorems, and topics, together with a list of page references suggested for each of the topics. The bibliography consists of some twenty books which are kept on reserve in the library. As you are no doubt aware, many of the methods of the course are practical only because the class is small. At first there was considerable difficulty in reading and understanding the references, but now the students are able to recognize progress in their increased ability to understand.

The actual results of the course will be difficult to measure since the aims are far reaching and the achievements not immediately visible. Indeed the course was planned not to teach theorems or facts but to give appreciations and attitudes toward the subject which the prospective teacher plans to teach.

6. A list of the principal topics of the course is given below. No attempt is made in the course to give an exhaustive treatment of the topics but a very definite attempt is made to bring into clear focus the importance of the topics for the teacher of secondary school mathematics.

(1) Division algorithm of Euclid. Unique factorization of natural numbers. The divisors of an integer.

(2) Infinitude of Primes. Conjectures on the distribution of primes.

(3) Pythagorean numbers. Fermat's last theorem.

(4) Concepts of "class," "belonging to a class," correspondence between classes.

(5) The class of positive integers and denumerable classes.

(6) The concepts of "group," "ring," and "field."

(7) Algebraic integers. Failure of unique factorization rule.

(8) Divisibility tests for rational integers. Congruences.

(9) Scales of notation. The relation between the polynomial and the integer.

# QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

*The department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

## A NOTE ON TANGENT PASCAL LINES

P. A. CARIS, University of Pennsylvania

It does not seem to be generally known that for a given conic a tangent line may also be a Pascal line associated with a hexagon inscribed in the conic. In other words, some of the Pascal lines associated with hexagons inscribed in a conic are tangent to the conic. For example, consider the points $A\equiv(-3,4,5)$, $B\equiv(5,12,13)$, $C\equiv(21,-20,29)$, $D\equiv(4,3,-5)$, $E\equiv(20,21,29)$, $F\equiv(91,-60,109)$, on the conic $x^2+y^2-z^2=0$. $AB$ and $DE$ intersect at $U\equiv(1,1,1)$; $BC$ and $EF$ intersect at $V\equiv(1,-2,1)$; $CD$ and $FA$ intersect at $W\equiv(17,-12,17)$. The equation of the line through $U$, $V$, $W$, is $x-z=0$ and this line is tangent to the conic at the point $(1,0,1)$.

The foregoing remark also has a pedagogical significance. Let $A$, $B$, $C$, $D$, $E$, $F$, be any six distinct points on a conic. Let $AB$ and $DE$ intersect at $U$ and let $BC$ and $EF$ interesect at $V$. Suppose one wished to prove Pascal's Theorem by projecting the conic into a circle and, at the same time, projecting the line $UV$ into the line at infinity. This method is suggested at one or two places in the literature. It lacks generality, however, since the line $UV$ might be a tangent line.

As a matter of fact, four distinct points $A$, $B$, $D$, $E$, can be taken arbitrarily on a conic and then two points $C$ and $F$ can be found so that the six points are distinct and the Pascal line associated with the hexagon $ABCDEF$ is tangent to the conic. Let $AB$ and $DE$ meet at $U$. Let $t$ be a tangent to the conic from $U$ and let $T$ be its point of contact. Choose a point $V$ on $t$ distinct from $U$ and $T$, not on the tangent at $B$ nor on the tangent at $E$, and not on one of the lines $AE$, $BD$, $BE$. Let $VB$ meet the conic again at $C$ and let $VE$ meet the conic again at $F$. Since $V$ is distinct from $U$, $C$ is distinct from $A$ and $F$ is distinct from $D$. Since $V$ is distinct from $T$, $F$ is distinct from $C$. Since $V$ is not on the tangent at $B$, $C$ is distinct from $B$. Since $V$ is not on the tangent at $E$, $F$ is distinct from $E$. Since $V$ is not on $AE$, $F$ is distinct from $A$. Since $V$ is not on $BD$, $C$ is distinct from $D$. Since $V$ is not on $BE$, $C$ is distinct from $E$ and $F$ is distinct from $B$. That $t$ then is the Pascal line associated with the hexagon $ABCDEF$ is obvious.

# RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

## BOOK REVIEWS

*Analisi Matematica Algebrica ed Infinitesimale.* By Beppo Levi. Bologna, Nicola Zanichelli, 1937. 7+541 pages..

In 1916 Beppo Levi published volume I of an *Introduzione alla Analisi Matematica*, an excellent treatise on algebra with the subtitle *Teorie Formali*. In this earlier work all questions of continuity and infinite processes were avoided—or, rather, postponed to later sections of the treatise, which have not been printed.

Rather than complete the work previously begun, Levi offers us the present book, which can probably be regarded as a somewhat briefer version of both the written and the unwritten parts of the *Introduzione*. In content it has much in common with American works on "advanced calculus," differing from them in the greater attention given to algebra and in the commendably comprehensive points of view. For instance, the "numbers" used are of a very general type, except where the contrary is stated. Definite integrals are defined, from the start, so as to include those of both Lebesgue and Riemann. Simpson's rule comes in as a special case of a much more general method.

Levi intended, it seems, to write a text requiring previous knowledge only of "algebra through quadratics" and of the elements of trigonometry. He takes no acquaintance with analytic geometry for granted. Of course it is a freshman of extraordinarily tough mind to whom the work is ostensibly addressed; and even this prodigy is forgotten when the author later assumes acquaintance with cross-ratios. There is a hint of an upper bound for the topics to be considered when Levi declines to give a proof of Stirling's formula, "which would be beyond the theoretical limits we have set ourselves."

Chapter I treats of integral rational functions. Among the topics considered are the Lagrange and Newton formulas for interpolation, that of Leibniz for the power of a polynomial, symmetric functions, discriminants, and the general concept of number. The definition of a function seems a little unsatisfactory. "We give the name 'function of certain variables' to an expression . . . able to assume values when a suitable value is assigned to each of the variables"—where the meaning of "value" is given, from case to case, by the context. Levi should have recalled the existence of many a function which "hadn't been given a name," and for which no "expression" existed.

Numbers are not defined; the properties of sets of numbers are alone essential. In his definitions of these sets ("modulo," "campo," "corpo," "anello") Levi deviates from those of other authors—and, indeed, from those of his own *Introduzione*. A ring (anello), for instance, may be neither closed nor commutative as to multiplication, and must not be closed under division.

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Chapter XII, on the direct definition of the integral, is one of the most substantial and stimulatingly original. After some pages on graphic and mechanical integration, Levi comes to the definition, already mentioned, covering Lebesgue and Riemann integration at the outset. After a good development of the important properties of the integral, we are introduced to that of Stieltjes—although not to the Lebesgue-Stieltjes integral.

Among the topics in the chapter on "Applications to analysis and to numerical calculation"—a chapter whose practical value does not occasion any lower theoretical standards—we may note the error in approximate computations, Newton's methods, the Budan-Fourier theorem, separation of roots, approximate calculation of definite integrals, criteria of convergence, infinite products, Fourier polynomials, series, and integrals.

An important feature of the chapter on ordinary differential equations is the variety of methods of graphical integration. In its forty pages it can bring only the most important methods of solution, but the study of these in uncommonly illuminating. The chapter on functions of several variables is noteworthy, for instance, because of the introduction of the Jacobian through a quite general theorem on implicit functions. A return to geometry brings us, finally, a glimpse of the differential geometry of $n$-space.

One of three amplifying postscripts treats of differential notation. Levi objects to the usual definition of the differential of the independent variable (only) as the increment of that variable. The chief virtue of an equation in differentials is its failure to set apart any particular variable, its homogeneity. He therefore refuses to define *a* differential of any variable, speaking only of a class of simultaneous differentials of related variables.

There are few diagrams in this book, but most of those given are very carefully done. Problems for the reader are wanting—the author, again thinking highly of his public, believes that most is gained when the student sets his own problems. There are, on the other hand, many illustrative examples. To a large extent they are not typical, but present exceptional cases—$x^\alpha \sin 1/x$ is a favorite function for discussion. Frequently the dates of mathematicians are given, but there are only a few specific references to the original works.

On two matters we could wish the author had abandoned common illogicalities. To our mind, the limits of a definite integral should never include the letter assigned to the variable of integration. In the treatment of complex numbers, $a+bi$, we need names for three kinds—those with $a$ and $b$ unrestricted, those with $a$ zero, those with $b$ different from zero. By using the name "complex" sometimes for the first kind, sometimes for the last, Levi inevitably falls into contradictions.

Of the errors not noted in the "errata" list (pp. v–vii) very few need trouble the reader. On page 95, line 9, the additive term for a logarithm should be $2\pi i$, not $2\pi$. In equation (XV. 14.14) the denominator on the right ought to be $\partial f/\partial y$ instead of $\partial f/\partial x$.

In conclusion, we would warmly commend this work of Beppo Levi, which

brings to important topics already frequently treated deep insight and a stimulating wealth of fresh and fertile ideas.

E. S. ALLEN

*Arithmetik.* By Paul B. Fischer. (Sammlung Göschen Band 47.) Berlin, Walter de Gruyter and Co., 1938. 152 pp. RM 1.62.

This new volume of the well-known Göschen series deals with the theory of elementary arithmetic with suggestions of more advanced topics. It furnishes a formal introduction suitable for junior college level, avoiding unnecessarily technical language and notation while developing the subject in a rigorous manner with well chosen illustrative material. Save for a few suggestions left for the reader to handle there are no problems listed—the subject furnishes its own incentive and the reader who needs drill must devise his own exercises. The six sections concern (I) *Numbers and counting*, with valuable historical and philosophical material; (II) *The domain of natural numbers*, with explicit statement of formal rules, and their geometric interpretation including briefly such topics as tests for divisibility, the uniqueness of factorization, number systems to bases other than 10; (III) *The domain of the integers*, introducing zero, negative integers and the division transformation for polynomials with integral coefficients; (IV) *The domain of the rational numbers*, discussing fractions, continued fractions, proportions, decimal fractions, periodic decimals; (V) *The domain of the real numbers*, handling continuity, extraction of square roots, roots of higher order, the continued fraction algorithm for the square root, Heron's method, logarithms, the slide rule; (VI) *The domain of the complex numbers* including an introduction to quaternions as illustrating hypercomplex number systems. An *Appendix* treats of arithmetic and geometric series, combinatorial analysis and the binomial theorem. This book presents in brief, simple, and reliable fashion material with which every advanced student and certainly every teacher of mathematics should be thoroughly familiar. But this subject has been almost crowded out of the American educational program. Being theoretical, it has no place in the grade schools; being arithmetic, it receives no fresh study in high school or college. It can be heartily recommended to the attention of all elementary and secondary school teachers for whom this foreign language is not a fatal obstruction. Would that there were as effective a book in English!

A. A. BENNETT

*Leitfaden der Algebra.* By H. Stohler. Dritter Teil. Zürich und Leipzig, Orell Füssli, 1938. 158+15 pages. Halbleinen Fr. 3.60. RM 2.20.

This little book is a part of a set that will apparently make up a complete course in elementary algebra. Furthermore, similar texts in other subjects are contemplated or have been published, all under the editorship of the Society of Swiss Teachers of Mathematics. The reviewer was unable to find reviews of any other parts of this work nor has he seen any of these other sections.

The book under review is reasonably self-contained, however. It deals, in an elementary fashion, with various types of sequences. Thus it begins with a discussion of sequences arising in statistics and gives a brief treatment of frequency curves, correlation, *etc.* Then the familiar "progressions" are taken up, as well as the sequences arising in combinatorial problems. A large part of the book is devoted to the theory of probability. Here the author first introduces *a priori* probability as a relative frequency in a finite sequence and then defines empirical probability as a limit when the sequence is infinite. It is admitted that there are difficulties here that cannot be discussed in an elementary text, but the author stresses the practical value of the approximations that arise in the limiting process. Then there is a section on mathematics of insurance and a final one on the theory of errors.

The book may seem rather unconventional to the reader who is used to our ordinary textbooks. Thus the sections on statistics, probability, *etc.*, are frankly empirical in character. The strictly mathematical sections, however, are rigorous, and the proofs are complete. There are various original touches throughout the book, many practical numerical illustrations (with data taken directly from Swiss statistical tables), and historical references. It should be stimulating to a student and can be read with profit by any teacher of algebra.

H. W. BRINKMANN

*Archimedes.* By E. J. Dijksterhuis. Groningen-Batavia, P. Noordhoff, 1938. 8+213 pages. 4.50 f.

At the present time historical studies on the evolution of mathematical science have found new centers of activity in the low countries. The Historical Library of the Exact Sciences, of which series this first part on Archimedes is the sixth volume, is distinguished by careful and scholarly editing, combined with fine printing and substantial and colorful binding.

The first five volumes deal with Euclid (I, III), Non-Euclidean Geometry (II), and Newton's Principia (IV, V), all six parts appearing over a period of ten years.

After giving a preliminary list of works to be cited in this volume on Archimedes, the author gives briefly an account of the life and the works of Archimedes, with sufficient and correct bibliographical references. Upon this follows an excellent summary (pp. 44–133), The Elements of the Work of Archimedes, employing modern symbolism.

The Dutch version of the texts of Archimedes, represented in this volume by the two books, On the Sphere and the Cylinder, concentrates on the modern interpretation, algebraic and geometric, of the propositions rather than the literal translation. This will make the work really more valuable for the Dutch teachers of mathematics as the historical connections are also clearly indicated.

L. C. KARPINSKI

*Higher Algebra*. By S. Barnard and J. M. Child. London, Macmillan and Co., Ltd. 1936; New York, The Macmillan Co., 1937. 15+585 pages. $6.00.

Even the two paragraphs on the book-jacket made the reviewer want to sit down at once and work through this interesting text, designed "for students working for higher school certificate, mathematical scholarships, and examinations of similar standard."

The authors promise to follow this text with an advanced algebra, or as we would call it, a treatise on modern higher algebra. We must bear this in mind when we miss such topics as matrices and linear sets. Yet, as it is, it seems that it might replace the stand-by of the Actuarial Society for students preparing for the society's examinations.

A preface of two pages gives a complete analysis of the subjects treated, pointing out the new material and new solutions or treatments, the sources of exercises, the texts to which references are made, suggested courses of reading, *etc.* There are eight pages of contents by chapters, and at the end fifteen pages of answers to exercises, and four double column pages of index. Thus the reader can readily find what he may want.

The claim of completeness of treatment of the topics selected seems well founded and makes us marvel at what can be demanded of students in England. It makes us wish that as much had been required of us, and were now required of those intending to teach in our schools and colleges. Starting with positive numbers, the number system is extended to include irrational and complex numbers, and the theory of irrationals based on Dedekind's definitions. Limits and continuity are fully discussed. The symbol $d/dx$ is introduced and the rules of differentiation are proved. The authors aimed to treat the theory of equations so fully that no separate text on that subject would be needed except by advanced students. The summation of series is fully discussed, leading to a rapid way of calculating Bernoulli's numbers. All the usual tests for the convergence of infinite series and products are presented, and the elementary properties of determinants, including the symmetric and skew-symmetric types, and also all the fundamental inequalities. The subject of the theory of numbers is carried further than in the usual textbooks; congruences and methods of solving them, primitive roots, and recurring decimals are considered. Elementary finite difference equations, the operators $\Delta$, $E$, $D$, and interpolation, curve-tracing, approximations, continued fractions, and probability are amongst the other topics treated.

On page 29, line 7, two typographical errors were noted: $a+b+c=0$ should be $a+b+c=2$, and $a+3b=-5$ should be $a+3b=5$.

On page 62 in #17, two signs should be minus, $-\cos 5\theta-\cos 3\theta+2\cos\theta$.

On page 123, Ex. 2 "Expanding . . . to first row" the upper $c$ in the second minor should be $f$.

The reviewer let one of his advanced students study the multinomial theorem from the text and found him annoyed and confused by finding $(\alpha+1)$ in

statement (ii) p. 354, without finding the new letter $\alpha$ defined before nor immediately following. Why not start with what the authors have at the end of page 354, the finding of the coefficient of $x^r$ in the expansion of a multinomial and then state that we have to obtain by trial all the positive integral values of $\beta, \gamma, \delta, \cdots$ which satisfy the given definition of $r$? At any rate, it seems that this presentation can be made clearer and smoother.

The text furnishes an admirable course for students going into the mathematics of statistics or actuarial science. Even when finishing the review, the reviewer wishes for the time and leisure to work through the whole text, every exercise from cover to cover.

CHARLES C. GROVE

*Essentials of Engineering Mathematics*. By J. P. Ballantine. New York, Prentice-Hall, Inc., 1938. 11+502+76 pages. $3.75.

This book, as the author states in the preface, keeps strictly to mathematics and scarcely treats of engineering subjects although some practical problems are found in the exercises. However, it is written for students of engineering.

The following are the chapter headings: "Coördinates," "Functions," "Angles," "Certain Elementary Functions," "Approximation Calculus," "Limits and Continuous Functions," "Derivatives and Differentials," "Integration," "Trigonometry," "The Straight Line and Circle," "Fundamental Properties of Curves," "Zeros of Functions," "Equations and Loci," "Natural Logarithms," "Trigonometric Calculus," "Differentiation," "Three Dimensions," "Maxima and Minima," "Tables."

The arrangement of the text is that of a unified course. The calculus has been introduced early and has been used throughout the rest of the book. The material included is a large part of that in the usual courses. Many topics have been introduced and developed in ways different from those usually used although standard practice is used many times too. There are many problems including those "class projects" on which the entire class can work. Tables are found at the end of the book.

Certain points are worthy of special note. Among these are the chapters on limits, the definition of the differential by means of the tangent line to a curve, the definition of an integral as an anti-differential, the method of setting up the formulas for areas, volumes, and the like by using the limits of the proper inequalities and the integration chart for $\int \sin^m x \cos^n x dx$.

The work on determinants does not always state carefully that the determinant of the coefficients cannot be zero in applying Cramer's rule nor is it proved that the results of this rule are actually solutions. The general case where this determinant is zero is not discussed. The section on mathematical induction is likely to be rather abstract for students unless adequate time is spent on the problems which follow it.

A few errors were noted: On page 103 in the derivative of a product and on

page 104 in the derivative of a quotient, $u$ and $v$ should be specified as differentiable functions of $x$ as in Art. 43b; on page 104, after $y_0=y=u/v$, there should be inserted that $v\neq 0$; equation 85, page 135 should read $V_x^{x+\Delta x}$ instead of $V_x^{x+\Delta}$.

The author has attempted to meet a real need in mathematics texts. The final test of a book must be in the success students have in learning from it. It seems difficult to estimate this without using the book in class. The book should appeal to teachers who have been teaching the traditional courses and who are looking for a different approach to the various subjects.

E. B. ALLEN

*Enciclopedia delle Matematiche Elementari.* Edited by L. Berzolari, G. Vivanti, and D. Gigli. Volume II, Part II. Milan, Ulrico Hoepli, 1938. 560 pages.

For a review of volume I, see the *Bulletin of the American Mathematical Society*, vol. 38, p. 157. The present part of the encyclopedia contains articles on geometric topics. They are elementary in the sense that they would not be beyond the understanding of a good undergraduate after two years of college mathematics. All the articles are noteworthy for frequent references to original sources, a wealth of interesting historical data, and very good bibliographic material. The book is worth reading for pleasure as well as using for a reference work. To the best of the reviewer's knowledge, it is unique in being an encyclopedic source of mathematical information readily accessible to those whose interests have carried them only through fairly elementary studies. There are nine articles, namely: Massimi e minimi, by E. G. Togliatti; Teoria elementare delle sezioni del cono e del cilindro rotondo, by G. Lazzeri; Elementi di calcolo vettoriale, by C. Burali-Forti; Geometria analitica, by B. Segre; Geometria proiettiva, by E. G. Togliatti; Geometria descrittiva e applicazioni, by A. Comessati; Curve e superficie speciali, by G. Loria; Geometrie non euclidee e non archimedee, by G. Fano; Geometria elementare e matematiche superiori, by O. Chisini. The first of these articles presents elementary methods of investigating extremes, with references going back to the Greeks. There follows an exposition of the calculus methods, with applications to algebraic functions as a preliminary to an interesting discussion of various problems involving polygons, conic sections, prisms, cones, and general polyhedra. The titles of most of the other articles are sufficiently suggestive of their content. The last two articles are perhaps of particular interest as affording an opportunity for a reader with limited background to obtain some insight into the existence of essentially different geometries and into their relationships. The author of the last article outlines the development of geometry according to Klein from the viewpoint of invariants under groups of transformations. He then presents, with illustrations, parts of the elementary theories of birational transformations, topology, projective geometry, analytic geometry, and various related questions of analysis.

S. S. CAIRNS

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.*

## CLUB TOPICS

The summary of club topics appearing in this department of the August–September 1938 MONTHLY has already brought interesting and suggestive subjects with accompanying bibliographies. We hope that this is just a beginning, and we should like to receive:

1) Additional references on topics previously published. See List, vol. 45, pp. 475–476.

2) References on new topics suitable for mathematics clubs. These latter references may include a brief introduction or summary as well as a bibliography which would encompass books and periodicals usually found in the average college mathematics library.

3) Additional topics for which bibliographies are desired.

Because of the rather large number of topics being suggested, this department believes that the discussion of them will be facilitated by numbering them consecutively. Since the last subject appearing on the list in vol. 45, pages 475–476 carries the number 26, additional subjects will receive numbers starting with 27.

Additional topics and bibliography.

27. *The Polar Planimeter. Proposed by Louis C. Mathewson, Dartmouth College.*

1. A. W. Larson, How a polar planimeter works, School Science and Mathematics, vol. 35, 1935, pp. 932–941.

2. W. Cox, The Polar Planimeter. Published by Keuffel and Esser, 1891, 1915.

3. J. Y. Wheatley, The Polar Planimeter and Its Uses. Keuffel and Esser. 1903.

4. F. Brooks, The Theory of the Polar Planimeter. Publications of the Association of Engineering Societies, vol. 3, 1884, No. 12.

Topics for which bibliographies are needed:

28. *Magic Squares, Cubes and Circles.*

29. *Nomographs.*

30. *Coördinate Systems.*

## CONFERENCES FOR SECONDARY SCHOOL MATHEMATICS CLUBS

Among the various activities suggested for mathematics clubs is that of developing an interest in mathematics among secondary school students. Several clubs have already successfully carried out a conference for high school mathematics clubs and other students particularly interested in mathematics. The plan of the Junior Mathematics Club of the University of Wisconsin Extension Division was presented in the August–September 1938 issue. Another such conference was held by the two mathematics clubs, Sigma Phi Mu and Aphesteon of Montclair State Teachers College, in May 1938. They entertained as their guests for the afternoon over 700 high school students and their teacher sponsors from various New Jersey schools. The program included a moving picture "Relativity"; the film "Parabola," accompanied by a talk by the designer, Rutherford Boyd; and an hour's entertainment based primarily on Mathemagic

by Royal V. Heath, a member of the New York Stock Exchange, whose hobby is magic from the mathematics standpoint. As a supplement to this program the clubs prepared a sixteen page magazine containing mathematical problems and amusements of interest to young mathematics students and presented a copy to each guest. There was also a carefully planned exhibit and demonstration of mathematical models and devices.

This department would appreciate knowing of other attempts to stimulate interest in mathematics among secondary school students. We shall be glad to make such information available to all clubs.

## CLUB REPORTS

## 1937–38

### *Mathematics Club, Brown University*

The program for the year included six meetings, four of which were devoted to undergraduate speakers who presented papers half an hour in length which showed all the finish and care that hours of preparation, rehearsal and coaching by various faculty members could effect. Such topics as Arctangent relations, One sided surfaces, Undetermined coefficients, Graphical solution of the cubic, The cell of the honey bee, The history of prime number theory, Quaternions, and Ruled surfaces, were presented. At the other two meetings Professor Gilman talked on Polygons and Professor Brinkmann of Swarthmore College spoke on A new kind of number. These two meetings were followed by a social hour to which each member could invite a guest.

### *Mathematics Club, Syracuse University*

Speakers from the college campus and the city who would tell the part mathematics played in their field of work presented the various monthly programs during the year. Most of the meetings were for members of the club but the March meeting was open to the public. At that time Carl Brower of the Actuarial Department of the Metropolitan Life Insurance Company spoke of the mathematics used in actuarial work. Some of the other subjects discussed were: The relationship of mathematics to political science; The relationship of mathematics to social studies; Marking system in Forestry; The future of theoretical physics; The theory and operation of calculating machines. A social hour followed each meeting.

### *Mathematics Club, Smith College*

The résumé of several books and published mathematical papers formed the basis for discussion at the monthly meetings. Several of the most interesting were: Dantzig's Number, the Language of Science; Pascal, the Life of Genius; Bragdon's The Frozen Fountain; and Elementary Aspects of Topology, by W. L. Ayres (this MONTHLY, February 1938).

### *Mathematics Club, Butler University*

The program for the year included several meetings at which topics were discussed; a Christmas party at which was presented the usual club Christmas play; and the annual picnic with which the year closed. Dr. Nakarai of the College of Religion spoke on the Semites and their numerical system. Demonstration of the fallacy of the trisection of the angle, and a talk by Mr. Davis on the various number systems, completed the program.

President, Mildred Rugenstein; Vice-President, Margaret Stump; Secretary, Roberta Conveaux; Treasurer, William Davis.

### *Mathematics Club, Cooper Union Institute of Technology*

The annual New York Intercollegiate Mathematics Club Contest was won in 1938 by the Cooper Union team which was comprised of and sponsored by members of the Cooper Union Mathematics Club. The prize of a polyphase duplex slide rule for excellence in first year mathematics was awarded by the club to S. Manson. The following papers were presented at the monthly meetings: The slide rule; Tensor analysis; Normals to curves and surfaces; Nomograms; Operational methods applied to aerodynamics.

President, T. Berlin; Vice-President, S. Aronow; Secretary-Treasurer, H. Robbins.

### *Mathematics Club, University of Buffalo*

Mathematical topics discussed by student members of the club and mathematical games or problems solved as an informal part of the meeting, made up the program for the year. The club annually sponsors the Wilfred Sherk Memorial Prize Contest for the most original mathematical paper by an undergraduate. It was won by Paul Civin for his paper on Absolute Values. Topics considered were: The mathematics of relativity; A solution to a problem on conjugate hyperbolae; Actuarial mathematics; Infinite sets; Mathematical fallacies; Semi-linear equations; The mathematics of extra-sensory perception; and a review of Three Moons in Mathesis.

President, R. Schafer; Secretary-Treasurer, Rhona Garvey; Faculty Advisor, Dr. Harriet Montague.

### *White Mathematics Club, University of Kentucky*

Topics reported for the year were the following: Indeterminate equations; The nine point circle; Trisection of an angle.

President, Sylvia Lavinson; Secretary-Treasurer, Ann White; Faculty Advisor, M. C. Brown

### *Square Circle, Woman's College of the University of North Carolina*

The subjects discussed at program meetings included: Reviews of popular books in mathematics; The significance of the number "*e*"; Mathematical themes in design; New names in mathematics; Survey of Chinese and Japanese mathematics; Magic squares. Social meetings included a Christmas party featuring mathematical games, puzzles, and a formula contest, and the annual club picnic.

President, Dorothy Lewis; Vice-President, Catherine Davis; Secretary-Treasurer, Helen Veasey; Faculty Adviser, Miss Emily Watkins.

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 356. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of enumeration there exists a two-place number with equal digits, whose square is a four-place number with equal digits. If each digit of the four-place number is itself a four-place number in the decimal system, determine the base of the unknown system of enumeration, and show that the solution is unique.

E 357. *Proposed by H. T. R. Aude, Colgate University.*

The equation, $x^2+xy+y^2=k$, is to be solved in positive integers, with $x<y$. Find the least value of $k$ for which there are exactly five distinct solutions.

E 358. *Proposed by L. S. Johnston, University of Detroit.*

Mr. Brown asked Mr. Smith to perform the following operations in the order named, without Mr. Brown's being able to see Mr. Smith's work:

(i) Write any integer, preferably of two digits, to save labor on the part of Mr. Smith.
(ii) Multiply this number by the next higher integer.
(iii) Multiply the result of (ii) by 225.
(iv) Add 56 to the result of (iii).
(v) Tell Mr. Brown all the result of (iv) except the two right-hand digits.

Mr. Smith gave 4064 in response to the request of (v), whereupon Mr. Brown, after a moment's computation, informed Mr. Smith that his result after step (iv) was 406406, and that the number he originally chose was 42. Mr. Smith confirmed these statements.

(A) How did Mr. Brown reach his conclusion?
(B) Generalize the problem.

E 359. *Proposed by Cezar Coşniţă, Focsani, Rumania.*

A quadrilateral, $ABCD$, is inscribed in a circle. A second circle is passed through $A$ and $D$, tangent to $CD$. A third is passed through $B$ and $C$, tangent to $CD$. A fourth is passed through $A$ and $D$, tangent to $AB$. A fifth is passed through $B$ and $C$, tangent to $AB$. Points $E$, $F$, $G$ and $H$ are the respective second intersections of the second circle with $AB$, the third circle with $AB$, the fourth

circle with $CD$, and the fifth circle with $CD$. Show that quadrilaterals $ABGH$, $CDEF$ and $EFGH$ are each inscriptible, and that the centers of the three new circles thus determined are concyclic with the center of the first circle.

E 360. *Proposed by J. E. Trevor, Cornell University.*

Let $S_{ij}$ be the set of $j$ consecutive positive integers, starting with the integer $i$. Form all possible combinations of these integers, taken $1, 2, 3, \cdots, j$, at a time. Add the integers in all these combinations into a single total, $T_{ij}$. Prove that $T_{ij}=j(2i+j-1)2^{j-2}$.

E 361. *Proposed by Virgil Claudian, Bucharest, Rumania.*

The medians of triangle $ABC$ cut the nine-point circle of that triangle again at $D$, $E$ and $F$, respectively. The tangents to this circle at $D$, $E$ and $F$ meet the corresponding sides of the orthic triangle (with vertices at the feet of the altitudes of $ABC$) at the points $P$, $Q$ and $R$ respectively. Prove $P$, $Q$ and $R$ collinear.

## SOLUTIONS

E 320 [1938, 118]. *Proposed by D. H. Lehmer, Lehigh University.*

A table of the binomial coefficients of the 48593rd power would begin as follows:

$$\begin{array}{r}
1\\
48593\\
1180615528\\
19122429707016\\
232289714865976860\\
2257344991124589930108\\
18279979738126929254014584\\
\cdots\cdots\cdots\cdots\cdots
\end{array}$$

Show that if the table were completed the first digits of these numbers would lie very nearly on a parabola, assuming equal spaces for all digits.

*Solution by K. W. Miller, Chicago.*

A binomial coefficient is of the form

$$(1)\qquad {}_KC_N = \frac{N!}{K!(N-K)!}, \qquad 0 \leqq K \leqq N,$$

where $N$ and $K$ are integers. Two cases require attention, namely when $N=2n$ is an even number and when $N=2n+1$ is odd. Referred to the central or maximum value, in either case we have

$$(2) \qquad C = \frac{(2n)!}{(n+k)!(n-k)!}, \qquad -n \leqq k \leqq n,$$

or

$$(3) \qquad C = \frac{\left[2\left(\frac{2n+1}{2}\right)\right]!}{\left[\left(\frac{2n+1}{2}\right)+\left(\frac{2k+1}{2}\right)\right]!\left[\left(\frac{2n+1}{2}\right)-\left(\frac{2k+1}{2}\right)\right]!}, \qquad |k| \leqq n,$$

both of which are now expressed in the general symmetrical form

$$(4) \qquad C = \frac{(2m)!}{(m+h)!(m-h)!}, \qquad -m \leqq h \leqq m, \quad i.e., \quad -\frac{N}{2} \leqq h \leqq \frac{N}{2}.$$

Using the principal value of the Stirling formula we have as a very close approximation to the factorial

$$(5) \qquad x! \doteq \sqrt{2\pi x}\left(\frac{x}{e}\right)^x.$$

Introducing the approximation into equation (4) and making some algebraic reductions we obtain

$$(6) \qquad C = \frac{2^{2m}/\sqrt{\pi m}}{\sqrt{\left[1-\left(\frac{h}{m}\right)^2\right]}\left[1+\frac{h}{m}\right]^{m+h}\left[1-\frac{h}{m}\right]^{m-h}}.$$

Now the number of digits $P$ in any number $C$, when expressed in a number system of base or scale $s$, is the whole number next greater than $P$, or, to the nearest integer, approximately

$$(7) \qquad P = (\log C)/(\log s),$$

where all logarithms are natural or to base $e$. Thus we have from equations (6) and (7)

$$(8) \qquad \begin{aligned} P \log s = \log\left[2^{2m}/\sqrt{\pi m}\right] &= \tfrac{1}{2}\log\left[1-\left(\frac{h}{m}\right)^2\right] \\ &\quad - (m+h)\log\left(1+\frac{h}{m}\right) - (m-h)\log\left(1+\frac{h}{m}\right). \end{aligned}$$

Expanding the logarithmic terms in power series of $h/m$ and collecting terms of equal power in $h$, we find all odd powers of $h$ vanish, whence equation (8) becomes

$$P \log s = \log \left[2^{2m}/\sqrt{\pi m}\right] - \sum_{p=1}^{p=\infty} \left(\frac{h}{m}\right)^{2p} \left[\frac{2m-(2p-1)}{2p(2p-1)}\right]; \tag{9}$$

or restoring $m$ in terms of $N$ we have finally

$$\begin{aligned} P = {} & \frac{(N+\frac{1}{2}) \log 2 - \frac{1}{2} \log N - \frac{1}{2} \log \pi}{\log s} \\ & - \frac{1}{\log s} \sum_{p=1}^{p=\infty} \left(\frac{2h}{N}\right)^{2p} \left[\frac{N-(2p-1)}{2p(2p-1)}\right], \qquad |h| \leqq \frac{N}{2}. \end{aligned} \tag{10}$$

This approximate expression for the number of digits in a binomial coefficient ${}_{(N/2\pm h)}C_N$ is from the nature of equation (5) increasingly accurate the larger the number $N$. In fact by trial it is found to give very close approximation to the binomial coefficients themselves (by taking antilog $P$) as well as giving the next exact number of digits even when $N$ is so small as 5. Moreover when $2h/N$ is small, *i.e.*, for values of $C$ in the central portion of the table, the series converges very rapidly and is closely approximated by the first term only. In this case equation (10) takes the form

$$P = a - bh^2, \tag{11}$$

where $a$ and $b$ are constants and $h$ is the position of the coefficient $C$ measured from the center of the tabulation. Thus the *central* portion of the tabulation will have the first digits lying closely on a parabola. However at the beginning and end of the table this is far from the case. If we take the number $N=48593$ proposed by Dr. Lehmer in the ordinary decimal system $s=10$, equation (10) becomes numerically

$$\begin{aligned} P = {} & 14{,}625.50924 \cdots \\ & - 10{,}551.6187 \cdots (2h/48{,}593)^2 \\ & - \phantom{0}1{,}758.5307 \cdots (2h/48{,}593)^4 \\ & - \phantom{0{,}}703.383 \cdots (2h/48{,}593)^6 \\ & - \phantom{0{,}}376.797 \cdots (2h/48{,}593)^8 \\ & - \cdots . \end{aligned} \tag{12}$$

For the extreme case of the first and last entries in the table which we know to be unity or one digit long we have $h=N/2$ or the ratio $2h/48{,}593$ becomes unity and the series converges very slowly so that the parabolic relation using the second degree term only gives a result incorrect by a number of 4,074 digits. At the center of the table when $h=\pm\frac{1}{2}$ the largest numbers or binomial coefficients have 14,626 digits. Therefore if we fit the vertex of the parabola to the group of first digits at the center of the table it will be off by 28% of these values in fitting the end values! One must take a liberal interpretation of Dr. Lehmer's statement that "these numbers would lie very nearly on a parabola assuming

equal spaces for the digits." One can of course produce an empirical formula, for example

$$P = 14{,}625.50924[1 - (2h/48{,}593)^2] \tag{13}$$

which will be correct at the center and ends of the table but depart by several integers at the intermediate values, but several integers is a large per cent error in absolute value!

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3900. *Proposed by J. E. Trevor, Cornell University.*

Two players carried out an experiment with a heap of freshly minted twenty-five cent pieces. The first player discarded one coin from the heap, and then one half of the remainder, which was easily done by use of an available balance. In the second move, beginning with the undiscarded coins, the second player discarded $2^r$ coins, and also one half of the coins then remaining. Here $r$ is an arbitrary positive integer. In general, in the $x$th move, $x^r$ coins and one half of those then remaining were discarded. If $n$ coins were given, and $y$ coins remained at the close of the $x$th move, $n$ is a function of $x$, $y$, $r$. For successive values $r=1, 2, 3, \cdots$ of the parameter $r$ obtain the formulation

$$n = 2^x[y + \phi_r(x)] + c_r,$$

where $\phi_r(x)$ is a polynomial of $r+1$ terms, and $c_r$ is a constant.

3901. *Proposed by V. Thébault, Le Mans, France.*

Given the positive integers $a$, $b$, $c$ such that $a^2=b^2+c^2$, the positive numbers $m$ and $n$ may be determined so that (1) $a=m+n+(2mn)^{1/2}$. Conversely, if $2mn$ is a perfect square, then $a$ in (1) is such that its square is the sum of two squares. (2) Show that, if $x=2(m+n)/mn$, the numbers

$$(bx+1)^2-1, \qquad (cx+1)^2-1, \qquad (ax-1)^2-1$$

are also the sides of a right triangle, the last term being the length of the hypotenuse. Determine the smallest integral value of $x$ so that the sides may be expressed as integers.

3902. *Proposed by V. Thébault, Le Mans, France.*

A point $M$ is chosen arbitrarily on the circumcircle of the triangle $ABC$, and the chords $MA'$, $MB'$, $MC'$ are drawn parallel to $BC$, $CA$, $AB$. Show that the orthopoles of the circumcircle diameters through $A'$, $B'$, $C'$ are the vertices of a triangle equal to the orthic triangle of $ABC$. Generalize.

## SOLUTIONS

3800 [1936, 501]. *Proposed by D. C. Duncan, Los Angeles City College.*

Solve the equation

$$Ax^4 + (x-1)^3 = 0, \qquad A \neq 0,$$

given that it has a double root. Also prove the generalized theorem:

*For just one set of real values of $A$ and $a_i$ the equation*

$$Ax^{2k} + (x-1)^3 \prod_{i=1}^{k-2} (x-a_i)^2 = 0, \qquad A \neq 0, \qquad k > 2,$$

*has $k-2$ double roots.*

*Editorial Note 1.* Problem 3549 [1932, 239] is similar to the above, and the theorem in this latter problem is used in the proposer's article *A plane elliptic curve of order $4k+2$, with singularities all real and distinct, and autopolar by $4k+4$ conics*, Bull. Amer. Math. Soc., vol. 39 (1933), p. 810. No solution of problem 3549 has been received.

*Editorial Note 2.* It will be shown that for $k=3$, $A\neq 0$, and $a$ real, the equation

$$(1) \qquad Ax^6 + (x-1)^3(x-a)^2 = 0$$

has two distinct double roots for two sets of values $a$ and $A$. For $a=1$ and the corresponding value of $A \neq 0$, it has one double root. For each value of $a$ other than the above there corresponds two values of $A\neq 0$ such that each of the two equations (1) has one double root. In each of the above cases there are no other multiple roots.

Setting the derivative of the left side of (1) equal to zero, and eliminating $A$ from the two equations, we obtain an equation of the fifth degree whose roots are given by

$$(2) \qquad x^2 - (3a+4)x + 6a = 0, \quad (3) \quad (x-1)^2(x-a) = 0.$$

If $a=0$, $x=0$ is a root of both (2) and (3); but one of these zero roots was introduced by the elimination in this case. If we remove this double root from (1) we have the equation of the fourth degree in the first part of the problem. This resulting equation has a double root for a suitable $A\neq 0$, if and only if the double root is 4, as shown by (2). It then follows that $A = -3^3/4^4$. The remaining two roots are then easily found to be $4[5\pm\sqrt{-2}]/27$. This disposes of the first part of the problem and the case $a=0$ for which (1) has two double roots 0 and 4.

There are two cases in which $a$ is a root of (2), $a=0$ and $a=1$. If $a=1$ the roots of (2) are 1 and 6. For $A\neq 0$ unity cannot be a root of (1). Hence for $a=1$ the equation (1) has 6 for a double root if $A=-5^5/6^6$. Since neither $a\neq 0$ nor unity can be a root of (1) for $A\neq 0$, we have no further use for (3). For $x=2$ the left side of (2) has the value $-4$, hence (2) has always a positive root greater than 2, and, moreover, the two roots are always distinct. For $x=1$ the left side of (2) has the value $3(a-1)$. These results will now be used in examining two intervals for $a$. First, if $a<0$, the negative root of (2) is a double root of (1) for a corresponding positive value of $A$, while the positive root is a double root of (1) for a corresponding negative value of $A$. For the interval $0<a<1$ the equation (2) has both roots positive; the smaller root is a double root of (1) for a positive value of $A$ since the root is less than unity, while the larger root is a double root for a negative value of $A$. For $a>1$ the smaller positive root of (2) is greater than unity, and hence the two values of $A$ are both negative.

We now consider the values of $a$ for which the smaller root $x_2\neq 0$ and the larger root $x_1$ determine the same value of $A$, that is so that each is a double root of the same equation (1). For this we must have

$$(x_1-1)^3(x_1-a)^2x_2^6-(x_2-1)^3(x_2-a)^2x_1^6=0. \tag{4}$$

The left side of (4) is obviously divisible by $x_1-x_2$, and the quotient is then symmetric in $x_1$ and $x_2$. The symmetric functions of $x_1$ and $x_2$ that appear after the division are easily expressed in terms of $a$. After a rather tedious computation (4) reduces to

$$243a^5-972a^4+1080a^3-1872a^2+1152a-256=0, \tag{5}$$

after discarding the other factor of (4) $a^2(9a^2+16)^{1/2}$. From the form of (5) we infer that it has no negative root, a fact we knew before getting (5). Also from the preceding results we know that it has no root between zero and unity. If the roots of (5) are reduced by unity the resulting equation has the first two terms positive and the remaining terms negative; hence (5) has only one real root $\alpha$, and this completes the proof. We shall give the approximate numerical results. It is easily seen by substitution of 4 in (5) that no real root is beyond 4, and then the root is seen to lie between 3 and 4.

$$\alpha=3.23154806791, \qquad A=-0.034273030,$$
$$x_1=12.09103563, \qquad x_2=1.60360857.$$

The extension to $k=4$ for the case of one zero double root is trivial.

The problem of determining whether the general equation can have $k-1$ distinct double roots, no one of which is zero, appears to be much more difficult. However, it is possible to find out something about the values of the $a_i$'s. If we set $x=t^{-1}$ and $a_i=b_i^{-1}$, the equation becomes

$$t(t-1)^3\prod_{i=1}^{k-2}(t-b_i)^2=B,$$

and we are to determine the $b_i$'s so that the curve represented by the left member is tangent to a horizontal straight line at $k-1$ distinct points. The curve for $y=t(t-1)^3$ is below the $t$ axis between 0 and 1; and, if there are $m$ of the $b_i$'s inside this interval, there are $m+1$ minimum points within the interval and $k-2-m$ maximum points outside. Hence all of the $b_i$'s must be inside the interval in order to give $k-1$ critical points of the same type; and in this case they are minimum points. By using the logarithmic derivative, we find that the values of $t$ which give multiple roots, $B\neq 0$, must satisfy

$$\frac{1}{t}+\frac{3}{t-1}+2\sum\frac{1}{t-b_i}=0.$$

For $0<b_1<b_2<\cdots<b_{k-2}<1$, we see that there are no multiple roots other than double roots. There will be precisely $k-1$ distinct double roots in this case provided that each of the roots of this last equation gives the same value for $B$ in the first equation above. We shall then have as many equations as unknowns. The question now arises: Does there exist one and only one set of real and distinct values for the $b_i$'s, no one of which is zero, for which these equations are true? For the discussion of this question in the general case we need a method less tedious than that given for $k=3$.

3808 [1936, 643]. *Proposed by N. A. Court, University of Oklahoma.*

$A$, $B$, $C$ are three given points on three given spheres $(P)$, $(Q)$, $(R)$. Find a point $D$ on the radical axis $u$ of the three given spheres such that, if $A'$, $B'$, $C'$ are the points of intersection of the lines $DA$, $DB$, $DC$ with the spheres $(P)$, $(Q)$, $(R)$, respectively, the plane $A'B'C'$ shall be parallel to the plane determined by the centers $P$, $Q$, $R$ of the three spheres.

*Note.* A similar problem relative to two circles was discussed in the *Nouvelles Annales de Mathématiques*, 1846, p. 260, and 1848, p. 231.

*Solution by J. Hadamard, Paris, France.*

It is clear that, generally, the problem has no solution, there being two conditions, *i.e.* parallelism of planes, for one unknown, the position of $D$ on $u$. More precisely, if we invert with $D$ as a pole and an inversive power equal to the power of $D$ with respect to the given spheres, the plane $A'B'C'$ will become the sphere $ABCD$, which ought to cut $u$ orthogonally and therefore to have its center on $u$ and also on the axis of the circle $ABC$. But these two lines in the general case will have no common point. The case is, of course, quite different from the plane one, which, however, can be treated by the same method.

Solved also by the proposer who stated that, if $u$ and the axis of circle $ABC$ intersect, there are two solutions for $D$.

3809 [1936, 643]. *Proposed by Don Wallace, Charlottesville, Va.*

Let $P$ be that point for which the product of its distances to the vertices of a triangle is an extreme. Then $P$ is the symmedian point of its pedal triangle with respect to the first triangle.

*Solution by C. M. Sparrow, University of Virginia.*

Let the vertices and sides of the triangle be $A_k$, $a_k (k=1, 2, 3)$; those of the pedal triangle of $P$, $B_k$, $b_k$. Let $PA_k = r_k$, $PB_k = h_k$. It was shown in the solution of Problem 3769 that the condition on $P$ is equivalent to the condition

$$\sin A_2PA_3 : \sin A_3PA_1 : \sin A_1PA_2 = 1/r_1 : 1/r_2 : 1/r_3.$$

But $a_1h_1 = r_2r_3 \sin A_2PA_3$, *etc.* This gives $h_ka_kr_k^2 = \text{const}$ for $k=1, 2, 3$. By the law of sines we can write this

$$(1) \qquad (r_k \sin A_k)^2 = c \sin A_k/h_k.$$

The double area of the triangle $PB_2B_3 = h_2h_3 \sin B_2PB_3 = h_2h_3 \sin A_1$. The areal coördinates of $P$ with respect to its pedal triangle are thus $\sin A_k/h_k$, or, by (1) $(r_k \sin A_k)^2$. $B_2$, $B_3$ are on the circle whose diameter is $PA_1 (=r_1)$. Hence $B_2B_3 = r_1 \sin A_1 = b_1$. Using this, we see that the areal coördinates of $P$ become $b_1^2$, $b_2^2$, $b_3^2$, so that $P$ is the symmedian point of its pedal triangle.

Solved also by the proposer.

*Editorial Note.* The problem apparently means that only points $P$ are to be considered which have an actual pedal triangle, since it says *that* point $P$ and since the exceptional cases are trivial.

## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

The Fifth International Congress for the Unity of Science is to be held at Harvard University, September 5–10, 1939. The theme of the Congress is *Logic of Science*. Interest will center upon the relation of concepts, laws, and methods of the various sciences. Attention will be devoted to general problems connected with the unification of science, and, in particular, with the logic of the physical sciences, the relation of the physical and biological sciences, and the relation of the biological and sociohumanistic sciences. Those who wish later notices of the Congress should send request to Professor C. W. Morris, University of Chicago, Chicago, Illinois.

Word has reached this country that the Editor of the *Zentralblatt für Mathematik und ihre Grenzgebiete*, Professor Otto Neugebauer, now of Copenhagen, has resigned. The resignation from this mathematical abstracts journal was occasioned by the action of the publisher, Julius Springer of Berlin, in dropping Professor Levi-Civita of Italy from the board without the knowledge of the Editor, as well as by the demand that the Editor give assurance that no emigrants would be allowed to referee articles by German authors. In consequence of this interference with editorial policies, the American associate editors, Professors Tamarkin and Veblen, have tendered their resignations as have also a number of associate editors and collaborators in other countries.

The Mathematics division of the Society for the Promotion of Engineering Education has appointed a committee for the purpose of collecting problems from various scientific and engineering fields. These problems are to be suitable for use in the courses in mathematics taken by freshmen and sophomores in the College of Engineering and those not found in the usual mathematics texts. Any one having at hand such problems is requested to send them to the chairman, Professor J. W. Cell, Box 5548, College Station, Raleigh, North Carolina.

The following persons are in residence at the Institute for Advanced Study for all or part of the current academic year: Professor H. E. Arnold, Dr. W. H. Barkas, Dr. P. G. Bergmann, Dr. Valentin Bergmann, Professor Niels Bohr, Dr. Herbert Busemann, Professor Claude Chevalley, Professor H. B. Curry, Professor Jesse Douglas, Dr. Paul Erdös, Dr. Kurt Gödel, Professor G. A. Hedlund, Dr. Witold Hurewicz, Professor B. W. Jones, Dr. Dorothy Manning, Professor Walther Mayer, Dr. A. P. Morse, Professor Tadasi Nakayama, Professor I. I. Rabi, Dr. H. E. Robbins, Professor Leon Rosenfeld, Dr. Hyman Serbin, Gertrude K. Stanley, Mrs. C. C. Torrance, Dr. C. C. Torrance, Dr. Henry Wallman, J. S. de Wet.

At the University of California, Jerzy Neyman has been appointed professor of mathematics, and Assistant Professor C. B. Morrey has been promoted to

the rank of associate professor. Also Associate Professor Pauline Sperry has been on leave of absence for the first half-year 1938–39, and Professor B. A. Bernstein has been granted a leave of absence for the second half-year 1938–39.

Dr. J. W. Calkin has been appointed an assistant professor at the University of New Hampshire.

R. E. Gaskell has been teaching the classes of Professor E. R. Sleight of Albion College during his leave of absence for the first semester 1938–39.

The following appointments to instructorships are announced:

University of California: Dr. F. W. Dresch
Stanford University: C. R. Bubb, C. D. Olds
Wilson Junior College, Chicago: Edna M. Feltges, Dr. Ruth B. Rasmusen, Jerome Sachs
Wright Junior College, Chicago: Dr. Leonard Tornheim

Dr. James Pierpont, professor emeritus at Yale University, died December 9, 1938. He had been a member of the Yale faculty since 1898 and at his retirement in 1933 was Erastus L. DeForest Professor of Mathematics.

Associate Professor W. E. Wilbur of the University of New Hampshire died September 3, 1938. He had been on the New Hampshire faculty since 1922.

The Editor wishes to express his appreciation to the following persons who have refereed papers or otherwise assisted in the work of editing the MONTHLY for the years 1937 and 1938:

R. C. Archibald, W. L. Ayres, J. P. Ballantine, R. W. Barnard, Walter Bartky, J. W. Bradshaw, B. H. Brown, F. L. Brown, O. E. Brown, Daniel Buchanan, W. H. Bussey, W. B. Carver, W. F. Cheney, Jr., E. W. Chittenden, N. A. Court, A. T. Craig, H. V. Craig, D. R. Curtiss, H. T. Davis, L. E. Dickson, L. L. Dines, H. L. Dorwart, Otto Dunkel, Arnold Emch, E. J. Finan, B. F. Finkel, Tomlinson Fort, Orrin Frink, Jr., T. C. Fry, R. E. Gilman, J. W. Givens, L. M. Graves, Lois W. Griffiths, V. G. Grove, W. L. Hart, E. H. C. Hildebrandt, T. H. Hildebrandt, R. R. Hitchcock, C. A. Hutchinson, Mark Ingraham, Dunham Jackson, R. L. Jeffery, R. A. Johnson, B. W. Jones, A. J. Kempner, J. F. Kenney, B. O. Koopman, E. P. Lane, C. G. Latimer, D. H. Lehmer, Walter Leighton, Mayme I. Logsdon, W. R. Longley, C. C. MacDuffee, W. D. MacMillan, W. L. Miser, U. G. Mitchell, J. R. Musselman, J. A. Nyswander, H. L. Olson, F. W. Owens, Helen B. Owens, W. C. Randels, W. T. Reid, H. L. Rietz, Robin Robinson, W. H. Roever, R. G. Sanger, H. A. Simmons, W. G. Simon, C. H. Sisam, D. E. Smith, Gabriel Szegö, W. J. Trjitzinsky, A. W. Tucker, H. S. Wall, G. C. Webber, R. M. Winger, and F. E. Wood.

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AUBREY JOHN KEMPNER

*WITH THE CO-OPERATION OF*



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NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.

## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, May 13.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 13.
MICHIGAN, Ann Arbor, March 18.
MINNESOTA.
MISSOURI, Springfield, April 28.
NEBRASKA, Lincoln, May 5.
OHIO, Columbus, April 8.
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 2–3.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

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# SOME TOPICS IN MATHEMATICAL STATISTICS

J. F. KENNEY, Northwestern University

**1. Fundamental notions and definitions.** First courses in statistics deal largely with descriptive characterizations of quantitative data. When $N$ observations on one or more variables are available, such characterizations consist in computing the numerical values of certain functions, such as the mean, variance, and correlation coefficient, denoted respectively by $\bar{x}$, $s^2$, and $r$, and defined as follows:

$$N\bar{x} = \sum x_i, \qquad Ns_x^2 = \sum (x_i - \bar{x})^2, \qquad Nrs_x s_y = \sum x_i y_i - N\bar{x}\bar{y}.$$

(The sign $\sum$ here and hereafter indicates in every case summation over values of the index from 1 to $N$.)

Both for practical and for interesting theoretical purposes it is important to construct a theory dealing with abstract notions. This theory, in the field of statistics, is commonly called mathematical statistics. It is an idealization of observed distributions and their summarizing functions, comparable to the idealization of the outlines of material objects into the figures of geometry.

The main object of the present paper is to give, in brief form, a nearly self-contained exposition of certain developments in this field that are not so well known as they should be to many teachers of college courses in statistics.

A fundamental notion in mathematical statistics is the concept of distribution function. It relates to a *population* or universe of discourse. A continuous variable $x$ is said to have the distribution function $f(x)$, which we take to be single-valued and non-negative, if the frequency of occurrence of $x$ in the range $a < x < b$ is measured by

$$\int_a^b f(x)dx. \tag{1}$$

A constant factor in the distribution function may be determined so that

$$\int_{-\infty}^{\infty} f(x)dx = 1, \tag{2}$$

and under this condition the integral (1) denotes the probability that $x$ lies in the interval $(a, b)$. If the actual occurrence of the variable is limited to a finite range, $f(x)$ is defined to be identically zero outside that range.

Corresponding definitions will now be given for a distribution function of two variables. The continuous variables $(x, y)$ have the joint distribution function $f(x, y)$ if the double integral of $f(x, y)$ over a region of the $(x, y)$-plane measures the frequency of occurrence of pairs of values $(x, y)$ in that region. It will be understood that $f(x, y)$ is single-valued and non-negative. If values of $(x, y)$ are restricted to a finite region we define $f(x, y)$ to be identically zero outside that region. In the extended region of definition, if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1 \tag{3}$$

then

$$\int_{a}^{b}\int_{c}^{d} f(x, y)dydx \tag{4}$$

represents the probability that $x$ lies between $a$ and $b$ at the same time that $y$ lies between $c$ and $d$.

In the case of joint distribution functions we distinguish between two cases: (a) when the variables are independent in the probability sense, and (b) when they are correlated. The variables $x$ and $y$ are independent in the probability sense if

$$f(x, y) \equiv g(x)h(y)$$

where $g(x)$ and $h(y)$, called marginal distributions, are defined as follows: $g(x)=\int f(x, y)dy$ and $h(y)=\int f(x, y)dx$, the integrals being taken over all admissible values of the variables. When $x$ and $y$ are correlated, $f(x, y)$ cannot be expressed as the product of the functions describing their distributions separately.

**2. Expected values.** Let the variable $x$ be subject to the distribution function $f(x)$ and let $\phi(x)$ be an arbitrary function of $x$. Then the expected value of $\phi(x)$, denoted by application of the operator $E$, is defined by

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)f(x)dx$$

provided this integral exists. In particular, if $\phi(x)=x^k$, $(k=1, 2, \cdots)$, we have

$$E(x^k) = \int_{-\infty}^{\infty} x^k f(x)dx. \tag{5}$$

For $k=1$, (5) defines the mean of the $x$'s in the universe represented by $f(x)$. This will be denoted by $\tilde{x}$ in contradistinction to the mean, $\bar{x}$, of a sample. Therefore we may write

$$E(x) = \tilde{x}. \tag{6}$$

If $\phi(x)=(x-\tilde{x})^2$, we have the variance of $x$

$$\begin{aligned} \sigma_x^2 &= E(x-\tilde{x})^2 \\ &= E(x^2) - \tilde{x}^2. \end{aligned} \tag{7}$$

The (positive) square root of $\sigma_x^2$ is called the standard deviation or standard error of the distribution of $x$. Analogous definitions hold, of course, for $y$.

If the variables $x$ and $y$ are simultaneously distributed in accord with the function $f(x, y)$, then

$$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dydx.$$

The correlation coefficient, $\rho$, between $x$ and $y$ in the bivariate universe represented by $f(x, y)$ is defined by

$$\rho\sigma_x\sigma_y = E(xy) - \tilde{x}\tilde{y}. \tag{8}$$

The quantities $\tilde{x}$, $\sigma$, $\rho$, and so on, relating to a universe, are called parameters.

The following propositions may easily be established, and so they are stated without proof.

I. *The expected value of the product of a variable and a constant is equal to the product of the constant and the expected value of the variable. That is,*

$$E(cx) = cE(x).$$

II. *The expected value of deviations of a variable from its expected value is zero. That is,*

$$E(x - \tilde{x}) = 0.$$

III. *The expected value of the sum of two or more variables is the sum of their expected values. In symbols,*

$$E(x + y + z) = E(x) + E(y) + E(z).$$

IV. *If $x$ and $y$ are mutually independent in the probability sense, then the expected value of their product is equal to the product of their expected values. That is,*

$$E(xy) = E(x)E(y).$$

V. *The expected value of the product of deviations of two mutually independent variables from their expected values is zero. That is,*

$$E[(x - \tilde{x})(y - \tilde{y})] = 0.$$

VI. *The expected value of the product of deviations of two correlated variables from their expected values is given by*

$$E[(x - \tilde{x})(y - \tilde{y})] = \rho\sigma_x\sigma_y.$$

**3. Standard error of a linear function of variables.** Let $w$ be a linear function defined by

$$w = c_1x_1 + c_2x_2 + \cdots + c_Nx_N, \tag{9}$$

where each variable $x_k$, ($k=1$ to $N$), is arbitrarily distributed and where the $c$'s are arbitrary constants. Let $\sigma_k$ represent the standard error of $x_k$ in the universe to which it belongs, $\sigma_w$ the standard error of the distribution of $w$, and $\rho_{ij}$ the correlation coefficient (if any correlation exists) between $x_i$ and $x_j$. We seek the standard error of $w$ in terms of $\sigma_k$ and $\rho_{ij}$, ($i=1$ to $N, j=1$ to $N$).

Case I. We will suppose first that the variables in the several universes are

correlated, that is, that $\rho_{ij}$ is different from zero for every combination of $i$ and $j$. From (9) and Proposition III we have

$$E(w) = c_1E(x_1) + c_2E(x_2) + \cdots + c_NE(x_N), \tag{10}$$

that is,

$$\tilde{w} = c_1\tilde{x}_1 + c_2\tilde{x}_2 + \cdots + c_N\tilde{x}_N. \tag{11}$$

Then

$$E(w - \tilde{w})^2 = \sum c_i^2 E(x_i - \tilde{x}_i)^2 + \sum_{i \neq j} c_ic_jE(x_i - \tilde{x}_i)(x_j - \tilde{x}_j),$$

which by definition (7) and Proposition VI becomes

$$\sigma_w^2 = \sum c_i^2 \sigma_i^2 + \sum_{i \neq j} c_ic_j\rho_{ij}\sigma_i\sigma_j. \tag{12}$$

If $c_1 = 1$, $c_2 = \pm 1$, and $N = 2$, we have as a special case

$$\sigma_w^2 = \sigma_1^2 \pm 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2. \tag{13}$$

Case II. Suppose the $x$'s in (9) are mutually independent in the statistical sense so that $\rho_{ij} = 0$. Then (12) becomes

$$\sigma_w^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_N^2\sigma_N^2. \tag{14}$$

**4. Theorems.** Relations (10)–(14) enable us to prove some interesting and useful theorems about the distribution of means of samples from an arbitrary universe. Let $(x_1, x_2, \cdots, x_N)$ be a set of $N$ independent variables each subject to the same distribution function $g$, so that their joint distribution function is

$$f(x_1, x_2, \cdots, x_N) \equiv g(x_1)g(x_2) \cdots g(x_N).$$

Then $(x_1, x_2, \cdots, x_N)$ is called a *sample* of $N$ from a universe with distribution function $g(x)$.

THEOREM I. *If samples of size N be drawn from an arbitrary universe and if $\bar{x}$ be the mean of a sample, then the mean of all possible such means equals the mean of the universe. That is,* $E(\bar{x}) = \tilde{x}$.

*Proof.* In (9), let $c_1 = c_2 = \cdots = c_N = 1/N$ and let $x_1, x_2, \cdots, x_N$, constitute a sample from a universe with mean $x$ and variance $\sigma_x^2$. Then $w = \bar{x}$. As a consequence of the definition of sample, $E(x_i) = \tilde{x}$ for each value of $i$ from 1 to $N$. Therefore (10) becomes, $E(\bar{x}) = \tilde{x}$.

THEOREM II. *The variance of the distribution of means of samples of size N from an arbitrary universe equals the variance of the universe divided by N. In symbols,*

$$\sigma_{\bar{x}}^2 = \sigma_x^2/N. \tag{15}$$

*Proof.* As in the proof of Theorem I let $w = \bar{x}$. Then (14) becomes

$$\sigma_{\bar{x}}^2 = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2)/N^2. \tag{16}$$

Since the $x$'s constitute a sample, $\sigma_i^2 = \sigma_x^2$ for each value of $i$ from 1 to $N$. So (16) reduces to (15).

In the next three theorems it will be understood that $x$ and $y$ are correlated variables which are simultaneously distributed in accord with $f(x, y)$ in which the parameters are $\tilde{x}$, $\tilde{y}$, $\sigma_x$, $\sigma_y$, and $\rho$.

THEOREM III. *Let* $(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)$ *be a sample of N pairs drawn independently from* $f(x, y)$ *and let* $(\bar{x}, \bar{y})$ *be the mean of a sample. The correlation coefficient, R, between the means of all possible such samples equals* $\rho$.

*Proof.* By definition

$$R = \frac{E(\bar{x}\bar{y}) - \tilde{x}\tilde{y}}{\sigma_{\bar{x}}\sigma_{\bar{y}}}. \tag{17}$$

And

$$\begin{aligned} E(\bar{x}\bar{y}) &= \frac{1}{N^2} E[(x_1 + x_2 + \cdots + x_N)(y_1 + y_2 + \cdots + y_N)] \\ &= \frac{1}{N^2} E(S), \end{aligned}$$

where

$$\begin{aligned} S = {} & x_1y_1 + x_1y_2 + \cdots + x_1y_N \\ & + x_2y_1 + x_2y_2 + \cdots + x_2y_N \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + x_Ny_1 + x_Ny_2 + \cdots + x_Ny_N. \end{aligned}$$

We will separate $S$ into two parts, conveniently called $u$ and $v$, where

$$u = x_1y_1 + x_2y_2 + \cdots + x_Ny_N,$$

and

$$v = \text{sum of } (N^2 - N) \text{ terms of the form } x_iy_j, \quad i \neq j.$$

Then

$$E(u) = E[\sum (x_iy_i)] = \sum [E(x_iy_i)] = NE(xy).$$

In $v$, $x_i$ must be uncorrelated with $y_j$ since $i \neq j$. Therefore

$$E(x_iy_j) = E(x_i)E(y_j) = \tilde{x}\tilde{y},$$

and

$$E(v) = (N^2 - N)\tilde{x}\tilde{y}.$$

So we have

$$E(S) = NE(xy) + (N^2 - N)\tilde{x}\tilde{y},$$

and therefore

(18) $$E(\bar{x}\bar{y}) = \frac{1}{N}\,[E(xy) + (N-1)\tilde{x}\tilde{y}].$$

Making use of Theorem II and (18) the right member of (17) reduces to the definition of $\rho$.

THEOREM IV. *Let $\bar{x}$ be the mean of a sample of $N$ from $g(x)$ and let $\bar{y}$ be the mean of a sample of $N$ from $h(y)$ where $g(x)$ and $h(y)$ are the marginal distributions of the universe $f(x, y)$ of correlated variables. Let $w = \bar{x} - \bar{y}$. The variance of the sampling distribution of $w$ is*

(19) $$\sigma_w^2 = \frac{1}{N}\left\{\sigma_x^2 - 2\rho\sigma_x\sigma_y + \sigma_y^2\right\}.$$

The proof follows from (13) and Theorem III.

THEOREM V. *Let $\bar{x}$ and $\bar{y}$ be the means, $s_x$ and $s_y$ the standard deviations, and $r$ the correlation coefficient of a sample of $N$ correlated items. Suppose $N$ is so large that $s^2$ is a good estimate of $\sigma^2$ and $r$ of $\rho$, so that we may write*

$$\sigma_{\bar{x}}^2 = s_x^2/N, \qquad \sigma_{\bar{y}}^2 = s_y^2/N, \qquad \rho = r.$$

*The variance of the sampling distribution of $w$, referred to in Theorem IV, may be computed from the sample by the formula*

(20) $$\sigma_w^2 = \frac{1}{N^2}\left[\sum (x_i - y_i)^2 - \frac{(\sum x_i - \sum y_i)^2}{N}\right].$$

The proof follows from (19).

**5. Expected value of sample variances.** The definition of the variance, $s^2$, of a sample may be written

$$s^2 = (x_1^2 + x_2^2 + \cdots + x_N^2)/N - \bar{x}^2.$$

Then the expected value of $s^2$ from repeated samples is

$$E(s^2) = E\left[\frac{1}{N}(x_1^2 + x_2^2 + \cdots + x_N^2)\right] - E(\bar{x}^2).$$

Since the $x$'s constitute a sample, we may write

$$E(x_1^2 + x_2^2 + \cdots + x_N^2) = NE(x^2),$$

and from (18), replacing $y$ by $x$ there, we have

$$E(\bar{x}^2) = \frac{1}{N}\,[E(x^2) + (N-1)\tilde{x}^2].$$

Therefore

$$E(s^2) = \frac{1}{N}[NE(x^2)] - \frac{1}{N}[E(x^2) + (N-1)\tilde{x}^2]$$

$$= \frac{N-1}{N}[E(x^2) - \tilde{x}^2],$$

and using (7) we obtain

$$E(s^2) = \frac{N-1}{N}\sigma^2. \tag{21}$$

Thus the mean of the sampling distribution of $s^2$ from an arbitrary universe is given by (21).

As Jackson [1] observes, it is to be anticipated that the expected value of $s^2$ is less than $\sigma^2$. The variance $\sigma^2$ refers to deviations from $\tilde{x}$, whereas any $s^2$ refers to deviations from an $\bar{x}$. For any sample, then, we may regard $\tilde{x}$ as an arbitrary origin. Since, in the case of any sample, the sum of the squares of deviations from its mean, $\bar{x}$, is less than the sum of the squares of the deviations of the same variates from an arbitrary point $\tilde{x}$ (unless the sample is one whose mean falls at $\tilde{x}$), it is to be expected that the mean of all the values of $s^2$ will be less than $\sigma^2$. Relation (21) measures the extent of this inequality.

**6. Unbiased estimates of population parameters.** A distribution function is not only a function of the variable involved, but it is also a function of the parameters, or hypothetical quantities, which are introduced to specify the universe sampled. For example, in the normal distribution function

$$f(x) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-(x-\tilde{x})^2/2\sigma^2}$$

the population mean $\tilde{x}$ and the variance $\sigma^2$ are the parameters.

An appropriate function of the variates given by a sample for estimating a parameter is called a *statistic*. If the expected value of a statistic equals the corresponding parameter, then the statistic is called an *unbiased estimate* of the parameter.

It is clear from Theorem I that $\bar{x}$ is an unbiased estimate of $\tilde{x}$. Let $\dot{\sigma}^2$ be an unbiased estimate. If this estimate* is based on a single sample, then from (21) we have

$$\dot{\sigma}^2 = \frac{N}{N-1}s^2 = \frac{\sum (x_i - \bar{x})^2}{N-1}. \tag{22}$$

If $n = N-1$ it is obvious that

$$s^2 = \frac{n}{n+1}\dot{\sigma}^2. \tag{22a}$$

* R. A. Fisher and some other writers use the symbol $s^2$ to denote this unbiased estimate of $\sigma^2$.

It is conventional [2, 3] to take

$$\dot{\sigma} = \left\{\frac{N}{N-1}\right\}^{1/2} s \tag{23}$$

as an estimate of $\sigma$. If $N$ is large the difference between unity and the coefficient of $s$ in (23) is negligible in numerical problems. With $N$ large it would not be invalid, to any appreciable extent, to use $s$ as an estimate of $\sigma$.

If two independent samples are available from the same universe, an unbiased estimate based on the two samples is given by

$$\dot{\sigma}^2 = q/(N-2), \tag{24}$$

where

$$q = N_1 s_1^2 + N_2 s_2^2, \qquad N = N_1 + N_2,$$

$s_1^2$ and $s_2^2$ being the variances of samples consisting of $N_1$ and $N_2$ variates respectively.

In case $k$ independent samples are available from the same universe, we may generalize (24) and write

$$\dot{\sigma}^2 = Q/(U-k), \tag{25}$$

where

$$Q = N_1 s_1^2 + N_2 s_2^2 + \cdots + N_k s_k^2,$$
$$U = N_1 + N_2 + \cdots + N_k,$$

and $s_i^2$ is the variance of the $i$th sample consisting of $N_i$ variates. In subsequent references to $\dot{\sigma}$ it will be clear from the context, if it is not stated explicitly, whether this estimate is based on 1, 2, or $k$ samples.

If $N_i = N$ is the same for every sample, (25) reduces to

$$\dot{\sigma}^2 = \frac{N(s_1^2 + s_2^2 + \cdots + s_k^2)}{U-k}, \tag{26}$$

where $U = Nk$. Clearly, (26) may be written in the form

$$\frac{N-1}{N}\dot{\sigma}^2 = \frac{1}{k}(s_1^2 + s_2^2 + \cdots + s_k^2). \tag{26a}$$

When $k$ is taken infinitely large so that $U$ becomes the universe, the right member of (26a) then refers to the expected value of $s^2$ and $\dot{\sigma}^2$ becomes $\sigma^2$ itself. Hence as $k \to \infty$ the limiting value of (26a) becomes

$$\frac{N-1}{N}\sigma^2 = E(s^2),$$

as given in (21).

As an alternate to (26), in the case where all samples contain the same number of variates, we may take

$$\dot{\sigma} = \frac{1}{b(N)} \times \frac{1}{k}(s_1 + s_2 + \cdots + s_k) \tag{27}$$

$$= \frac{1}{b(N)} \times \text{mean value of standard deviations,}$$

where $b(N)$ is a function of $N$ which approaches unity as $N$ increases. The exact expression for $b(N)$ will be derived in §9. We have as an approximate value $b(N) = 1 - 3/4N$. The limiting value of (27) as $k \to \infty$ is

$$\sigma = E(s)/b(N). \tag{28}$$

In §9 we shall show that $b(N)\sigma$ is the expected value of the sampling distribution of $s$ from a normal universe whose standard deviation is $\sigma$. Values of $b(N)$ have been tabulated by E. S. Pearson [**4**] and others [**5**].

As an alternate to (23) we have from (27) when $k = 1$,

$$\dot{\sigma} = s/b(N). \tag{29}$$

**7. Degrees of freedom.** In §6 we have proved, essentially, that the expected value of $\Sigma(x_i - \bar{x})^2$ is $(N-1)\sigma^2$, where the $N$ values of $x$ in the sample are subject to the linear restriction $\Sigma x_i = N\bar{x}$. This is equivalent to proving that the expected value of $\Sigma x_i^2$ is $(N-1)\sigma^2$ when the $x$'s are subject to the linear restriction $\Sigma x_i = 0$. Suppose, however, that there are $k < N$ linear restrictions on the $x$'s. What then is the expected value of $\Sigma x_i^2$? A. T. Craig [**6**] has proved analytically that if $x_1, x_2, \cdots, x_N$ are $N$ independent values of a variable which is normally distributed about zero with variance $\sigma^2$ and if the $N$ values of $x$ are subject to $k < N$ homogeneous linear restrictions, then the expected value of $\Sigma x_i^2$ is $(N-k)\sigma^2$. The number $n = N - k$ is frequently called the number of "degrees of freedom."

**8. "Student's" distribution.** We have shown (Theorems I and II) that the means of samples of $N$ from an arbitrary universe with mean $\tilde{x}$ and variance $\sigma^2$ are distributed about $\tilde{x}$ with variance $\sigma^2/N$. It is well known and can readily be proved (see below) that if the universe is normal then the means are themselves normally distributed. In fact, if the $x$'s in (9) are independent and normally distributed, then $w$ is normally distributed. This property of a linear function of normally distributed variables being normally distributed is sometimes called the *reproductive property* of the normal law.

On the side of applications, $\sigma^2$ is seldom available and must be estimated from the available data. If we use the estimate defined in (22) we may replace $\sigma^2/N$ by $s^2/(N-1)$. In testing a null [7] hypothesis that a sample of $N$ comes from a proposed normal universe, using the difference between the observed and hypothetical means as a criterion of judgment, it had long been the custom to refer the calculated value of

$$t = \frac{\bar{x} - \tilde{x}}{s/\sqrt{N-1}} \tag{30}$$

to a normal probability scale. While Helmert obtained the distribution of $s^2$ as early as 1876, it seems that "Student" [8] was the first (1908) to recognize the importance, when $N$ is not large, of taking account of the variability of $s$ in (30). By means of a remarkable intuition he obtained, somewhat empirically, the simultaneous distribution function for $\bar{x}$ and $s$ from a normal universe. Later writers, notably Fisher [9], established his results rigorously and proved that

$$f(\bar{x}, s) = g(\bar{x})h(s), \tag{31}$$

where

$$g(\bar{x})d\bar{x} = k_1 e^{-N(\bar{x}-\tilde{x})^2/2\sigma^2} d\bar{x}, \qquad -\infty \leqq \bar{x} \leqq \infty,$$
$$h(s)ds = k_2 e^{-Ns^2/2\sigma^2} s^{N-2} ds, \qquad 0 \leqq s \leqq \infty,$$
$$k_1 = (2\pi\sigma^2/N)^{-1/2}, \qquad k_2 = \frac{2\left(\frac{N}{2\sigma^2}\right)^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)}.$$

From these results it is easy to show that (30) is distributed in accordance with what has come to be called the "Student" curve

$$F_n(t) = K_n(1 + t^2/n)^{-(n+1)/2}, \tag{32}$$

where $1/K_n = n^{1/2}B(n/2, 1/2)$, $B$ being the Beta function and $n$ the number of degrees of freedom in the estimate of $\sigma^2$. It is important to observe that the criterion (30) and its probability function (32) are completely expressible in terms of the observations. Such criteria have been called by Wilks "Studentized" functions. Tables [10] of the probability $P = 1 - P_n(t)$, where

$$P_n(t) = 2\int_0^t F_n(t)dt, \tag{33}$$

are available for assigned values of $n$ and $t$.

Fisher showed (*loc. cit.*) that the "Student" curve has a much wider range of applications than the problem for which it was designed. He demonstrated that (32) can be used in testing the significance of the difference between two sample means and in testing the significance of regression coefficients and of certain curvilinear regressions. The scheme by which the "Student" curve is made available to these other problems consists in building a variable in the nature of a fraction

$$t = \frac{v}{\dot{\sigma}_v} \tag{34}$$

whose numerator is a statistic normally distributed about zero and whose denominator is an independently distributed and unbiased estimate of the standard deviation of the statistic in the numerator. It is imperative that the numerator and denominator be independent.

As an example of (34), let $v = \bar{x}_1 - \bar{x}_2$ be the difference between the means of two independent samples consisting of $N_1$ and $N_2$ items, respectively, from a normal universe with variance $\sigma^2$. Then we have

$$\dot{\sigma}_v = \dot{\sigma}\left[\frac{N_1 + N_2}{N_1 N_2}\right]^{1/2},$$

where $\dot{\sigma}$ is defined in (24), and (34) is distributed in accord with (32) for $n = N_1 + N_2 - 2$. In this connection, the writer [**11**] has called attention to a more appropriate formula than the one frequently given in textbooks for $\dot{\sigma}_v$ when the samples are large.

Recently, Rider [**12**] and Rietz [**13**] have given excellent surveys of the contributions of "Student," Fisher, and others to the theory of sampling. Jackson (*loc. cit.*) has explained, at the level of "ordinary courses and textbooks in mathematics," how the establishment of "Student's" results by Fisher, which involves $N$-dimensional geometry, can be put in completely analytical form.

**9. The distribution of** $s$. The distribution of the standard deviations of samples of $N$ from a normal universe is given by $h(s)$ in (31). So its mean value is

$$E(s) = \int_0^\infty h(s) s ds$$

which upon integration yields

$$E(s) = \frac{(2/N)^{1/2}\Gamma(N/2)\sigma}{\Gamma(\overline{N-1}/2)}. \tag{35}$$

The coefficient of $\sigma$ in (35) is the function we denoted by $b(N)$ in (28). Romanovsky [**14**] showed that

$$b(N) \to \left(1 - \frac{3}{4N} - \frac{7}{32N^2} - \cdots\right).$$

The modal value of $s$, found by making $h(s)$ a maximum, is

$$s = \sigma(\overline{N-2}/N)^{1/2}.$$

Romanovsky deduced the standard deviation of $s$ to be

$$\sigma_s = \sigma\left(\frac{1}{2N} - \frac{2}{8N^2} - \frac{3}{16N^3} - \cdots\right)^{1/2}.$$

The approximate value $\sigma_s = \sigma(2N)^{-1/2}$ is often used in practice, and this is the basis for the common statement that the standard error of a standard deviation is $1/\sqrt{2}$ times that of a mean.

**10. The ($\bar{x}$, $s$)-frequency surface.** Since

$$\int_0^\infty \int_{-\infty}^\infty f(\bar{x}, s) d\bar{x} ds = 1,$$

the volume under the surface over a closed contour in the $\bar{x}s$-plane represents the proportion or percentage of samples whose means and standard deviations fall simultaneously within the ranges defined by the boundary of the given contour. In depicting this surface it is convenient to let $\bar{u} = \bar{x} - \tilde{x}_1$ so that the origin of $\bar{u}$ is at $\bar{x} = \tilde{x}$. In a comprehensive paper by Deming and Birge [**15**] two such frequency surfaces are represented. These are reproduced in Figure 1, one for a small value of $N$ and the other for a comparatively large value of $N$.

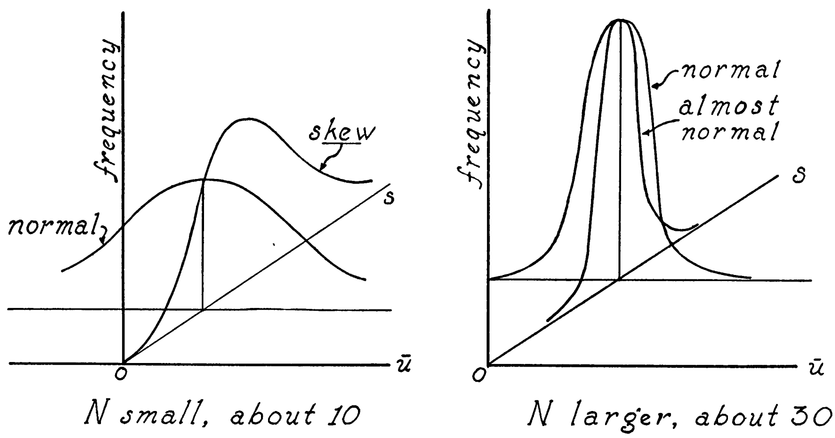


FIG. 1. The frequency surface $f(\bar{u}, s) = Ke^{-N(\bar{u}^2+s)/2\sigma^2}$

As the authors point out, the highest point of the surface has the coördinates $\bar{u} = 0$, $s = \sigma(\overline{N-2}/N)^{1/2}$. Because of the independence of $\bar{x}$ and $s$, all plane sections $s$ = constant will be normal curves with standard deviation equal to $\sigma/\sqrt{N}$. The $\bar{u}$ = constant sections will be skew curves whose equations are given by $h(s)$. They will all have the same mean and mode. As $N$ increases, their mean and mode approach coincidence with the value $\sigma$ while the curves lose their skewness and become normal with center at $s = \sigma$ and standard deviation equal to $\sigma/\sqrt{2N}$. As $N$ increases, the surface becomes more and more concentrated about the point $\bar{u} = 0$, $s = \sigma$.

**11. The $s^2$ and $\chi^2$ distributions.** Since $ds^2 = 2sds$ the distribution of $s^2$, which we will denote by $H(s^2)$, can be found at once from $h(s)$. Thus we obtain

$$H(s^2)ds^2 = \frac{e^{-Ns^2/2\sigma^2}(Ns^2/2\sigma^2)^{(N-3)/2}ds^2}{(2\sigma^2/N)\Gamma(\overline{N-1}/2)} \cdot \tag{36}$$

In §5, the mean of the sampling distribution of $s^2$ from an arbitrary population

was obtained. It is interesting to verify that result in the present case where the distribution function is known. The mean of the distribution of variances of samples of $N$ from a normal universe is given by

$$E(s^2) = \int_0^\infty H(s^2)s^2ds^2 = \sigma^2(N-1)/N.$$

The standard deviation of the $H(s^2)$ distribution is, approximately,

$$\sigma_{s^2} = \sigma^2\sqrt{2/N}.$$

If, in (36), we let $\chi^2 = Ns^2/\sigma^2$ we get the $\chi^2$ distribution

$$T_n(\chi^2)d\chi^2 = \frac{e^{-\chi^2/2}(\chi^2/2)^{(N-3)/2}d\chi^2}{2\Gamma(\overline{N-1}/2)}. \tag{37}$$

This $\chi^2$, although not the same as the one [16] used in tests of goodness of fit, has essentially the same distribution.

12. **Fiducial inference and confidence limits.** A method of inverse argument by which values of population parameters are inferred from samples "randomly drawn" from populations of known functional form has recently been developed. The limits placed on the unknown parameters in such an inverse argument have been called *fiducial* or *confidence* limits. In arguing from a sample to the population, any inference must be attended with some degree of uncertainty. But uncertainty should not be confused with lack of rigor. Statements can be made about population parameters, subject to risks of being wrong, where the error is precisely expressed in terms of probability theory. In other words, the nature and degree of the uncertainty can be rigorously expressed. The modern method of expressing the reliability of a statistical estimate of a parameter in terms of fiducial limits seems likely to replace the traditional but often misleading method of expression involving "probable error."

(a) *For the mean.* Let $\bar{x}$ and $s$ be the mean and standard deviation of a sample of $N=n+1$ items from a normal universe with unknown mean $\tilde{\bar{x}}$. The problem is to determine an interval surrounding $\bar{x}$ in which we may assume, with a certain degree of confidence, that $\tilde{\bar{x}}$ is contained. Suppose we make the claim

$$\bar{x} - t_\alpha s/\sqrt{n} < \tilde{\bar{x}} < \bar{x} + t_\alpha s/\sqrt{n}.$$

Let the probability of an error in a statement of this sort be equal to or less than $\alpha$, $0<\alpha<1$, $\alpha$ being chosen in advance. The quantities $\bar{x} \pm t_\alpha s/\sqrt{n}$ are the fiducial limits for $\tilde{\bar{x}}$ corresponding to a given value of $\alpha$. Our measure of confidence in such a claim is called *fiducial probability.* We can obtain from tables [10] the value of $t$, $t_\alpha$, corresponding to assigned values of $n$ and $\alpha$, by assigning to $P=1-P_n(t)$ a value of $\alpha$ appropriate to the level of confidence chosen, solving for $t$, and then determining the limits for $\tilde{\bar{x}}$ from the relation

$$\pm\, t = (\bar{x} - \tilde{\bar{x}})\sqrt{n}/s.$$

It is conventional among certain workers to take $\alpha=.01$ since they wish to determine values of $\tilde{x}$ in an interval dividing hypotheses that will be rejected from those acceptable under a null hypothesis at the 1% level of significance.

To illustrate, for $n=15$, we find from the tables that $t=\pm 2.947$ when $P=.01$. Then we have

$$(\bar{x}-\tilde{x}) = \pm\, 2.947s/\sqrt{15} = \pm\, .76s,$$

and the claim

$$\bar{x} - .76s < \tilde{x} < \bar{x} + .76s$$

will be correct 99% of the time.

It is clear from the above procedure that our confidence in the fiducial limits $\bar{x}\pm t_\alpha s/\sqrt{n}$ is measured by the area under the $F_n(t)$ curve inside $t=\pm t_\alpha$, that is, by $P_n(t_\alpha)$ (Figure 2). This means that if we could observe all possible samples, the proportion represented by $P_n(t_\alpha)$ would yield values of $\bar{x}$ and $s$ for which the claim is true, while the remaining proportion, $P=1-P_n(t_\alpha)$ would yield values of $\bar{x}$ and $s$ for which the claim is false.

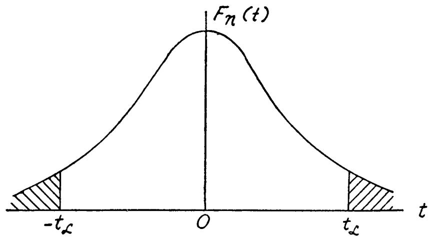


FIG. 2.

If we were testing a hypothetical value of $\bar{x}$ we would say that $\bar{x}$ is *not significant* at the 1% level of significance if $\tilde{x}$ has any value in the $\bar{x}\pm t_\alpha s/\sqrt{n}$ interval, $\alpha=.01$. If $\tilde{x}$ does not lie in this interval we say that $\bar{x}$ is *significant* at this level.

Obviously, values of $t$ satisfying the equation $P=.01$, that is, $P_n(t)=.99$, vary with $n$. When the sample is large an alternate method may be used which avoids the trouble of entering a table. The variable

$$t = (\bar{x}-\tilde{x})\sqrt{N}/s$$

is approximately normally distributed for $N>30$. The area under the normal curve outside $t=\pm 2.576$ is .01. Therefore, the 99% fiducial range of $\tilde{x}$ is

$$\bar{x} \pm 2.576s/\sqrt{N}$$

and the range gets smaller as $N$ increases.

(b) *For the difference between two means.* Let $\bar{x}_1$ and $s_1^2$ be the mean and variance of a sample of $N_1$ from a normal universe with unknown mean $\tilde{x}_1$ and let

$\bar{x}_2$ and $s_2^2$ be the mean and variance of a sample of $N_2$ from a normal universe with unknown mean $\tilde{x}_2$. It is assumed that the two universes have a common variance $\sigma^2$. For brevity, let

$$\bar{w} = \bar{x}_1 - \bar{x}_2, \qquad \tilde{w} = \tilde{x}_1 - \tilde{x}_2, \qquad N = N_1 + N_2,$$

$$\sigma_{\bar{w}} = \left[\frac{N_1 s_1^2 + N_2 s_2^2}{N - 2} \times \frac{N_1 + N_2}{N_1 N_2}\right]^{1/2}.$$

Then

$$t = \frac{\bar{w} - \tilde{w}}{\sigma_{\bar{w}}} \tag{38}$$

is distributed in accord with $F_n(t)$ for $n = N - 2$. From (38), upper and lower fiducial values of $\tilde{w}$ can be found by assigning to $t$ the solutions of $P_n(t) = .99$, that is, of $P = .01$. If the value $\tilde{w} = 0$ falls outside the fiducial interval thus established, the conclusion is that the difference between the means is significant at the 1% level. That is, $\tilde{w} \neq 0$ and hence $\tilde{x}_1 \neq \tilde{x}_2$.

If the two samples are equal in number so that the variates can be paired in some way we may compute (38) by a different method. Let $N = N_1 = N_2$, $w = x_1 - x_2$, and compute $\bar{w}$ and $\Sigma(w_i - \bar{w})^2$. Then

$$t = \frac{\bar{w} - \tilde{w}}{\sigma_w} = \frac{\bar{w} - \tilde{w}}{s_w/\sqrt{N - 1}}$$
$$= \frac{\bar{w} - \tilde{w}}{\left[\sum (w_i - w)^2/N(N - 1)\right]^{1/2}}.$$

The last expression is sometimes called *Bessel's Formula.*

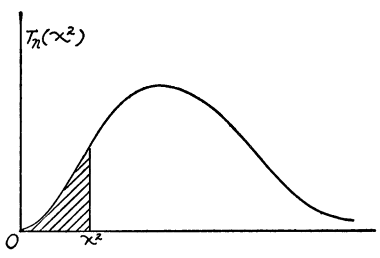


FIG. 3.

(c) *For the variance.* To determine the fiducial limits of $\sigma^2$ we first observe from (22a) that $Ns^2 = n\dot{\sigma}^2 = \Sigma(x_i - \bar{x})^2$, and therefore we may write $\chi^2 = n\dot{\sigma}^2/\sigma^2$. If now we make the claim

$$n\dot{\sigma}^2/\chi_2^2 < \sigma^2 < n\dot{\sigma}^2/\chi_1^2,$$

where $\chi_1^2$ and $\chi_2^2$ are arbitrarily chosen constants ($\chi_1^2 < \chi_2^2$), then our measure of

confidence in the correctness of this claim is given by $I_n(\chi_2^2)-I_n(\chi_1^2)$, where (Figure 3)

$$I_n(\chi^2) = \int_0^{\chi^2} T_n(\chi^2)d\chi^2.$$

Tables of $I_n(\chi^2)$ are easily available [**17**].

Rietz [**18**] has explained this recent advance in statistical inference in a paper specially designed for teachers of collegiate courses in statistics. References are given there to the contributions in this connection of Fisher, Neyman, E. S. Pearson, Wilks, and others.

*Notes and References*

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**7.** See R. A. Fisher, The design of experiments, Oliver and Boyd, 1937.

**8.** "Student" was the pseudonym of W. S. Gosset. Interesting biographical sketches of Mr. Gosset will be found in the Journal of the American Statistical Assoc., vol. 33, 1938, pp. 226–28; and the Journal of the Royal Statistical Society, vol. CI, Part I, 1938, pp. 248–251.

**9.** R. A. Fisher, Applications of "Student's" distribution, Metron, vol. 5, 1925, no. 3, pp. 90–93.

**10.** R. A. Fisher, Statistical methods for research workers, Oliver and Boyd. See also Yule and Kendall's new text.

**11.** J. F. Kenney, A note on certain formulas used in sampling theory, this MONTHLY, vol. 45, 1938, pp. 456–458.

**12.** P. R. Rider, A survey of the theory of small samples, Annals of Mathematics (2), vol. 31, 1930, pp. 577–628.

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**14.** V. Romanovsky, On the moments of standard deviations and of correlation coefficient in samples from normal., Metron, vol. 5, no. 4, 1925, pp. 3–46.

**15.** Deming and Birge, On the statistical theory of errors, Reviews of Modern Physics, vol. 6, 1934, pp. 119–161. (Reprints available at the Graduate School of the United States Department of Agriculture).

**16.** For a definition and discussion of the $\chi^2$ used in tests of goodness of fit, see T. C. Fry, Probability and Its Engineering Uses. D. Van Nostrand Co. 1928.

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# JOSIAH WILLARD GIBBS*

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In the great glass of time the grains of sand are even now stirring which by their fall will soon mark the lapse of a century since the birth of Josiah Willard Gibbs, a man who must be placed at once among the greatest and the least generally known of American scientists.

To us, in this scientific group, it is a matter as we will it—the unexalted choice of a perspective—whether the century be regarded as a unit great or small. The geologist, the astronomer, the biological evolutionist, thinks effortlessly in eons by comparison with which the century is an insensible instant, and the scholar of the arts, the science, the philosophy or the mathematics of antiquity reaches aptly back over centuries, be they fifty or more. And yet, as students of modern science we remember that only three centuries or less have rolled by since Galileo paid in discouragement and anxiety the penalty of his heroic thought; since Descartes preached his scientific doctrine and founded modern mathematics; and since the mechanics of the world of today took shape in the great mind of Newton. The life of our national identity is not yet to be measured by two centuries, and even a single century ago the soil we tread was still in all truth that of a new land.

The American scientist of a hundred years ago stood forth, therefore, very much in the rôle of the pioneer. He won out in his achievements over many handicaps; without traditions at his back, with little in the way of schools and libraries or laboratories to his hand, and often in the face of small sympathy or encouragement from his neighbors, who were for the most part engrossed in the busy incidents peculiar to the industrial and commercial and agricultural enterprise of the new nation. The issues which were soon to lead to the test whether the bonds uniting the nation were tender or tough, were already inflaming the passions of men, and were all too well designed to overshadow the inconspicuous scholarly searchings and deductions, in the tiny impulses of which the revolutions of scientific discovery are generally born.

I need not recall to you the pomp and circumstance of a few years ago by which a great American city saw fit to mark the consummation of a century of scientific progress. There were exhibits to swell with pride the hearts of the disciples of science, and there was much that called to all mankind for homage on behalf of those rare men, who, facing the questions which the facts of nature constantly pose to the human mind, were of that superior wisdom that they could see the principles behind the facts, and grasping these place into the hands of their fellows the machineries of modern science for the control and understanding of nature and the consequent amelioration and enrichment of life. Of these rare men of extraordinary intellect was Josiah Willard Gibbs. An ancient proverb notwithstanding, the world-wide scientific authority of today

* An address to the Minnesota section of the Mathematical Association of America, October 28, 1938; also given at Northwestern University on November 30, 1938.

is agreed that in his own field he was one of those uncommon men who could see tomorrow's sun.

Josiah Willard Gibbs was born on the eleventh day of February in the year 1839, in the city of New Haven, Connecticut. His family had made its homes upon American soil from almost the earliest of colonial days, and had been identified over many generations with the finest in American thought and culture. Thus one finds among Gibbs's direct though more remote ancestors a President Willard of Harvard College, and a President Dickenson of the College of New Jersey which later became Princeton College, while Gibbs's father, also named Josiah Willard, was a professor at Yale, and an eminent scholar widely noted for the profundity of his thought in the fields of sacred literature and the classics. From men such as these Gibbs inherited genius and those traits of industry, power of concentration, and intellectual integrity, keenness, and modesty which were to characterize him throughout his life.

Though records of personal incidents or anecdotes of Gibbs's boyhood or youth are scant or utterly missing, it is not hard to reconstruct in essence the atmosphere of the environment in which he lived. New Haven was even then the seat of Yale College, and so was a center of American learning. From early colonial times it had been a community built largely around the church, and possessed of a strongly local and very individual character. Traditionally its code of preference was inclined in all matters toward a general and extreme conservatism, and this attitude, especially in matters of politics and religion, had been unwaveringly maintained. Although as a seaport it had cherished, and in those earlier days of the steamboat perhaps still did cherish, some dreams of commercial greatness, the life and ambitions of the city, as they expressed themselves through its more influential citizens, were focused in large measure upon the College. Yale was the acknowledged center of civic pride and ambition. It requires little in the way of imagination to conceive of how Gibbs, born into the family of a scholar and professor in such surroundings, must during his boyhood have been saturated with the atmosphere of collegiate life, and how he must have had instilled into him a thorough sense of the prime importance of things academic.

During these years the community of New Haven merited with far more justice a description as an overgrown village than as a city. Its connection with New York by railroad was not materialized until the year 1848, and in that same year street lighting by gas lamps was first introduced upon the main streets of the town. Until 1854 there were no graded schools, and for years beyond that fire protection was on a volunteer basis, and policing in the hands of a Department of the Watch. By common custom scavenger duty was delegated to swine and was allocated to the gutters of the city streets, and not until 1861, when Gibbs had already attained his majority, were pig-pens banished from the city, and a law passed abating the nuisance of horses and cows pasturing in the streets. On the main thoroughfares sidewalk paving was an innovation which had been bitterly opposed, and it is recorded that a prominent citizen

still strode over this object of extravagance only when necessity compelled him to, and then in haste, and walked by preference in the street, fortified in his conviction that God's soil was still good enough for him.

Gibbs prepared himself for college at the Hopkins Grammar School in his home city. He was a distinguished student who divided his interests between mathematics and the classics. At the age of fifteen he entered Yale, and continued there to live up to his earlier promise. His interests were exclusively cultural, and so following his graduation he remained at Yale for five years more to receive his doctorate at the end of that time, in 1863. Yale had established the degree of doctor of philosophy only three years earlier, and in doing so had taken the lead among American institutions. With his degree Gibbs received an appointment as tutor in the College, and was assigned to teach classes at first in Latin, and then in Mathematics and Science. He found this assignment no easy one, nor one for which he was temperamentally well equipped, or in which expectations of brilliant success could be his. Whatever his genius, it was not that of the college teacher.

In physical aspect Yale presented at that time the picture of a row of old and homely buildings of brick, which were flanked round about by a campus noted for the distinction of its archways of great and stately elms. Along the street this campus was edged by a long rail fence around which much of Yale tradition entwined itself, and which from immemorial time had been the favorite roost of the students. As a barrier to ingress the fence was a negligible matter, and the campus, therefore, assumed much the character of a village green, where tramps, and beggars, and pedlars, and organ-grinders, and in fact all who would, mixed freely in the college life. As at noon the elms lent shade, they at night increased the darkness, and so unwittingly conspired to enhance the lure and convenience of the unlighted campus for all such as were bent upon purposes of academic mischief. The fugitive student could depend here at once upon ready access to the city streets and upon the asylum of many handy refuges and concealments, temptations which, it has been said, rarely failed to convert original saints into undergraduate sinners.

To the common creature comforts of the students the dormitories of the old brick row afforded only the rudest of satisfaction, and of this sort also were the class-rooms. The beams sagged, the floors were billowy and the ceilings cracked; and the walls were gouged and furrowed, for it had long been the prime ambition of every student to leave some lasting mark of his upon pillar or doorway. The rooms were illy lighted and seemed, therefore, steeped in permanent gloom. There was a general air of mustiness about and an appearance of roughness, and the sanitation was far from modern. Such details, however, seemed of small effect toward mitigating the intensity of life for the five hundred students of the College, and faculty vigilance enjoyed but rare respite from its appointed task of keeping pace with the students' inventive genius for pranks and insubordination.

With all this it was an academic era noted as one in which professors were,

or at least were popularly regarded to be, highly individualized beings. Among them oddities of personality or habit were not regarded the exception but the rule, and even the younger tutors and teachers generally shared in these distinctions. All too commonly custom and their own expanding dignity placed them under the incumbency of assuming an aspect of austerity which made them appear as nothing so much as arch-foes to their charges. It was their first duty to match their discipline vigilantly against their opponent's ever-present urge for fun, and a tutor in actual physical pursuit of a fleeing undergraduate was by no means an uncommon sight.

For this sort of thing Gibbs could hardly have been more poorly endowed by nature. He was shy and retiring, gentlemanly, contemplative and reserved. The subjects he taught, moreover, Latin and Mathematics, were at that time required of all students throughout their first and second years, and, as is the way with requirements generally, these bred neither a desire to spare the tutor, nor a fondness for the subjects. Quite the contrary we may infer, for of the many campus customs at Yale in those times none appears to have been entered into with so much gusto and zest as the annual farcical pageant of the Burial of Euclid, with which the sophomore class was wont to celebrate its mathematical emancipation.

There are many records of this ceremonial in the Yale archives, and though in its details it naturally varied with the genius of the class, it maintained its identity in form over a period of generations. The sophomore class having been summoned to gloat over Euclid's death, assembled in some college hall which was bedecked suitably to the occasion. The scene was dominated by a large and lurid cartoon which bristled in detail with fire and fury, and depicted how in the presence of Jupiter demon stokers were assisting at the consumption of Euclid's remains in a sea of blazing tar. A dismal forest with embattled demons filled the remoter parts of the scene, while in the foreground a student visibly filled with despair lent company to a weeping crocodile. Under this aspect Euclid's volume was perforated with a glowing poker, each man of the class thrusting the iron through in turn to signify that he had gone through Euclid. Following this the book was held for a moment over each man to betoken that he had understood Euclid, and finally each man passed the pages under foot that he might say thereafter that he had gone over Euclid.

These preliminaries accomplished, the funeral cortege was formed, and proceeded lugubriously, with grotesque garb and blazing torchlights to the chosen place of interment. At times Euclid himself was impersonated, dressed in classic raiment and pressing his beloved volume to his breast, and at others the book alone was borne suitably shrouded at the head of the procession. At the pyre the celebration waxed in boisterousness and assumed more the aspects of revelry. There was elaborate mock lamentation, a funeral oration was held, and dirges more or less derisive were sung.

"No more we gaze upon that board
Where oft our knowledge failed,

As we its mystic lines ignored,
On cruel points impaled."

* * * *

"We're free! Hurrah! We've got him fast
Old Euk is nicely caged at last."

* * * *

"Black curls the smoke above the pile
And snaps the crackling fire:
The joyful shouts of Merry Sophs
With wails and groans conspire.
May yells more fiendish greet thy ears,
And flames yet hotter glow;
May fiercer torments rack thy soul
In Pluto's realms below."

As a student at Yale, Gibbs must have participated in such a rite, and during his life he must have witnessed it many times. For him personally, however, Euclid never died in any but a metaphoric sense; for geometry and geometric imagination were the corner-stone and buttress of his genius.

At the expiration of his appointment as tutor Gibbs went to Europe for further study. The Civil War, which had been running its course, ravaging the country and depressing personal incentive, had meanwhile come to its close, and in departing this country Gibbs figuratively became one of a brilliant troop of young intellectuals who were destined to play a decisive rôle in the cultural development of America. These men went to Europe in search of learning, and they found the fulfillment of their quest at the great German universities. There they found great minds in numbers and under conditions which made those minds accessible to others less mature. They found there also a breadth of academic viewpoint, a freedom of research, and an insistence upon productive scholarship which they had not theretofore known. Upon their return they brought back with them these ideals and perspectives, together with an abundance of enthusiasm; and under the spur of this inspiration they became the institutors of the system of post-graduate instruction and research which is the essence of the modern American university, and became the founders of the many learned societies which are today conspicuous forces in the intellectual and cultural life of the nation.

Gibbs spent three years in Europe, studying for a time at Paris, but principally at Heidelberg, as a student of those great teachers, Helmholtz and Kirchhoff, and at Berlin under the influence of the supreme rigorist genius of the great Weierstrass. Subsequent to his return to America, Gibbs was elected, in 1871, to the Professorship of Mathematical Physics at Yale. He was then thirty-two years old, and was to hold this professorial chair without interruption for precisely that many years again until his death. The distinction which had thus been bestowed upon him was apparently a cheap one for Yale, and one which for Gibbs remained long empty of all but honor. During many years

it carried no remuneration at all, and during many more only a fragment of an otherwise customary salary.

Had Gibbs been under the necessity of maintaining himself, his position at Yale would, of course, have been untenable. Fortunately this was not so. His father had bequeathed him a competence which, though small, was matched by the smallness of his need. He was, and always remained unmarried. Throughout his life he retained the occupancy of rooms in the old family house in which he had spent all the boyhood he could remember, and which stood in close proximity to the College campus. This house sheltered now the family of a married sister, and in this family he found his own permanent home. His life was routine and uneventful; for social contacts he felt but little need, and the craving to see and hear, which impels one to travel, was not his. He had few aesthetic needs, was abstemious in his habits, and seemed to find in his work all that he sought from life.

The study which claimed the initial interest of Gibbs as a professor was that of thermodynamics, the science which treats of heat as a form of energy, and concerns itself with the laws governing the transformations of heat into energy in different forms. This was then a new science, and for Gibbs it was an absorbing one. The results of his first two years of research were given out by him in the form of two papers which were respectively entitled "Graphical Methods in the Thermodynamics of Fluids," and "A Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces." The papers were remarkable. By using as mathematical coördinates such physical quantities as volume and pressure, energy, temperature and entropy, they derived, on the one hand, heat diagrams of various types which later became instruments of great importance for the thermal engineer, and, on the other hand, showed how with any physical body there might be associated a graph or so-called thermodynamical surface, from the geometrical configuration of which could be recognized the many relations between volume, energy, temperature, pressure and entropy, the conditions for stability and equilibrium, and the passage from the liquid to the solid or gaseous states. In these papers Gibbs revealed himself at once as possessed of a rare imagination in the domain of abstract geometry, and as a master in its application.

These initial works of Gibbs's genius were followed in 1876 and 1878, that is, in his thirty-seventh and thirty-ninth years, by his greatest memoir, "On the Equilibrium of Heterogeneous Substances." Here Gibbs rose to the pinnacle, and revealed himself to be a true intellectual giant. The work is a monumental one, immense in its scope, and one which shows, as almost no other scientific work does, the sheer power of human thinking. In the science of physics the law of the conservation of energy, under the transformation of mechanical work into heat and vice versa, was even in Gibbs's time a familiar instrument, and one of the most effective, in the hands of the theoretical investigator, to his purpose of deducing from observed phenomena an intelligible picture of nature. In this respect the science of chemistry was far behind, for

the relations between the energies of chemical reactions and heat had almost completely eluded all attempts to bring them within the realms of scientific law. It was to the difficulties of this problem that Gibbs had bent his thought, and these difficulties he had at one stroke subjugated with an astounding completeness.

The paper stands as a great model of the rôle which mathematics rightly plays in its relations with the sciences. Assuming command over a bewildering welter of apparently unrelated facts, it imposes upon them a few fundamental laws, and reduces the whole to rule and order. Gibbs based his authority upon the first two laws of thermodynamics, namely, the law of the conservation of energy and the law that heat will not of itself flow from a colder body to a hotter one. To these laws he adjoined a few experimentally determined primary chemical facts, and from this basis proceeded by mathematical deduction alone, with unbending rigor, to clear his intellectual way step by step, and to uncover again and again a basic principle and intrinsic likenesses between things in which such had superficially seemed remote. Specifically, Gibbs deals in this great work with the statics of chemical substances which are in contact with each other, and derives for them conditions for their co-existence, their equilibrium, or their stability as solids, liquids, vapors, or gases, or as liquid films, gaseous mixtures, solutions, or crystals, and discusses the effects upon them of osmosis or gravity, of electromotive or capillary or catalytic forces. In its reasoning it is a true unfolding of nature's law, and in its results it laid the foundation of a new science, a science of great present-day vitality, the science of Physical Chemistry.

Gibbs's great achievement was slow to attain its deserved and destined influence, and for this there were many reasons. As a scientist he was a thoroughly solitary figure, and the researches of his paper constituted a scientific departure which in its originality had been entirely unforeshadowed by the work of others. He had had no helpers, and as a true innovator he had no rivals. His paper was modestly published in the *Transactions of the Connecticut Academy of Arts and Sciences*, and was, therefore, in large measure obscure, and certainly in Europe almost inaccessible. Finally his paper was of a most forbidding aspect, as Gibbs was certainly no easy writer to read. His style was bare and concise, the cast of his ideas was severe, and his reasoning prompt, unerring and rigorous. Of emphasis he gave little, and a thing once said was done with. Finally, but critically, the great treasures he uncovered were treasures for the chemist, whereas his paper barely mentioned as many as five or six chemicals, all of them simple, and at the same time extended over three hundred printed pages which are covered with some seven hundred mathematical formulas. Such a memoir, one may venture, would be no mean test of mettle for the prospecting chemist of today. Small wonder that it remained largely unexplored some three score years ago.

The past half century has spoken in emphatic term for the brilliance and profundity of Gibbs's achievement. His paper is rated as a preeminent document

of scientific prophecy, for many phenomena which it predicted, which were at the time unknown and unsuspected, have since been discovered or rediscovered by experimental means. The principles which Gibbs laid down have led to a wealth of original and fruitful researches which even now are apparently far from exhaustion. Emerson might well have said of them, "The creation of a thousand forests is in one acorn."

There is every reason to believe that Gibbs was himself fully aware of the great ultimate importance of his work. Nothing would have been more foreign to him, than to have lent word or action of his own to further its recognition or acclaim. It was a conspicuous characteristic of his nature to mantle his own personality with a distinctive cloak of reserve. In his personal contacts he seems to have been always friendly and considerate, kindly and affable and of a ready and spontaneous if somewhat subdued sense of humor. As an intellect, however, he withdrew into himself. Though his associates and colleagues easily recognized him as having in full measure those traits of character which Francis Bacon held characteristic of the true scholar, "the desire to seek, the patience to doubt, fondness to meditate, slowness to assert, readiness to reconsider, carefulness to dispose and set in order, and repugnance to every kind of imposture", they were permitted to recognize beyond this none of the more intimate things which filled his mind. He loved his work, and was possessed of an impelling enthusiasm for it, but these matters he kept in concealment. He worked without either the stimulus of conversation or that of criticism, and never spoke of his ideas until they were rounded out and in every way ready for publication.

Herein undoubtedly lay Gibbs's greatest failing. By profession he was a teacher, by temperament he lacked entirely the teaching spirit. Silent and reserved men have often been great as teachers, but this was not so with Gibbs. He taught only graduate students, and his students were never more than a few, but even to these he never confided the matters which at any time were at the focal point of his interest. To the few who had the will and the ability to follow him he was inspiring. All found his lectures difficult. They were invariably well prepared, but their thought was heavily concentrated, the progression of ideas often precipitate, and exercises, if such were included at all, were all too often and readily brushed aside.

The repertoire of courses from which Gibbs lectured from year to year seems to have been essentially the following: Vector Analysis, Capillarity, The Wave Theory of Light and Sound, Least Squares, The Theory of Potential, The Mathematical Theory of Electricity and Magnetism, and Multiple Algebra, and later,—not until fifteen years after his great memoir,—Thermodynamics, Statistical Mechanics, and The Computation of Orbits.

Much of the theory set forth in these lectures was original. This was so, for instance, with the Vector Analysis, a creation of Gibbs's for which almost all scientists of the present day are indebted to him. The Cartesian coördinate geometry, powerful mathematical tool though it be, becomes involved and unwieldy in the extreme when it is applied to elaborate space relations, or to the

study of strains, twists, spins, or other common aspects of rotational motion. The desirability of an emancipation from it had been recognized by many mathematicians before Gibbs, and had led the German mathematician Grassmann to his "Ausdehnungslehre," and the Englishman Hamilton to his Theory of Quaternions. This last as a theory is concise, consistent and elegant. As an instrument, however,—and though it had been used with masterly effect by Maxwell,—it had generally been found unfortunately artificial. The mathematician Cayley speaking to this point, compared Quaternions to a pocket-map, which to be used has to be unfolded. The quaternion formula, he felt, had to be retranslated into coördinates to be really understood.

These defects Gibbs sought to overcome through the medium of a sort of fusion of the German and English works. The result was his creation of Vector Analysis, which during the lapse of half a century has established itself as an indispensable tool of the theoretical scientist. With customary modesty Gibbs refused to regard his work here as properly original. He said of it: "The notions are only those which he who reads between the lines will meet on every page of the great masters of analysis, the only difference being that the vector analyst, having regard for the weakness of the human intellect, does as the early painters did who wrote beneath their pictures, 'This is a tree and this a horse'." He had an account of the Vectors printed privately for the use of his students in 1881. Only in 1901 did he consent to the publication of a book on the subject.

During the years 1882 to 1889 Gibbs published a series of papers on the theory of light, and an important paper for the astronomer, "On the Determination of Elliptic Orbits from Three Observations." Of these the latter has become classical, and has achieved an immense saving in astronomical calculations. The former constitute together what is generally regarded as the simplest and most conclusive argument on purely theoretical grounds for the acceptance of the electromagnetic theory of light. At this day the correctness of his contentions has, of course, been long established, by those experimental results which meant the triumph of Maxwell's theory.

The genius of Gibbs had meanwhile received recognition, not alone in this country but throughout the world. Though to the layman, and even to many of his immediate colleagues in other fields, he remained unknown, and though he has been called an author whose books no one of his generation was ready to read, he had been elected to membership in the National Academy of Sciences, and had been awarded the Rumford Medal of the American Academy of Arts and Sciences at Boston. Before his death he was to hold honorary degrees from universities in three different countries, and to become a corresponding member of fifteen of the world's great learned societies. He was to receive from the Royal Society of London its Copley Medal, the highest distinction for research in any land, and to be elected to a vice-presidency of the American Association for the Advancement of Science.

In his final work, "The Principles of Statistical Mechanics," Gibbs was again to step forth as the innovator, and to open a most fertile field for subse-

quent extended scientific investigation. The belief had long been common among scientists that heat in a substance was due to motion of the molecules, but simple as this conception was, it had persistently defied all efforts aimed toward its theoretical demonstration. This failure was stigmatized by the Physicist Kelvin as "a cloud upon the history of science in the nineteenth century." To remedy it Gibbs undertook the study of mechanical systems which are composed of vast aggregates of particles. Though these particles individually were to be regarded as obeying the classical Newtonian laws of motion, a consideration of them, in the face of their great number, would have been hopeless. He took, therefore, as cornerstones upon which to build, the laws of averages and of probability, and with none but the simplest of mechanical assumptions as tools proceeded to erect his theory. From it emerged in fine succession all the basic laws of heat as they are embodied in the science of thermodynamics.

Among American men of science Gibbs holds a preeminently high place. His was one of those rare intellects from which the race obtains its pictures of the world as a cosmic universe. His mind was one in which disjointed phenomena were organized, and such generalized statements of scientific law as mark epochs in the advance of exact knowledge were thought out. Such minds are very rare, and their thoughts are incalculable treasures. The mathematician finds peculiar satisfaction in the work of Gibbs, for in it is revealed the quintessential power of mathematics for "spreading its net over the Cosmos and calling forth from it order, abstract form, and the law of science."

Gibbs died suddenly in 1903, in the sixty-fourth year of his life. He left little in the way of notes, for he had always been accustomed to carry his unfinished work only in his head. It was known that he planned an extension of his great work in thermodynamics, but his thought in that connection will never be known. He waited with them too long.

"They do not die who leave their thought
Imprinted on some deathless page,
Themselves may pass: the spell they wrought
Endures on Earth from age to age."

# THE DELTOID REGARDED AS THE ENVELOPE OF SIMSON LINES

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**1. Introduction.** Recently C. E. Van Horn (this MONTHLY, 1938, p. 434) has called attention to the envelope of the Simson lines of a triangle and proved by an ingenious use of coördinates the theorem that it is a hypocycloid of three cusps, or deltoid, circumscribed about the ninepoint circle. This and his other results, the location of the cusps and the points of contact on the sides and altitudes of the given triangle, will be treated in this paper by synthetic methods. The other questions considered in this paper concern the construction of tangents to the deltoid from points of the inscribed circle, the equal but oppositely directed rotation of tangents whose intersection moves along a given tangent, and the fact that the lengths of two tangents are in the same ratio as the sines of the angles which they make with the third tangent through their intersection. The purely geometrical method for treating the envelope and constructing the tangents from points of the ninepoint circle seems not to be generally known.*

**2. The Envelope.** To identify the envelope of the Simson lines we need only note that each such line passes through a point $P'$ of the ninepoint circle midway between the orthocenter $H$ and its pole $P$, and that if $P$ and $P'$ describe arcs determined by central angles $2\theta$, the Simson line rotates through the angle $-\theta$. (Johnson, p. 206.) But this is exactly the behavior of the tangent to the deltoid. To verify this we may note that if the point of contact $C$ of the rolling circle $(O')$ with the fixed circle $(O)$ is determined by the parametric angle $2\theta$, the angle $CO'D$, then $D$ is the point of the deltoid, is $6\theta$. The line $P'D$ is tangent to the curve, where $P'$ is the point of contact of the rolling circle with the inscribed circle. The point $D$ is the foot of the perpendicular dropped from $C$ on the tangent, an observation which is useful in proving Van Horn's theorems concerning the contacts of the sides and altitudes. To sketch the proof, it will be sufficient to note further that for an altitude the point $P'$ lies midway between the orthocenter and the corresponding vertex, and for a side it is the midpoint of that side. To conclude the present remarks about the rotation of the tangent to a deltoid, we note that the angle $CP'D$ is $3\theta$ or that the line $P'D$ has rotated through the angle $-\theta$ while $OP'$ rotates through the angle $2\theta$.

To locate the cusps, we note that a cusp tangent passes through the ninepoint center. Consider the behavior of the Simson line as its support $P'$ moves from the position $M$, midway between $H$ and a vertex, towards the point $K$ at the extremity of the radius of the ninepoint circle parallel to, and in the same direction as, $HM$. If $K$ also moves towards $P'$ but only half as fast as $P'$, the Simson line and the radius will remain parallel and will coincide when $P'$ has come to a position two-thirds of the way from $M$ to the initial position of $K$.

* See R. A. Johnson, Modern Geometry, 1929, p. 212.

**3. Construction of Tangents.** The books by Altshiller-Court* and Johnson both give methods for constructing the Simson line parallel to a given line. After the cusp tangents have been drawn, we may give another very simple construction using again the relation that the Simson line which makes the angle $-\theta$ with a cusp tangent $NX$ passes through a point $P'$ of $(N)$ such that the angle $XNP'$ is $2\theta$. In addition to the points at infinity from which tangents can be constructed, as noted by Johnson, we can construct the tangents from $P'$ of $(N)$ using the angles between $NP'$ and the three cusp tangents in turn.

The problem of finding the tangents from a general point to the deltoid can be reduced to finding the points of intersection of the ninepoint circle and a certain cubic having three asymptotes parallel to the cusp tangents of the deltoid. This curve is determined as the intersection of $NP'$ and a line through the given point parallel to the Simson line through $P'$. The portion of the cubic generated by the positive ray $NP'$ cuts the ninepoint circle in one or three points.

**4. Theorems Concerning Tangents.** It is known† that the orthopole of a line cutting the circumcircle at $P$ and $Q$ is the intersection of the Simson lines of $P$ and $Q$, and the locus of the orthopole as the line moves parallel to itself is another Simson line. Since $P$ and $Q$ move over equal but oppositely directed arcs, we can announce for the deltoid the following theorem:

THEOREM. *If two tangents to a deltoid rotate through equal but oppositely directed angles, their intersection describes a third tangent, and conversely.*

This result can be applied very simply to obtain another. Consider the two triangles formed respectively by the initial and terminal positions of the two moving tangents and having the side along the fixed tangent in common, whose length we shall call $a$. Let the equal angles at $A$ and $A'$ opposite the side $a$ be $\alpha$, and let the initial and terminal points of $a$ be called $B$ and $C$ respectively. Let $y$ be the angle $ACB$ and $y'$ the angle $A'CB$. Then

$$CA/\sin(\alpha + y) = a/\sin\alpha = CA'/\sin(\alpha + y').$$

Let $C$ move back into coincidence with $B$, whereupon $\alpha$ approaches zero and $A$ and $A'$ approach the points of contact on the tangents $BA$ and $BA'$. Thus we arrive at the following theorem:

THEOREM. *The segments from the point of intersection of two tangents to a deltoid to the respective points of contact are proportional to the sines of the angles which the two tangents make with the third tangent through the point.*

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* College Geometry, 1925.

† Johnson, p. 247.

# AN ACCURATE METHOD FOR OBTAINING THE DERIVATIVE FUNCTION FROM OBSERVATIONAL DATA

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**1. Introduction.** In the case of observational data proceeding at constant increments of the independent variable, Rutledge [1] has given a method for the determination of the derivative function. The formulas which he gave may also be obtained as follows.

We fix attention upon the values of $y$ (the dependent variable) corresponding to $x=x_0, x_1, \cdots, x_8$ (where $\Delta x$ is constant) and assume a fourth degree polynomial approximation for $y=f(x)$ in the interval $x_0 \leqq x \leqq x_4$. Newton's interpolation formula [2] gives

$$y = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \binom{u}{3}\Delta^3 y_0 + \binom{u}{4}\Delta^4 y_0, \tag{1}$$

where

$$\binom{u}{j} = \frac{u(u-1)\cdots(u-j+1)}{j!}, \quad \text{and where } x = x_0 + u\cdot\Delta x.$$

We differentiate with respect to $u$ and evaluate at $u=0, 1, 2, 3, 4$. In a similar manner we apply the corresponding formula for the five points starting with $x_1$, then with $x_2$, then with $x_3$, and finally with $x_4$.

We thus obtain five determinations for $dy/du$ at $u=4$ as follows:

$$\begin{aligned}
z_0 &= \Delta y_0 + \frac{7}{2}\Delta^2 y_0 + \frac{13}{3}\Delta^3 y_0 + \frac{25}{12}\Delta^4 y_0,\\
z_1 &= \Delta y_1 + \frac{5}{2}\Delta^2 y_1 + \frac{11}{6}\Delta^3 y_1 + \frac{1}{4}\Delta^4 y_1,\\
z_2 &= \Delta y_2 + \frac{3}{2}\Delta^2 y_2 + \frac{1}{3}\Delta^3 y_2 - \frac{1}{12}\Delta^4 y_2,\\
z_3 &= \Delta y_3 + \frac{1}{2}\Delta^2 y_3 - \frac{1}{6}\Delta^3 y_3 + \frac{1}{12}\Delta^4 y_3,\\
z_4 &= \Delta y_4 - \frac{1}{2}\Delta^2 y_4 + \frac{1}{3}\Delta^3 y_4 - \frac{1}{4}\Delta^4 y_4.
\end{aligned} \tag{2}$$

We make use of familiar relations such as $\Delta y_0 = y_1 - y_0$, $\Delta^2 y_0 = y_2 - 2y_1 + y_0$, to obtain:

$$\begin{aligned}
12z_0 &= 3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4,\\
12z_1 &= -y_1 + 6y_2 - 18y_3 + 10y_4 + 3y_5,\\
12z_2 &= y_2 - 8y_3 + 8y_5 - y_6,\\
12z_3 &= -3y_3 - 10y_4 + 18y_5 - 6y_6 + y_7,\\
12z_4 &= -25y_4 + 48y_5 - 36y_6 + 16y_7 - 3y_8.
\end{aligned} \tag{3}$$

Either set of formulas may be used although the second set is much easier to use if a computing machine is available.

Similar sets of formulas could be obtained by the use of "sliding" curves of other degrees than the fourth.

**2. The weighted average.** In formulas (1) we drop the hypothesis of a fourth degree polynomial approximation, continue the series, evaluate the derivatives, and obtain as error series for the formulas (2) the following:

$$
\begin{aligned}
R_0 &= \frac{1}{5}\Delta^5 y_0 - \frac{1}{30}\Delta^6 y_0 + \frac{1}{105}\Delta^7 y_0 - \frac{1}{280}\Delta^8 y_0 + \cdots, \\
R_1 &= -\frac{1}{20}\Delta^5 y_1 + \frac{1}{60}\Delta^6 y_1 - \frac{1}{140}\Delta^7 y_1 + \frac{1}{280}\Delta^8 y_1 - \cdots, \\
R_2 &= \frac{1}{30}\Delta^5 y_2 - \frac{1}{60}\Delta^6 y_2 + \frac{1}{105}\Delta^7 y_2 - \frac{1}{168}\Delta^8 y_2 + \cdots, \\
R_3 &= -\frac{1}{20}\Delta^5 y_3 + \frac{1}{30}\Delta^6 y_3 - \frac{1}{42}\Delta^7 y_3 + \frac{1}{56}\Delta^8 y_3 - \cdots, \\
R_4 &= \frac{1}{5}\Delta^5 y_4 - \frac{1}{6}\Delta^6 y_4 + \frac{1}{7}\Delta^7 y_4 - \frac{1}{8}\Delta^8 y_4 + \cdots .
\end{aligned} \tag{4}
$$

We form $\sum_{i=0}^{4} w_i(z_i+R_i)$ and then so choose these weight parameters $w_0, w_1, \cdots, w_4$ that, if differences of order nine are zero, $\sum_{i=0}^{4} w_i R_i = 0$. An easy computation yields four homogeneous equations in these weight parameters, and the general solution is $w_0 = w_4 = k$, $w_1 = w_3 = 16k$, $w_2 = 36k$. *Hence a weighted average of the five determinations from formulas* (2) *or* (3), *using weights* 1, 16, 36, 16, 1 *respectively, yields a value for $dy/du$ at $u=4$ which is exact if $y=f(x)$ is a polynomial of degree eight or less.*

If the middle three determinations are utilized and if we assume differences above the sixth order to be zero, a similar procedure shows the proper weights to be 1, 3, 1. *This resulting weighted average is precise if $y=f(x)$ is a polynomial of degree six or less.*

Let the former weighted average be $W_5$ and the latter be $W_3$. Then

$$
\begin{aligned}
W_5 &= (z_0 + 16z_1 + 36z_2 + 16z_3 + z_4)/70, \\
W_3 &= (z_1 + 3z_2 + z_3)/5.
\end{aligned} \tag{5}
$$

We substitute from equations (3) to obtain:

$$
\begin{aligned}
W_5 &= (3y_0 - 32y_1 + 168y_2 - 672y_3 + 672y_5 - 168y_6 + 32y_7 - 3y_8)/840, \\
W_3 &= (-\, y_1 + 9y_2 - 45y_3 + 45y_5 - 9y_6 + y_7)/60.^*
\end{aligned} \tag{6}
$$

* *Note*: Either of these two formulas could be used for calculating $dy/du$ at $u=4$ directly from the data. The advantage in the former method lies in the fact that one can then see how much of the final result is reasonably certain, *i.e.*, common to the several estimates [1].

Also $W_5$ is the formula for $dy/du$ at $u=4$ that one would obtain by assuming an eighth degree polynomial approximation for $y=f(x)$ which passes through the points with abscissas $x_0, x_1, \cdots, x_8$, and by evaluating the derivative of this polynomial at $u=4$. A similar statement holds for $W_3$.

**3. Estimates for errors.** The errors in the estimates $z_i$ and in the weighted averages $W_3$ and $W_5$ will be due to two sources: errors in the original data, errors due to method.

Let $E(z)$ denote the maximum possible error in $z$ due to a maximum possible error of $k$ in each of the original values of $y$. Then one can easily find, using formulas (3) and (6):

$$\begin{gathered} E(z_0) = E(z_4) = \frac{32}{3}k, \qquad E(z_1) = E(z_3) = \frac{19}{6}k, \qquad E(z_2) = \frac{3}{2}k, \\ E(W_5) = \frac{25}{12}k, \qquad E(W_3) = \frac{11}{6}k. \end{gathered} \tag{7}$$

Let $q$ denote the probable error of each entry $y$ in the original data. Let $H(z)$ denote the probable error of $z$. We use formulas (3) and (6) to obtain:

$$\begin{aligned} &H(z_0) = H(z_4) = \frac{1}{12}q(3^2 + 16^2 + 36^2 + 48^2 + 25^2)^{1/2} \approx 5.58q;^* \\ &H(z_1) = H(z_3) \approx 1.81q; \qquad H(z_2) \approx 0.95q; \\ &H(W_5) \approx 1.17q; \qquad H(W_3) \approx 1.08q. \end{aligned} \tag{8}$$

Estimates for the error, due to method, in each $z_i$ are given by the first terms of the series in (4). We substitute these first terms in the first equation in (5) to obtain an estimate for the error, due to method, in $W_5$, namely $G(W_5)$:

$$\begin{aligned} G(W_5) &= \frac{1}{350}\Delta^5(y_0 - 4y_1 + 6y_2 - 4y_3 + y_4), \\ &= \frac{1}{350}\Delta^5(1 - E)^4 y_0, \quad \text{where} \quad Ey_0 = y_1, \quad etc., \\ &= \frac{1}{350}\Delta^9 y_0, \quad \text{since} \quad \Delta = E - 1 \ [3]. \end{aligned} \tag{9}$$

Similarly,

$$G(W_3) = -\frac{1}{100}\Delta^7 y_1. \tag{10}$$

We compare the results of formulas (8), (9), (10) and observe that on the basis of errors in the original data, $z_2$ is a better estimate than $z_0$, $z_1$, $z_3$, $z_4$, $W_3$, or $W_5$. On the basis of errors due to method, $W_5$ is the best estimate and $W_3$ is next. Since

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{\Delta x}\frac{dy}{du},$$

the errors in $dy/dx$ are $1/\Delta x$ times the errors in $dy/du$. Hence in a particular

* The symbol $\approx$ is used to mean "approximately equal."

problem, $\Delta x$ should be large enough to make the probable error negligible and small enough to make the error due to method negligible.

Since the computation of $W_3$ involves the use of only the middle three determinations, there is a saving of labor in the use of $W_3$ in place of $W_5$.

**4. A numerical example.** To illustrate the method, we utilize the same illustration as given by Rutledge, namely, the data in Table I obtained by rounding off a twenty-three place table to ten places of decimals [**4**].

TABLE I

| $x$ | $\sin x$ | $x$ | $\sin x$ | $x$ | $\sin x$ |
|---|---|---|---|---|---|
| 0.6 | 0.56464 24734 | 0.9 | 0.78332 69096 | 1.2 | 0.93203 90860 |
| 0.7 | 0.66421 76872 | 1.0 | 0.84147 09848 | 1.3 | 0.96355 81854 |
| 0.8 | 0.71735 60909 | 1.1 | 0.89120 73601 | 1.4 | 0.98544 97300 |

We apply formulas (3) and (5) to obtain (at $x=1$):

$$z_0 = 0.05402\ 88880\ 5, \qquad z_4 = 0.05402\ 94454\ 3,$$
$$z_1 = 0.05403\ 05344\ 7, \qquad W_3 = 0.05403\ 02302\ 5,$$
$$z_2 = 0.05403\ 00507\ 4, \qquad W_5 = 0.05403\ 02306\ 3.$$
$$z_3 = 0.05403\ 04645\ 4,$$

Since $\Delta x = 0.1$, the value of $dy/dx$ is obtained by multiplying each of these estimates by 10. The correct value is 0.54030 23059.

Table II is a section of a larger table. The second column gives the value of $\sin x$ correct to the nearest fifth decimal. The third, fourth, and fifth columns give the estimates $z_1$, $z_2$, $z_3$, each divided by $\Delta x$. The sixth column gives $W_3/\Delta x$. The seventh column gives the values of $dy/dx = \cos x$, correct to the nearest fifth decimal.

TABLE II

| $x$ | $\sin x$ | $z_1$ | $z_2$ | $z_3$ | $W_3$ | $\cos x$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.47943 | 0.87759 | 0.87755 | 0.87750 | 0.87755 | 0.87758 |
| 0.6 | 0.56464 | 0.82527 | 0.82532 | 0.82537 | 0.82532 | 0.82534 |
| 0.7 | 0.64422 | 0.76493 | 0.76488 | 0.76488 | 0.76489 | 0.76484 |
| 0.8 | 0.71736 | 0.69670 | 0.69671 | 0.69672 | 0.69671 | 0.69671 |

In this example, the probable error $q$ of each entry is 0.000,002,5. We use the formula

$$\Delta^n(\sin x) = \left(2 \sin \frac{\Delta x}{2}\right)^n \sin\left(x + n\,\frac{\Delta x + \pi}{2}\right)$$

and easy computations to show that $|G(z_1)| < 0.000,000,38$; $|G(z_2)| < 0.000,000,25$; $|G(z_3)| < 0.000,000,38$; $|G(W_3)| < 0.000,000,000,7$; $H(z_1) = H(z_3) \approx 0.000,004,5$; $H(z_2) \approx 0.000,002,4$; $H(W_3) \approx 0.000,002,7$. To obtain estimates for the errors in $dy/dx$, each of these numbers must be multiplied by $1/\Delta x = 10$. Hence the final accuracy of $10W_3$ is about one-tenth that of $y = \sin x$, since $H(10W_3) \approx 10H(y)$ where $H(10W_3) \approx 0.000,027$ and $H(y) = 0.000,002,5$.

**5. A general weight theorem.** The two sets of weights previously obtained form two illustrations of the general theorem:

THEOREM: *Let the values of $y=f(x)$ be given for $x=x_0, x_1, \cdots$, where $\Delta x$ is constant. Compute $n+1$ determinations for $dy/du$ at $u=n$ (where $x=x_0+u\Delta x$) by the use of a "sliding" n-ic and let these be $z_0, z_1, \cdots, z_n$ respectively. Form a weighted average of these $z_k$, using as weights the coefficients in*

$$\frac{\partial^{p+q}}{\partial\alpha^p\partial\beta^q}\sum_{j=0}^{n}\binom{n}{j}^2\alpha^{n-j}\beta^j, \qquad p+q<n,$$

(the weight for $z_0$ is the coefficient of $\alpha^n$, and so on). *The weighted average will be the precise value of $dy/du$ at $u=n$ if $y=f(x)$ is a polynomial of degree $2n-(p+q)$ or less.*

*Proof*: The Newton interpolation formula, for $y=f(x)$ a polynomial of degree $2n$, is

$$y = \sum_{j=0}^{2n}\frac{u^{(j)}}{j!}\Delta^j y_0, \qquad x = x_0 + u\Delta x, \tag{11}$$
$$u^{(j)} = u(u-1)(u-2)\cdots(u-j+1).$$

Then

$$\frac{dy}{du} = \left\{\sum_{j=0}^{n} + \sum_{j=n+1}^{2n}\right\}\left\{\frac{\Delta^j y_0}{j!}\frac{du^{(j)}}{du}\right\}.$$

By sliding indices on $\Delta^j y_0$, we obtain, for $k = 0, 1, \cdots, n$,

$$z_k = \sum_{j=0}^{n}\frac{\Delta^j y_k}{j!}\frac{du^{(j)}}{du}\bigg]_{u=n-k}, \tag{12}$$

and

$$\begin{aligned} R_k &= \sum_{j=n+1}^{2n}\frac{\Delta^j y_k}{j!}\frac{du^{(j)}}{du}\bigg]_{u=n-k} \\ &= \sum_{j=n+1}^{2n}(-1)^{k+j-n-1}\frac{(k+j-n-1)!(n-k)!}{j!}\Delta^j y_k. \end{aligned} \tag{13}$$

Each of the values $z_k+R_k (k=0,1, \cdots, n)$ is a precise value for $dy/du$ at $u=n$, if $y=f(x)$ is a polynomial of degree $2n$ or less.

Since differences of order higher than $2n$ are zero,

$$\Delta^j y_k = \sum_{l=j}^{p}\binom{k}{l-j}\Delta^l y_0, \tag{14}$$

where $p$ is the smaller of $k+j$ and $2n$. We form

$$\psi(\alpha, \beta) = \sum_{k=0}^{n} \binom{n}{k}^2 R_k \alpha^{n-k}\beta^k. \tag{15}$$

The theorem will be established if we can show that

$$\frac{\partial^{p+q}\psi(1, 1)}{\partial\alpha^p\partial\beta^q} = 0, \qquad (p, q = 0, 1, \cdots, p + q < n).$$

We substitute equations (13) and (14) in (15) to obtain

$$\psi(\alpha, \beta) = \sum_{k=0}^{n} \sum_{j=n+1}^{2n} \sum_{l=j}^{p} \frac{(n!)^2(-1)^{k+j-n-1}(k + j - n - 1)!\alpha^{n-k}\beta^k\Delta^l y_0}{(l - j)!(k)!(n - k)!(k - l + j)!}.$$

We rearrange this finite summation and obtain:

$$\begin{aligned} \psi(\alpha, \beta) &= \sum_{l=n+1}^{2n} A_l(\alpha, \beta)\Delta^l y_0, \\ A_l(\alpha, \beta) &= \sum_{j=n+1}^{l} \sum_{k=l-j}^{n} \frac{(-1)^{k+j-n-1}(n!)^2(k + j - n - 1)!\alpha^{n-k}\beta^k}{(k)!(j)!(n - k)!(l - j)!(k - l + j!)}. \end{aligned} \tag{16}$$

If, for brevity, we set $\gamma = (\alpha - \beta)^n$, we can write this in the form

$$\begin{aligned} A_{n+m}(\alpha, \beta) = \frac{n!}{(n + m)!}\Bigg\{&\binom{n + m}{m - 1}\beta^{m-1}\frac{\partial^{m-1}\gamma}{\partial\beta^{m-1}} - \binom{n + m}{m - 2}\beta^{m-2}\frac{\partial^{m-1}(\beta\gamma)}{\partial\beta^{m-1}} \\ &+ \cdots + (-1)^{m-1}\binom{n + m}{0}\frac{\partial^{m-1}}{\partial\beta^{m-1}}(\beta^{m-1}\gamma)\Bigg\}. \end{aligned}$$

Then

$$\frac{\partial^{p+q}A_{n+m}(1, 1)}{\partial\alpha^p\partial\beta^q} = 0, \qquad \begin{aligned} m &= 1, 2, \cdots, n - p - q; \\ p, q &= 0, 1, 2, \cdots, n; p + q < n, \end{aligned}$$

since each term contains a partial derivative of $\gamma = (\alpha - \beta)^n$. If $y = f(x)$ is a polynomial of degree $2n - (p+q)$, the limits on the summation, in equation (16), will be from $n+1$ to $2n - (p+q)$, since $\Delta^l y_0$ will be zero for $2n - p - q < l \leqq 2n$. Hence the theorem is established.

*References*

1. George Rutledge, Reliable method of obtaining the derivative function from smoothed data of observation, Physical Review, vol. 40, no. 2, 1932, pp. 262–268.

2. Whittaker and Robinson, Calculus of Observations, page 12.

3. Boole, Finite Differences, 1931, page 17.

4. C. E. Van Orstrand, Tables of the exponential function and of the circular sine and cosine to radian argument, Memoirs of the National Academy of Sciences, vol. 14, 1921, Fifth Memoir.

# RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

## NEW BOOKS RECEIVED

*Advanced Mathematics for Engineers.* By H. W. Reddick and F. H. Miller. London, Chapman and Hall, Limited, 1938. 10+473 pages. $4.00.

*Elementary Mathematical Statistics.* By W. D. Baten. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Limited, 1938. 10+338 pages. $3.00.

*Statistical Methods.* By George W. Snedecor. Second Edition. Ames, Iowa, Collegiate Press, Inc., 1938. 13+388 pages. $3.75.

*Band III, Astronomia Nova.* By Johannes Kepler, Gesammelte Werke. (Herausgegeben von Max Casper.) München, C. H. Beck'sche Verlagsbuchhandlung, 1937. 488 pages.

*Introductory Quantum Mechanics.* By Vladimir Rojansky. New York, Prentice-Hall, Inc., 1938. 10+544 pages. $5.50.

*A New Geometry.* By A. W. Siddons and K. S. Snell. Cambridge, The University Press, 1938. 16+304 pages. $1.32.

*Diophantische Gleichungen.* By Th. Skolem. Ergebnisse der Mathematik und ihrer Grenzgebiete, 1938. 10+120 pages. RM. 15.

*Introduction to Fourier Series and Boundary Value Problems.* By Ruel V. Churchill. Ann Arbor, Michigan, Ruel V. Churchill, 1938. 4+94 pages. $2.50.

*Leitfaden der Stereometrie.* By W. Benz. (Mathematisches Unterrichtswerk für höhere Mittelschulen herausgegeben vom Verein schweizerischer Mathematiklehrer.) Zürich und Leipzig, Orell Füssli Verlag, 1938. 219 pages. fr. 3.80.

*Archimedes.* By E. J. Dijksterhuis. (Historische Bibliotheek voor de Exacte Wetenschappen, Deel VI, Eerste Deel.) Groningen-Batavia, P. Noordhoff, N.V., 1938. 213 pages. f 4.50.

*Arithmetik.* By Paul B. Fischer. (Sammlung Göschen, Band 47.) Berlin, Walter de Gruyter and Company, 1938. 152 pages. RM 1.62.

*The Book of Time.* By Gerald Lynton Kaufman. New York, Julian Messner, Inc., 1938. 16+287 pages. $3.00.

*Calculus.* By H. W. March and H. C. Wolff. Third Edition. (Modern Mathematical Texts, edited by C. S. Slichter.) New York and London, McGraw-Hill Book Company, 1937. 17+424 pages. $2.50.

*Algebras and Their Arithmetics.* By L. E. Dickson. Reprint of 1923 edition. New York, G. E. Stechert and Company, 1938. 12+241 pages. $2.50.

*The Calculus.* By R. D. Carmichael, J. H. Weaver and L. LaPaz. Revised Edition. Boston, Ginn and Company, 1937. 16+384 pages. $3.00.

## REVIEWS

*A New Geometry*. By A. W. Siddons and K. S. Snell. Cambridge, The University Press, 1938. 16+304 pages. \$1.32.

This new book is based on an older one by Godfrey and Siddons, but it is arranged according to an entirely original plan. It furnishes a complete course in geometry, but it would be better if it were preceded by a preliminary course of one year in length.

The first part of the book is intended to give the pupil a knowledge of geometrical facts together with developing the power to appreciate geometrical arguments and to use the methods of deduction. These aims are to be realized by means of worked examples and calculations, constructions, and originals. Theorems are arranged by groups in chapters; the results are stated and the proofs are left as originals. Questions on solid geometry are included and for convenience are grouped separately.

In view of all of the work that has been done in England in recent years to improve the teaching of geometry in the schools and in view of the standing of the authors, teachers of geometry everywhere will want to study the organization and methods of presenting geometric materials set forth in this new book. They will be particularly interested in seeing how plane and solid geometry are taught together in the same course.

W. D. REEVE

*Introduction to the Theory of Equations*. By Louis Weisner. New York, The Macmillan Company, 1938. 9+188 pages. \$2.25.

In this distinctive text the author employs the concept of a field to effect the unification of the various topics of Theory of Equations. The result is a book in which the chapters are not separate entities but are related in a satisfactory manner.

The chapter titles are

I. Complex Numbers.
II. Division and Factorization of Polynomials in a Field.
III. Further Properties of Polynomials in a Field.
IV. Theory of Equations in the Field of Rational Numbers.
V. Theory of Equations in the Field of Real Numbers.
VI. Elimination. Resultants. Symmetric Functions.
VII. Algebraic Extensions of a Field.
VIII. Algebraically Closed Fields.
IX. Constructions by Ruler and Compasses.

It will be noted that chapters on determinants and systems of equations are not included. However, the author has compressed, without any apparent effort, an amazing amount of information in a book of this size. Much of this has been accomplished by more than 500 exercises satisfactorily distributed over the wide range from drill problems to extensions of theory. Readily acces-

sible references, given with some exercises in the miscellaneous set of 115 in the back of the book, should stimulate the student. The style is pleasing, the explanations clear, and there are numerous illustrative examples.

The reviewer finds much to praise and little to criticise. Even in a brief review attention should be called to the generality with which Budan's Theorem and Descartes's Rule of Signs are stated, the elegance of Chapter VI, and the proof of the Fundamental Theorem of Algebra in Chapter VII.

There are a few minor points where the reviewer is in disagreement with the author. No careful distinction is made between equations and identities, the word "equation" being used for both and the conventional symbol for an identity is never used. The word "equation" is also used for an equality between constants. At the top of page 61 we find "multiplying the equation" rather than the more precise "multiplying the coefficients of the equation."

The few typographical errors noted are found on pages 15, 32, 46, 59, 64, 123. This carefully planned and well executed book should prove a welcome addition to a field that is certainly not overcrowded.

F. A. LEWIS

*The Reverse Notation*. By J. Halcro Johnston. London and Glasgow, Blackie and Son Limited, 1937. 10+74 pages. 3s. 6d.

In *The Reverse Notation* the author finds a place for proposals of reforms of all sorts, from the calendar and standards of weights and measures, to international finance. However, his chief argument, as well as his most convincing and that most likely to be of interest to the mathematicians, is for a change in our system of expressing numbers. This change is more sweeping than those usually proposed by advocates of a duodecimal system in that it combines the substitution of 12 for 10 as the basis of counting with the introduction of negative digits as well as positive.

The advantages of these negative digits are convincingly expounded. Subtraction is entirely eliminated and approximate additions, multiplications, and divisions are made easy since the neglect of the last few figures makes errors which tend to cancel each other rather than add up. In places where positive and negative numbers are likely to enter, as in the keeping of accounts, only one column is necessary to enter and sum numbers of both sorts. The practical advantages of using as a base a number rich in small factors are also given a detailed exposition. The arithmetic of the new notation is carefully explained with numerous examples, and tables of addition, multiplication, conversion from the present system, and logarithms are given.

Although the considerations advanced are of necessity mainly practical and have little bearing on mathematics, the mechanics of the reverse notation is interesting to explore for its own sake and the book certainly deserves the attention of all those interested in possible, though perhaps unattainable, improvements in our present system.

W. R. TRANSUE

*Theory of Equations*. By J. M. Thomas. New York and London, McGraw-Hill Book Company, 1938. 10+211 pages. $2.00.

In writing this textbook suitable for a term's course the author states in the preface that he "has been guided by a two-fold purpose: to make the treatment agree in spirit and terminology with what is called modern algebra, and to lead up to the Galois theory." Although courses in theory of equations have been offered regularly, no new text written in English has entered the field in over fifteen years. In view of the developments of modern algebra, it would appear that there is a decided need for just such a book. In fact, the reviewer is in sympathy with the author's purpose, and he exceedingly regrets that a critical reading of the book reveals significant imperfections.

In Chapter I a brief account of the number system is followed by elementary number theory. Properties of linear congruences are developed and used as a tool in succeeding chapters. Chapter II contains a discussion of permutations including a new proof of the theorem on the decomposition of a given permutation into transpositions. This will be of particular interest to teachers of group theory to whom the use of the alternating function in this connection seems artificial. The symmetric and alternating groups are defined, but the general definition of a permutation group is not given. Modern notation in Chapter III enables the author to give elegant proofs of some properties of determinants and matrices, the latter being written without the usual parentheses or double bars. The notation of Chapter III enables the author to give excellent proofs of the usual theorems on linear equations and ranks of matrices in Chapter IV.

In Chapter V entitled "Polynomials in a Single Indeterminate," the ordinary synthetic division appears (without proof) as well as a more general form. There are sections on the remainder and factor theorems, derivatives, multiple roots, highest common factor, reducibility, Gauss's Lemma and Eisenstein's Theorem. Chapter VI on "Graphical Methods" contains, in addition to theorems relating to polynomials, a brief treatment of complex numbers. There is also included a definition of continuity, accompanied by a few related exercises which the reviewer regards as essential for the student who is meeting the $\epsilon$, $\delta$ language for the first time. Chapter VII contains an adequate discussion of roots of unity including theorems on the irreducibility of the cyclotomic and a binomial equation. The conventional symbol for an identity is used only in case of congruences.

The first seven chapters occupy almost the first half of the book, and Chapter VIII entitled "Single Equation in Single Unknown" takes up almost half the remainder. This chapter contains several interesting contributions to standardized material, among them being "the statement of Budan's and Sturm's theorems so as to include the upper end point of the segment considered" and "an elementary discussion of a limit to the error in Horner's method." Some teachers will regret that Newton's method is not included.

The last four chapters are brief but contain what should be considered ample treatment of the following topics, which are also the respective chapter titles:

Symmetric Functions, Constructibility, Resultants and Discriminants, and Simultaneous [non-linear] Systems.

The type is relatively large and well spaced. There are less than 400 exercises mostly of a routine nature supplemented by a miscellaneous set of 60 at the end of the book. It would seem that practically all of the material could be covered by advanced undergraduates in one term.

It is unfortunate that a book which otherwise is satisfactory should be marred by many careless slips on the part of the author and those who helped him see the book through the press. On page 59 the author defines "a *root* of the polynomial $f(x)$ and of the equation $f(x)=0$." While this is not objectionable *per se*, in the opinion of the reviewer this definition will increase the number of students who use the terms polynomial and equation synonymously. The definition may not be the "root" of all the following evils, but some statements from the text are offered to substantiate the reviewer's contention. On pages 96 and 99 the reader is asked to find equations having certain roots, but the answers are polynomials. On page 121 the reader is asked to locate the real roots of certain equations which are in reality polynomials. On page 188 he is asked to reduce certain systems of equations, but the systems in question are polynomials. On page 160 the reader is directed to substitute $\dfrac{y_4+y_5}{y_1+y_3}$ in $x^3+x+1$ and "clear of fractions." On page 114 we find "If these three equations are transposed to the left and multiplied together, a polynomial $F$ of degree 3 results, . . . ."

In the interest of clearness, precision, and faultless style, some sentences should be rewritten. For example, on page 30 we find "Similarly for subscripts, that is, we ignore $i$ as a superscript and $j$ as a subscript." On page 54 the following sentence appears as a paragraph: "To emphasize the difference between them and indeterminates complex numbers will be called constants." On page 28 listed as an elementary transformation appears "(ii) multiplying every element in a fixed row by the same non-zero quantity and the determinant by its reciprocal." On page 157 in discussing the intersection of two circles the author writes: "If $A$-$A'$ and $B$-$B'$ are not both zero, this is the equation of a straight line passing through the points of intersection, that is, it is the equation of the common chord, and the intersections of the circles are its intersections with either circle." Certain other involved constructions may be found on pages 11, 19, 37, 42, 78, 121, 144.

In conclusion the reviewer would like to point out a few needed corrections of a somewhat different nature. The word "associated" is omitted on page 69. On page 75 the word "other" is needed in the sentence "If a circle be drawn with center at one of them, say $a$, and passing through the nearest of the others $b$, there is no root of the polynomial within the circle." The word "minus" should be omitted on page 106. Certain inequalities are reversed on page 122. Some mention should be made of convergence of infinite series on page 151.

F. A. LEWIS

*Carl Friedrich Gauss—Inaugural Lecture on Astronomy and Papers on the Foundations of Mathematics.* Translated and edited by G. W. Dunnington. Baton Rouge. Louisiana State University Press, 1937, 11+91 pages. $1.00.

This little book contains the substance of a lecture on the early life of Gauss, which Professor Dunnington gave the Kansas Section of the Mathematical Association of America in 1933 (see this MONTHLY, July 1933), and in addition English translations of a number of little known Gaussian fragments on the foundations of mathematics and on astronomy. The report on the early life of Gauss (pp. 1–32) deals with Gauss's activities from his boyhood (he was born in 1777) till 1803. Then follows a paper written by Gauss in his earlier years, which has only recently been brought to light (pp. 38–42, *Werke*, vol. XII, pp. 57–61); this contains eleven theses on the foundations of mathematics. Then follow a few questions on foundations, taken from Gauss's 19th manual, begun May 1809 (pp. 136–137), which follows directly after the paper dated Dec. 4, 1825, vol. VIII, p. 444 of Gauss's collected works (*Werke*, vol. X, pp. 396–397). The remaining part of the book is devoted to Gauss's inaugural lecture on Astronomy, probably written in 1807 (*Werke*, vol. XII, pp. 177–198). Professor Dunnington adds a number of explanatory notes to this as well as to the other papers of Gauss. Added is an interesting youth portrait of Gauss, from the Heimats museum at Göttingen.

Though this edition does not seem to present us with any startling discovery concerning Gauss's work, we may be grateful to Professor Dunnington, whose work allows a larger number of students in the English-speaking countries to obtain first-hand knowledge of some of Gauss's fundamental ideas. The report on Gauss's early activities is also especially welcome.

The book is dedicated, on the occasion of the bicentennial jubilee of the University of Göttingen, not only to Professors Hasse and Riesle, but also to the rector, dean and curator of this university "who today are ably and worthily guiding the destinies of the University of Göttingen."

D. J. STRUIK

*Introduction to College Mathematics.* By M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Company, 1938. 373 pages + 93 pages of tables. $2.40.

This book is based upon First Year College Mathematics by the same authors, which has been reviewed by Frink.*

The earlier book consisted of parts: (I) Algebra and Trigonometry, (II) Analytic Geometry, and (III) Mathematics of Finance. The book under review is divided into Part I: Algebra and Trigonometry; Part II: Analytic Geometry and Calculus. Part I of the newer book is identical with the first part of the older one; the Analytic Geometry of the later book is, except for a slight rearrangement of topics and the omission of rotation of axes and a discussion of

* This MONTHLY, vol. 44, 1937, p. 651.

the conic in general position, the same as the second part of the first book. The Mathematics of Finance (90 pages) of the earlier book has been replaced by 54 pages of Calculus. Consequently we consider here only the section devoted to the Calculus.

Formulas are given for the differentiation of algebraic functions only. Applications include tangents and normals to curves, maxima and minima, and rates. There are numerous examples illustrating the processes involved, but, too frequently, these examples are used to give *rules* of procedure rather than to clarify concepts. Occasionally, the examples are used to give a "proof" by plausibility when a simple proof is available.

The discussion of differentials, the writer feels, would confuse a student. For instance, having defined $dx$ as the differential of $x$, the reader is told to take $dx$ equal to $\Delta x$ (p. 316). Why and how is left an open question. Again, since the differential is a quantity defined in terms of a function $f(x)$ at $x=x_0$, corresponding to an increment $\Delta x$, $\Delta f$ and $df$ should be compared for this value of $x$. Whereas, in order to make $\Delta f=df$ (p. 317), $\Delta f$ is evaluated at $x=x_0$ and $df$ at $x_0+\frac{1}{2}\Delta x$. This hardly clarifies the distinction between differential and increment.

Integration is considered as the inverse of differentiation only. Since this is emphasized it seems strange that the formula for $\int x^n\,dx$ is not obtained from the corresponding differentiation formula instead of merely suggesting the plausibility of the general formula by verifying it for $n=1, 2, 3, 4$, (p. 336).

Applications of integration are made to the finding of areas, falling bodies, work, and momentum.

On the whole, a student should be able to read the book with understanding. Those instructors who are content to have their students learn rules of procedure in the elements of the Calculus should find this text fairly satisfactory.

C. A. NELSON

*Procedures and Metaphysics*. By E. W. Strong. A study in the philosophy of mathematical-physical science in the sixteenth and seventeenth centuries. Berkeley, University of California Press, 1937. 7 + 301 pages. $2.50.

This book is an important source of information on the motives which inspired the founders of modern mathematical-physical science. The author has carefully studied the prefaces and commentaries in which the great investigators and teachers of the sixteenth and seventeenth centuries considered the methods and nature of their inquiry. These texts afford a direct testimony by the scientists themselves concerning the character of their work, their opinions of the use of geometry in physical investigation and demonstration, the relation between pure and practical geometry and arithmetic, their ideas on points and lines, and their use in practical mensuration compared to the concepts of Euclid. Prominent in this research is one particular topic. In how far were these scientists influenced by philosophical, by metaphysical speculations, especially by the Neo-Platonist and Neo-Pythagorean thought of their days, with its idealization of mathematics?

The author did not raise this question without some provocation. Professor E. A. Burtt, in his book *The Metaphysical Foundations of Modern Physical Science* (Harcourt, Brace & Co., New York, 1925) has suggested that the Platonic and Pythagorean tradition, asserting the cosmological status of mathematics, has provided a foundation and a justification for modern science. "The Neo-Platonic background of the mathematical and astronomical development of the time of Galileo," asserts Burtt, "has strongly penetrated the mind of the Italian scientist, as in the case of so many lesser figures."

Mr. Strong's book aims at the refutation of Burtt's thesis. He insists that the metamathematical tradition which regarded the study of mathematics mainly as a preparation for the study of the mystical nature of the universe was without any considerable influence on the new mathematical and mechanical research. The roots of the work of Tartaglia, Cardan, Galileo, Kepler and the other productive mathematicians did not lie in the Platonic tradition, but in the study of Euclid, or more general, in the operational approach exemplified by Euclid. Euclidean geometry offered their scientists a scientific structure which first establishes the meaning of its fundamental concepts and then develops by strictly logical procedure its consequences without any appeal to metaphysical speculations.

The task is accomplished in the following way. First we have an exposition of the function of mathematics in the Platonists and Pythagoreans of antiquity, as Nicomachus, Theon, Proclus. These opinions are compared to those professed by Tartaglia, Cardan, Cataneo, and other Italian and French mathematicians in the sixteenth and seventeenth centuries as Vieta, Clavius, Leonardo da Vinci, and others. Then comes an investigation of the status of the science of mechanics in Italy prior to Galileo, followed by procedures and metaphysics in Galileo and Kepler. The final chapter deals with the metamathematical tradition in the early modern period, and discusses the Platonic schools of Ficino, Pico, Reuchlin and Bungus with their influence upon mathematical writers as Zamberti, Domenico, Henrion and Dee. An appendix on the operational meaning of point and line in Euclid's *Elements* presents us with a discussion on the opinions of Mach and Poincaré. Strong's conclusion is: "Euclid offers us no evidence that he began his geometry by first entering into a theory of knowledge. . . . If Euclid's geometry needs any foundations for the meaning of its definitions, the subject-matter and procedures of the geometry itself would appear to be a sufficient basis. We do not need to appeal to the so-called more ultimate grounds of mind and nature. . . . A task of operational method is to keep clear the domain of reference intended by the questions we ask. Questions of relevancy or meaning are kept straight when we do not confuse our categories. fields, contents or universes of discourse. No geometrician needs metaphysical cement to connect pure and practical geometry" (pp. 243–244).

According to Strong, Galileo and the other productive mathematicians and physicists of his period are examples of the operational approach to science. They do not assert that "mathematical demonstrations in science must be based

on metamathematical doctrines or principles." "The methodological position brought out in the study of the Italian investigators and the analysis of the methodological insight in Kepler culminating in the *Epitome*," he concludes, "constitute an appreciable weight of evidence for the thesis that operational considerations existed in the early modern period without being based upon metamathematical foundations and without requiring subsequent metaphysical arguments, distinctions, and sanctions" (p. 183).

Kepler shows more influence of the metaphysical tradition, but Mr. Strong gives ample evidence that Kepler, in places where this tradition came in conflict with the evidence of an operational approach, does not hesitate to desert the metaphysical speculation for a scientific reasoning in the sense of Euclid.

Mr. Strong's case is very good, but it seems to the reviewer that he has gone somewhat too far in his attempts to separate the two traditions. In Kepler's case he mainly bases his conclusions upon the *Epitome*, one of Kepler's later books (1620), but there is a wealth of material in Kepler's work which shows how deeply his productive work was stimulated by his Platonic way of thinking. He was always looking for geometrical and arithmetical harmonies, because of his deep conviction of the metaphysical importance of mathematics. The cogency of a scientific hypothesis never was for him independent of metaphysical beliefs. We see in Kepler more than in Galileo the transition from scientific reasoning influenced by platonical metaphysics to a more "operational" way of thinking.

We also think of Copernicus, or Nicolaus Cusanus, not mentioned by Mr. Strong, as cases where metaphysical reasoning influenced the productive use of mathematics in physical study. We may even take Giordano Bruno, who has not lent his name to any particular theory or theorem in science, but who has influenced further mathematical and physical thought to a considerable extent.

It is certain, however, that Mr. Strong has shown that we cannot simply assume that the platonic revival of the sixteenth century is the cause of the approval of mathematical methods in the subsequent study of mechanics and physics. There was a wide gap between the two traditions. The question now takes a new form, which Mr. Strong has not attempted to investigate in his book. Why did philosophy begin to platonize in the early modern period? Why did, shortly afterwards, mathematics also enter into the study of nature? The answer seems to lie in a study of the social and economic forces of the period. The great social and technological changes affected both the philosophers and the scientific workers. University and practical work, however, were largely separated. Men like Tartaglia and Stevin had their education even entirely outside of the university halls. We seem to have currents due to similar causes in fields rather widely separated. Where university and practice joined hands, however, the platonic tradition let its force be felt, largely as an antidote to the aristotelian tradition. For this reason we still must take it into consideration as a secondary factor in the development of modern natural science.

D. J. Struik

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.*

## TOPICS FOR CLUBS

The following subjects have appeared on a number of club programs and if bibliographies could be sent to us for them, they will perhaps furnish additional material for discussions and papers.

31. Continued fractions.
32. Maps and map projection.
33. Constructions with compass alone.
34. Constructions with ruler alone.

## CLUB REPORTS, 1937–1938

### *Pi Mu Epsilon, Washington University*

At the regular meetings the following topics were discussed by members of the faculty: The aesthetics of science; The application of mathematics to meteorology; Problems in modern dynamics; Steiner's ellipse and point; reviews of *Men of Mathematics* and *Mathematics for the Million*. An open meeting was held in conjunction with the local chapter of Sigma Xi with Dr. John von Neumann of the Institute for Advanced Study speaking on "Foundation Problems of Quantum Mechanics," and "Foundation Problems of Mathematics." At the initiation banquet, Dean A. S. Langsdorf spoke on "The Method of Inversion as Applied to Electrical Problems."

Director, Dr. Gabriel Szegö; Vice-Director, Bernice Dunie; Secretary, Candace Wisbrock; Treasurer, Prof. E. Stephens.

### *Pi Mu Epsilon, Oklahoma A. and M. College*

The organization, formerly the Mathematics Club, was installed last year as a chapter of Pi Mu Epsilon, with twenty charter members. The papers presented during the year were: Cryptographs; Solution of equations of the fourth degree; The mathematical meaning of infinity and some infinite collections; Why nine is a magic number; The weather; Japanese mathematics; Complex numbers; Mathematics for aerial photography; Prime numbers; The slide rule; Mathematical fallacies.

Director, Wilma Meacham; Vice-Director, C. E. Abraham; Secretary-Treasurer, Charles G. Cruzan; Sponsor, Dr. E. F. Allen.

### *Mathematics Club, Stanford University*

The Mathematics Club as an organization attended the two lectures at Stanford by Dr. T. C. Fry of the Bell Telephone Laboratories, (1) On locating roots of polynomials by matrix iteration, and (2) Applications to network design. The second talk was accompanied by a motion picture of a machine for calculating the roots of polynomials in operation. Such subjects as Researches in higher geometry; An elementary proof of R. Hall's theorem on soluble groups; Lower bounds for the degree of multiply transitive group in terms of its degree of transitivity; A geometric proof of Minkowski's theorem on the product of two non-homogeneous linear forms in two variables; On the duration of play; On the luminosity of stars; were the basis of papers and discussions during the year.

Executive Council, Dorothy Evans, D. Belle Rundle, Edward Bewley; Adviser, C. D. Olds.

## *Pi Mu Epsilon, Hunter College*

The chapter held eight program meetings, an initiation banquet in the fall with Professor Courant of New York University as guest speaker, an initiation tea in the spring with Professor Mayme I. Logsdon of the University of Chicago as guest speaker, and a reunion supper in June.

The subject for the first semester was geometrical topics *via* complex numbers. The first two meetings concentrated on geometry growing out of the addition of complex numbers, the last two on geometry growing out of the multiplication of complex numbers. The reports of the second semester dealt with algebraic plane curves. A meeting each was devoted to singular points and lines, poles and polars, Cremona transformations, the theorem of Noether.

Prizes were awarded to the best speakers. The awards for the first semester were won by Estelle Poppiani and Helene Parnass, and for the second semester by Jeannette Miller.

Director, Prof. Marguerite D. Darkow; Vice-Director, Rita Mohnkern; Recording secretary, Wilma Szabo; Corresponding secretary, Selma Goldstein; Treasurer, Pearl Goldman.

## *Mathematics Club, Hunter College*

The year's program was planned to include topics from the mathematics of statistics, since a number of the club members were particularly interested in that field. Dr. Helen Walker of Columbia University addressed the club on "Statistical concepts which should be familiar to every educated person." Other papers presented: Calculus of variations; A game of logic; Some aspects of the Social Security Program; Algebra of logic; The four color problem; The trisection of an angle; A curve of learning; Some statistical measures; A geometric interpretation of the coefficient of correlation; The application of complex numbers to the solution of geometric problems; Puzzles and fallacies.

In addition to several social meetings, the annual party given by the club to all students of the department included for entertainment the play *It Can't Happen Here* written by Dr. Whelan of the department faculty.

Two competitive examinations were held. An integration contest was won by Malene Anderson, and a special mathematics examination with a prize of twenty-five dollars was won by Helene Parnass.

President, Lillian Leight; Vice-President, Vivian Fruchtbaum; Secretary, Justine Schmertz; Treasurer, Bessie Hochberg. Advisers, Dr. Jewell H. Bushey, Mrs. Helen Kutman.

## *The Mathematical Club, Regis College*

This club sponsored a contest for secondary school students on applications of mathematics. The six prize winners were guests of the club at a spring tea. Papers read at the meetings were: Construction of a beehive; Amicable numbers; Different radices; Women in mathematics.

President, M. Melanson; Vice-President, M. White; Secretary, M. Gahan; Treasurer, E. Mahoney.

## *Sigma Phi Mu, New Jersey State Teachers College*

The club observed the memorial year in honor of Dr. Slaught in a twofold way. Professor Davis, a former student of his, spoke on the life of this modern mathematician, and the famous play by Dr. Slaught *The Evolution of Numbers* was presented. Each of the classes in the club presented a mathematical play suitable for presentation before high school students. They were: Mock trial of B versus A; The evolution of numbers; Flatlanders; The clue of the little 3. Topics discussed at the remaining meetings were: Extracting cube roots mentally; Interesting sidelights on mathematics; Properties of the catenary; Logarithmic spirals in nature; Magic squares; Crystallography; Famous Americans as mathematicians; Mathematical induction; Mathematics in navigation; Repeating decimals. One of the guest speakers was ninety-five-year-old Dr. William Vail, author of *Div-A-Let*, who spoke on Division by letters, and on the Perpetual calendar.

President, Norman Chinoy; Vice-President, Marie Albers; Secretary, Marjorie MacInnes; Treasurer, Sheva Crystal; Adviser, Dr. E. H. Hildebrandt.

### *Kappa Mu Epsilon, University of New Mexico*

This chapter cooperates with several other institutions in the Southwest, in sponsoring an annual lecture given by a member of the Southwestern Section of the Mathematical Association. Dr. Wexler of Arizona State College was the traveling lecturer last year. One monthly program consisted of a talk on properties of numbers and the other meetings included talks on the life of the mathematicians: François, Viète Archimedes, Gauss, and Descartes.

President, Louis York; Vice-President, Clara May Mathew; Secretary, Osborn Keller; Treasurer, Charles Barker; Adviser, Dr. H. D. Larson.

### *Mathematics Club, University of Alberta*

Papers presented before this club are entered in competition for the Dr. Cook prize for the best student paper. It was awarded to Allen Gibb for his presentation of the calculus of variations. Other papers were: Design by geometry; Mathematics of insurance; Graphical representation; Projective coordinates; The discriminating cubic; The philosophy of Dr. A. N. Whitehead. One meeting was devoted to the solution of a set of problems distributed earlier in the year, and involving a bit of mathematical brain teasing. At the annual banquet, Professor Keeping spoke on the book *Philosophy and the Physicists* by Dr. Susan Stubbings.

President, D. R. Crosby; Secretary, Marjorie Stockwell; Treasurer, L. Palleson; Adviser, Dr. A. J. Cook.

### *Euclid's Circle, Mount St. Scholastica College*

The main topic for discussion for the year's programs included a study of the contributions of the various nations to the fields of mathematics. Other topics were: The life and work of Euclid; Culture and mathematics; and a review of *Mathematics for the Million*. Part of each meeting was given over to trick problems, mathematical games and puzzles. One meeting featured the play *A Problem for Pythagoras*.

President, Miriam Powers; Vice-President, Mary Schirmer; Secretary-Treasurer, Mary Donahoe; Sponsor, Dr. A. J. Reardon.

### *Mathematics-Physics Club, College of Saint Theresa*

Reports presented by faculty and student members included: Science of musical sounds; Beauty of a triangle; Teaching the subtraction of signed numbers; Mathematics in Germany; A billion; The spider problem. One meeting commemorated the Geometry of Descartes. At another a great deal of interest was aroused in a debate on the subject: The study of mathematics should have practical applications as its sole aim. Social meetings were held before Christmas and at the close of the year.

President, Mary Payant; Vice-President, Mary McKeown.

### *Pi Mu Epsilon, University of California*

Papers presented during the year were: Uniformly almost periodic functions; Secondary schools and secondary teachers of mathematics in England and France; The isoperimetric problem; The application of mathematics to the earthquake problem; Certain quadric, cubic and quartic loci in space of three dimensions; Mathematical games; Mathematical economics; An application of the complex variable to a geometric problem; Selected topics from the theory of relativity; Oddities of mathematics.

Director, Raymond Wakerling; Vice-Director, Virginia Wood; Treasurer, Charles Hayes; Secretary, Margaret Seegmiller.

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 362. *Proposed by V. Thébault, Le Mans, France.*

If $X$ and $Y$ are consecutive positive integers, and $Z$ is an integer, and if $X^2+Y^2=Z^2$, show that each of the two numbers, $(X+Y\pm Z)^2+1$, is the sum of the squares of two consecutive integers, and that of these last four integers the two odd ones are squares, and the even ones are the doubles of squares.

E 363. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

Construct triangle $ABC$, given $A$, $a$, and $(h_a+c-b)$.

E 364. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

If $(1+x)^n/(1-x)^3=a_0+a_1x+a_2x^2+\cdots$, show that

$$a_0+a_1+a_2+\cdots+a_{n-1}=\frac{n}{3}(n+2)(n+7)\cdot 2^{n-4}.$$

E 365. *Proposed by Virgil Claudian, Bucharest, Rumania.*

Prove that

$$\int_0^1 x^n(1-x^n)^n dx=\frac{n^n(n!)}{(n+1)(2n+1)(3n+1)\cdots[(n+1)n+1]}.$$

E 366. *Proposed by C. O. Oakley, Haverford College.*

Two ferry boats ply back and forth across a river with constant speeds, turning at the banks without loss of time. They leave opposite shores at the same instant, meet for the first time 700 feet from one shore, continue on their way to the banks, return and meet for the second time 400 feet from the opposite shore. As an oral exercise, determine the width of the river.

E 367. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

The point $P$ moves on the circumcircle of triangle $ABC$, and the bisectors of angles $APB$ and $APC$ meet $AC$ and $AB$ at $Q$ and $R$ respectively. Show that $QR$ passes through the center of the circle inscribed in triangle $ABC$. Show also that if $PS$ and $PT$ are perpendicular to $PQ$ and $PR$ and cut $AC$ and $AB$ at $S$ and $T$ respectively, then $ST$ passes through the center of the escribed circle which touches side $BC$ between $B$ and $C$.

E 368. *Proposed by W. B. Campbell, Drexel Institute.*

A clock has its hour, minute, and second hands turning about the common center, and all together at noon. They will not be all three coincident again for another twelve hours, although there will be several earlier instants when two of the three hands are superimposed. At which of these instants will the remaining hand make the least angle with the two which are superimposed?

E 369. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

Find the form of $n$ if $1^4+2^4+3^4 \cdots +n^4$ is exactly divisible by $1^2+2^2+3^2+ \cdots +n^2$.

E 370. *Proposed by V. Thébault, Le Mans, France.*

Locate the point $P$ within the irregular tetrahedron $ABCD$ so that each of the six planes, each through $P$ and an edge, will bisect the surface of the tetrahedron.

**SOLUTIONS**

E 321 [1938, 185]. *Proposed by V. Thébault, Le Mans, France.*

In what system of enumeration may a four-digit number of the form $aabb$ be the square of a number of the form $cc$, provided furthermore that if expressed in the decimal system, $a$ and $b$ consist of the same digits in opposite order?

*Solution by C. W. Trigg, Los Angeles City College.*

The given data may be expressed as $ar^3+ar^2+br+b=(cr+c)^2$, or as $a(r-1)+(a+b)/(r+1)=c^2$. Also, $a=10m+n$ and $b=10n+m$, with $m$ and $n$ each less than ten, $a$ and $b$ less than $r$, and $r$ greater than ten.

But $(a+b)/(r+1)=11(m+n)/(r+1)$ must be an integer, and since $1\leqq m+n\leqq 18$, the eligible values of $r$ are restricted to twenty-four, namely, 11, 12, $\cdots$, 17, and $11k-1$, where $1<k\leqq 18$. The latter group yields no solutions, since $a<r$. The further restrictions imposed by $c^2$ limit the solutions to $r=43$, with $40\ 40\ 04\ 04=(41\ 41)^2$, and $04\ 04\ 40\ 40=(13\ 13)^2$, or $r=65$, with $15\ 15\ 51\ 51=(31\ 31)^2$, or $r=109$, with $28\ 28\ 82\ 82=(55\ 55)^2$, or $r=131$, with $48\ 48\ 84\ 84=(79\ 79)^2$.

Also solved by E. P. Starke and the proposer.

E 322 [1938, 185]. *Proposed by J. E. Trevor, Cornell University.*

A manufacturer makes all possible sizes of brick-shaped blocks of molded building material, which are such that the lengths of the edges are integral multiples of the unit of length, and that the number of units in the total length of the twelve edges of a block is equal to two-thirds of the number of units of volume in the block. How many and what sizes does he make?

*Solution by E. P. Starke, Rutgers University.*

Let the numbers of units of length in the three edges meeting at a vertex be $x$, $y$, and $z$. Then, by hypothesis, we have

$$4x + 4y + 4z = 2xyz/3,$$

or

$$6(x + y + z) = xyz.$$

Evidently one letter is a multiple of 3. Let $x$ be that one, and put $x=3u$. Then $6u+2y+2z=uyz$, or

$$u = 2(y + z)/(yz - 6). \tag{1}$$

Thus $u=2(yz+y^2)/[y(yz-6)]=(2w+2y^2+12)/yw$, where $w=yz-6$ is a divisor of $2y^2+12$. If we put $y\leqq z$, (1) implies $yz-6\leqq 2(y+z)\leqq 4z$, or $y\leqq 4+6/z\leqq 10$. For assigned values of $y=1, 2, 3, \cdots, 10$, possible values of $w=yz-6$ may be easily determined as divisors of $2y^2+12$. If the corresponding $z$ and $u$ are integers, we have a solution. In this way we find eight essentially distinct solutions for $(x, y, z)$, namely (48, 1, 7), (27, 1, 8), (12, 1, 13), (9, 1, 20), (18, 2, 4), (6, 2, 8), (12, 3, 3) and (3, 4, 7).

Also solved by W. E. Buker, W. B. Clarke, Michael Goldberg, A. V. Richardson, C. W. Trigg and the proposer.

E 323 [1938, 185]. *Proposed by J. R. Musselman, Western Reserve University.*

If $D$, $E$ and $F$ are the mid-points of the sides $BC$, $CA$ and $AB$ respectively of the triangle $ABC$, show that the centers of the nine-point circles of triangles $AFE$, $BDF$ and $CED$ form a triangle homothetic with triangle $ABC$ in the ratio 1:2.

*Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

For the respective triangles, $ABC$, $AFE$, $BDF$ and $CED$, denote the nine-point centers by $K$, $L$, $M$ and $N$, the orthocenters by $O$, $P$, $Q$ and $R$, and the circumcenters by $S$, $T$, $U$ and $V$.

It is clear that since $K$, $L$, $M$ and $N$ bisect respectively the segments, $OS$, $PT$, $QU$ and $RV$, and since the triangles $AFE$, $BDF$ and $CED$ are each similar to triangle $ABC$ and of one-half its linear dimensions, then $KA=2KL$, $KB=2KM$, and $KC=2KN$. Hence the triangle $LMN$ is homothetic with the triangle $ABC$, the ratio 1:2, and the homothetic center $K$.

In his solution, E. S. Smith of Boulder, Colorado, points out that $K$ is also the nine-point center of triangle $LMN$, and investigates the center of symmetry for the congruent triangles $DEF$ and $LMN$. If the circumcenter and orthocenter of triangle $LMN$ are denoted respectively by $H$ and $J$, he finds $S$, $H$, $K$, $J$ and $O$ equally spaced in that order on a straight line. If the common mid-point of $HK$ and $SJ$ is denoted by $G$, he finds that $G$ is the center of perspective of triangles $DEF$ and $LMN$.

Also solved by W. B. Clarke, T. C. Esty, D. L. MacKay and C. W. Trigg.

E324 [1938, 185]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Right triangles with integer sides proportional to 3:4:5 are well known. The sides of such a triangle form an arithmetic progression. Show that these con-

stitute the only right triangles having integer sides in arithmetic progression, and furthermore that no right triangle exists having integer sides in geometric progression or in harmonic progression.

*Solution by Fred Discepoli, New York, N. Y.*

If $0<d<a$, and $a-d$, $a$ and $a+d$ are the sides of a right triangle, then $(a-d)^2+a^2=(a+d)^2$. This reduces to $a(a-4d)=0$. The only admissible solution here is $a=4d$, and the sides of the triangle are $3d$, $4d$ and $5d$, and hence in the required ratio.

To investigate the possibility of a right triangle having sides in geometric progression, let the sides be $a$, $\sqrt{ab}$ and $b$, so that $a^2+ab=b^2$. This gives the solution, $a=b(\sqrt{5}-1)/2$, which cannot be satisfied if $a$ and $b$ are both integers.

To investigate the possibility of a right triangle having sides in harmonic progression, let the sides be $1/(a+d)$, $1/a$ and $1/(a-d)$. The Pythagorean relation, $1/(a+d)^2+1/a^2=1/(a-d)^2$, reduces to the quartic equation, $a^4-4a^3d-2a^2d^2+d^4=0$. Since this does not factor rationally, $a$ is not expressible rationally in terms of $d$, and hence $a=d$ times an irrational expression. So no triangle can have integer sides in harmonic progression.

*Note by the Proposer.* If the sides of a right triangle are in arithmetic progression, they can only be in the ratio of 3:4:5, even though they be irrational. If the sides are in geometric progression, they must be in the ratio $1:r:r^2$, where $r^2$ is the limit of the ratio of any term to its successor in the Fibonacci series, $1, 1, 2, 3, 5, 8, 13, \cdots$. If the sides are in harmonic progression, two shapes are possible, with incommensurable sides, since the equation, $x^4-4x^3-2x^2+1=0$, has two positive, irrational roots and two imaginary roots.

Also solved by E. F. Allen, Lois E. Bell, J. H. Edmonston, H. L. Lee, M. Y. Luke, D. L. MacKay, A. V. Richardson, E. P. Starke, C. W. Trigg, Paul Weisz and the proposer.

E 325 [1938, 185]. *Proposed by W. W. Bigelow, Beloit College.*

Show that the radius of the sphere which can displace the most liquid from a filled conical wineglass is given by the formula, $r=hn/(n^2+n-2)$, where $h$ is the altitude of the cone, and $n$ is the cosecant of its generating angle.

*Solution by D. L. MacKay, Evander Childs High School, New York, N. Y.*

If $x$ is the altitude and $y$ the radius of the base of the segment of the sphere protruding above the glass, then $y^2=2rx-x^2$, $h-r+x=rn$, $x=r(1+n)-h$, and $dx/dr=1+n$.

The displaced volume, $V=4\pi r^3/3-\pi xy^2/2-\pi x^3/6=\pi(8r^3-3xy^2-x^3)/6=\pi(8r^3-6rx^2+2x^3)/6$. The volume $V$ is a maximum when

$$24r^2-6x^2-12rx\cdot dx/dr+6x^2\cdot dx/dr=24r^2-6x^2-12rx(1+n)+6x^2(1+n)=0,$$

or when $4r^2-2rx(1+n)+x^2n=0$. But this factors into

$$(2r-x)(2r-nx)=0; \qquad \text{so} \qquad r=x/2 \qquad \text{or} \qquad nx/2.$$

When $r=nx/2$, since $x=r(1+n)-h$, $r=hn/(n^2+n-2)$ and $V=(4\pi h^3/3) \div (n-1)(n+2)^2$, the maximum displaced volume. When $r=x/2$, $r=h/(n-1)$ and $V=0$. Then the sphere is above the liquid and tangent to its surface. This case does not satisfy the condition that the sphere touch the glass.

E. P. Starke calls attention to the fact that this problem appeared in Granville's *Elements of the Calculus*, 1911, p. 118, with the answer in the form, $r=h\sin\alpha/(\sin\alpha+\cos 2\alpha)$.

Also solved by W. E. Buker, J. H. Edmonston, V. E. Pound, S. A. Singer, Alvin Spiro, R. H. Urbano and the proposer.

*Note by the Editor.* A full discussion of this problem was recently given by W. R. Longley in a note entitled "An example of a continuous function with finite discontinuities in its second derivative," this MONTHLY, vol. 44, 1937, pp. 467–470.

E 326 [1938, 185]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Find all positive integer solutions, with $x<y$, of $x^2+y^2+xy=20461$.

*Solution by H. T. R. Aude, Colgate University.*

Write $x^2+y^2+xy=f(x, y)$. It is seen that $f(1, 2)=7$, $f(3, 4)=37$, and $f(3, 7)=79$, and that $20461=7\cdot 37\cdot 79$. It is readily verified that the product of $f(a, b)$ by $f(c, d)$ has the two forms, $f(bc-ad, ad+bd+ac)$ and $f(bd-ac, ac+bc+ad)$. It follows that the product, $f(1, 2)\cdot f(3, 4)$ is equal to both $f(2, 15)$ and $f(5, 13)$. Again the product formulas with the use of the identities, $f(-a, b)=f(a, b-a)$, and $f(a, b)=f(b, a)$, show readily that the products $f(3, 7)\cdot f(2, 15)$ and $f(3, 7)\cdot f(5, 13)$ yield respectively, $f(31, 125)$ and $f(65, 99)$, and $f(4, 141)$ and $f(76, 89)$. Consequently it is possible to write

$$20461 = f(1, 2)\cdot f(3, 4)\cdot f(3, 7) = f(31, 125) = f(65, 99) = f(4, 141) = f(76, 89).$$

Hence the required solutions $(x, y)$ are $(4, 141)$, $(31, 125)$, $(65, 99)$ and $(76, 89)$.

Also solved by W. E. Buker, Wm. Douglas, A. V. Richardson, E. P. Starke, C. W. Trigg and the proposer.

E 310 [1938, 47]. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form $aabb$ which is the square of $bb$. Show that the numbers, $b$ and $a^2+4(a-1)^2$, are each perfect squares. Determine the base of such a system and the values of $a$ and $b$, knowing that $a$ is also a perfect square.

*Partial Solution by Irene Price, State Teachers College, Oshkosh, Wisconsin.*

Let $B$ be the base of the number system for which $aabb=(bb)^2$. Then

$$aB^3 + aB^2 + bB + b = (Bb + b)^2.$$

Dividing by $(B+1)^2$, which is not zero, we obtain

$$b^2 = \frac{aB^2 + b}{B + 1} = a(B - 1) + \frac{a + b}{B + 1}\cdot \tag{1}$$

Since $b$ is an integer, $a+b$ must be a multiple of $B+1$. But $a$ and $b$ are each less than $B$, so $a+b=B+1$, and $b=B+1-a$. Substituting this value for $b$ in (1), we have

$$(B + 1 - a)^2 = a(B - 1) + 1. \tag{2}$$

The resulting quadratic equation in $B$ must have rational roots; so its discriminant, $5a^2-8a+4=a^2+4(a-1)^2$, must be a perfect square, $x^2$. Similarly, since $5a^2-8a+4-x^2=0$ has just one positive integer root $a$, which equals $[4+\sqrt{5x^2-4}]/5$, we see that $(5x^2-4)$ must be a perfect square $y^2$. Then $a=(4+y)/5$, and $y=5a-4$. Substituting the squares of the successive integers for $a$, we find that the first which satisfies all requirements is $a=16$. Then $y=76$, $x=34$, $B=40$, $b=25$, and the number $aabb=16\ 16\ 25\ 25=(25\ 25)^2$. Moreover, $b=5^2$ and $a^2+4(a-1)^2=34^2$ are perfect squares.

*Editorial Note.* No proof has been received that $b$ is always a square.

Also partially solved by G. W. Wishard and W. E. Buker.

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## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTIONS

3903. *Proposed by Simon Mowshowitz, New York, N. Y.*

Denote $[(n!)!]!$ by $n(!)^3$, etc., $n(!)^0=n$. Prove that for $k\geqq 2$

$$\frac{n(!)^k}{(n!)^{[n-1]![n!-1]![n(!)^2-1]!\cdots[n(!)^{k-2}-1]!}}$$

is an integer.

3904. *Proposed (July 1937) by the late R. P. Baker, University of Iowa.*

$ABC$ is a given triangle; find the condition that a point $P$ may be constructed in the plane of $ABC$ such that

$PA:PB:PC = p:q:r$, ($p$, $q$, $r$ real positive constants).

3905. *Proposed by V. Thébault, Le Mans, France.*

A hexagon $(H) \equiv A_1A_2A_3A_4A_5A_6$ with its opposite sides parallel is inscribed in a circle $(O)$, and the orthogonal projections of a point $M$ of its plane on its sides $A_1A_2$, $A_2A_3$ etc. are $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$ forming in this order a hexagon $(H')$. Show that (1) the sides of $(H')$ in sets of three are parallel to two determined directions; and that (2) the locus of the points $M$ such that the area of $(H')$ is constant is a circle concentric with $(O)$, and conversely.

3887 [1938, 482]. *Correction.* The equation in the last line should be $\rho_1+\rho_2 = r \sec^2\theta$.

## SOLUTIONS

3810 [1936, 643]. *Proposed by Oystein Ore, Yale University.*

One may define a new "multiplication" in the system of all positive real numbers by putting

$$[a, b] = a^b.$$

Determine all positive *rational* numbers for which this multiplication is:

I) *Commutative* $[a, b] = [b, a]$.
II) *Associative* $[a, [b, c]] = [[a, b], c]$.
III) *Right-hand* or *left-hand distributive*

$$[(a + b), c] = [a, c] + [b, c], \quad [c, (a + b)] = [c, a] + [c, b].$$

*Solution by E. P. Starke, Rutgers University.*

(I) If $b=a$, we have an obvious solution. Otherwise take $b>a$ and put $b=ra$ to get $a^{r-1}=r$ or $a=r^{1/(r-1)}$, $r>1$. Put also $1/(r-1)=u/v$, where $u$ and $v$ are integers with no common divisor $\neq 1$. Since $a$ and $r$ are rational, $r=(v+u)/u$ must be an exact $v$-power. Then $u$ and $v+u$, being relatively prime, are integral $v$-powers. But any two integral $v$-powers have a greater difference than $v$ unless $v=1$. Hence $1/(r-1)=u$ is an integer. Then $a=[(u+1)/u]^u$ and $b=[(u+1)/u]^{u+1}$.

(II) $a^{(b^c)}=(a^b)^c$ is evidently satisfied by $a=1$, $b$ and $c$ arbitrary, or by $c=1$, $a$ and $b$ arbitrary. If then $a\neq 1$, $c\neq 1$ we must have $b^c=bc$ or $b^{c-1}=c$. Thus by the above argument, if $u$ is any integer $c=(u+1)/u$, $b=[(u+1)/u]^u$, $a$ arbitrary.

(III) In $(a+b)^c=a^c+b^c$ without loss of generality we may take $a \geqq b$ and put $a=rb$. The given equation then reduces to $(1+r)^c=1+r^c$. For the moment consider $r$ as varying continuously. Then put $f_1(r)=(1+r)^c$ and $f_2(r)=1+r^c$. The derivatives are $f_1'(r)=c(1+r)^{c-1}$ and $f_2'(r)=cr^{c-1}$. Thus for all positive values of $r$, $f_1'(r)$ is greater than, equal to, or less than $f_2'(r)$ according as $c>1$, $c=1$ or $c<1$, respectively. But $f_1(0)=f_2(0)$. Hence $f_1(r)$ is greater than, equal to, or less than $f_2(r)$ according as $c>1$, $c=1$ or $c<1$. Thus the complete solution is given by $c=1$, $a$ and $b$ arbitrary.

In $c^{a+b}=c^a+c^b$ we may take $a \geqq b$. Let $a=br$ and put $c^b=v/u$, $r=p/q$, with

$p$, $q$, $u$, $v$ positive integers and each fraction in its lowest terms. Also $p \geqq q$. The given equation can then be written $v^{p-q}(v-u)^q = u^p$. If $v=1$, then $(1-u)^q = u^p$. Since $p \geqq 1$, $q \geqq 1$, every prime factor of $u$ is a divisior of $1-u$, hence of 1. Thus $u=1$, but $u=1$ does not satisfy $(1-u)^q = u^p$. Hence $v \neq 1$. Since $v$ is not a divisor of $u^p$, we must have $p-q=0$. But then $b=a$ and $v-u=u$, or $u=1$, $v=2$. Thus the complete solution is given by $a=b=1/n$, $c=2^n$, $n$ any integer.

We consider now the determination of sets of elements such that the given property of the multiplication in each case is true for any subset of the elements in a chosen order and where a given element may be repeated. As shown above for (I) the set has only one element, which is arbitrary; or the set has two distinct elements as indicated.

For (II) suppose first that the set has an element unity and an element $a \neq 1$. Then from $[[a, 1], a] = [a, [1, a]]$ we have $a^a = a$; and therefore $a=1$. Hence there is only one element, unity. Suppose no element is unity, then from $[a, [a, a]] = [[a, a], a]$ we have $a^a = a^2$ and $a=2$. Thus there is only one element in the set, either 1 or 2.

For III(a) $(a+a)^a = 2a^a$, or $2^a = 2$. Hence $a=1$; and 1 is the only element in the set. For III(b) $a^{(a+a)} = 2a^a$ or $a^a = 2$. Since this is impossible for a rational $a$, the set is empty.

Solved also by W. Penney and J. Rosenbaum.

*Editorial Note.* Penney stated that (I) was treated by H. L. Slobin in this MONTHLY, 1931, p. 444.

3811 [1937, 54]. *Proposed by N. A. Court, University of Oklahoma.*

A plane parallel to the base $ABC$ of the tetrahedron $DABC$ meets the edges $DA$, $DB$, $DC$ in the points $P$, $Q$, $R$; the same edges are met in the points $U$, $V$, $W$ by a plane antiparallel to the plane $ABC$. Show that, if the plane $PQR$ varies, the radical axis of the three spheres $AQRV$, $BRPW$, $CPQU$ remains fixed.

*Note.* For a definition of antiparallel see Court, *Modern Pure Solid Geometry*, p. 247. The Macmillan Co., 1935. The corresponding problem in the plane was considered in *Educational Times*, Reprints, vol. 60, 1894, p. 107, Q. 12001.

*Solution by the Proposer.*

The two lines $QR$, $VW$ are antiparallel with respect to the angle $BDC$; hence the four points $Q$, $R$, $V$, $W$ are concyclic. Similarly for the points $R$, $P$, $W$, $U$ and for $P$, $Q$, $U$, $V$.

The spheres $(B) = BRPWU$, $(C) = CPQUV$ have the points $P$, $U$ in common. On the other hand the four points $V$, $W$, $B$, $C$ being concyclic, the point $X = (BW, CV)$ has equal powers with respect to the spheres $(B)$, $(C)$, hence the plane $XUP = XDA$ is the radical plane of these two spheres.

Likewise, if $Y = (CU, AW)$, $Z = (AV, BU)$, the planes $YDB$, $ZDC$ are the radical planes of the pairs of spheres $(C)$, $(A) = AQRVW$; $(A)$, $(B)$. Now if $X' = (DX, BC)$, $Y' = (BY, CA)$, $Z' = (CZ, AB)$, the three lines $AX'$, $BY'$, $CZ'$ have a point $L$ in common, namely, the trilinear pole, with respect to the tri-

angle $ABC$, of the trace of the plane $UVW$ in the plane $ABC$. Hence the three planes $DAXX'$, $DBYY'$, $DCZZ'$ have the line $DL$ in common, this line is therefore the radical axis of the three spheres considered. Now the line $DL$ is determined by the tetrahedron $DABC$ and the plane $UVW$; hence the proposition.

3812 [1937, 55]. *Proposed by N. A. Court, University of Oklahoma.*

The vertex of a variable cone is fixed, the base being the circle of intersection of a fixed sphere with a variable plane passing through a fixed straight line. The cone cuts the sphere in a second circle: show that the plane of this second circle passes through a fixed line.

*Solution by the Proposer.*

Let $(S)$ be the given sphere, $V$ the fixed vertex, and $(p)$ the circle of intersection of $(S)$ with the plane $(P)$ passing through the given fixed line $a$. The line of intersection $m$ of the plane $(P)$ with the plane $(Q)$ of the second circle, which the cone $V-(p)$ and the sphere $(S)$ have in common, lies in the polar plane $\Sigma$ of $V$ with respect to $(S)$.

The traces $b$, $x$ of the fixed plane $V-a$ on the planes $\Sigma$, $(Q)$ pass through the fixed point $L=(am)=(a, \Sigma)$. The flat pencil of rays $a$, $b$, $x$, $LV$ in the fixed plane $V-a$ is harmonic, and the three lines $a$, $b$, $LV$ are fixed, hence the fourth ray, $x$, is also fixed, hence the proposition.

*Note.* An analogous problem in the plane was considered in the *Nouvelles Annales de Mathématiques*, 1882, p. 426, Q. 1381.

Solved also by R. Seamons and C. E. Springer.

3813 [1937, 55]. *Proposed by V. Thébault, Le Mans, France.*

Given a circle $(O)$ and two orthogonal axes $xx'$ and $yy'$; a variable point $P$ on $(O)$ is projected orthogonally into $M$ and $N$ on $xx'$ and $yy'$; then, on the perpendicular from $P$ to $MN$, two points $V$ and $W$ are taken so that the lengths of $VP$ and $PW$ are each in a given ratio $k$ with the distance $MN$. (1) On $VW$ as diameter a circle $(\Sigma)$ is described. Show that the radical axis of any two circles $(\Sigma)$ passes through a fixed point. (2) Determine the envelope of the circles $(\Sigma)$. (3) Examine the special case $k=1$.

*Solution by C. E. Springer, University of Oklahoma.*

Let $xx'$ and $yy'$ be taken as axes for rectangular coordinates, with $xx'$ passing through the center of the given circle $(O)$. Then the coordinates of $P$, a variable point on the fixed circle $(x-a)^2+y^2=r^2$ may be taken as $(a+r\cos\theta, r\sin\theta)$. The variable circle $(\Sigma)$ has the equation

$$[x-(a+r\cos\theta)]^2+[y-r\sin\theta]^2=k^2\sigma^2, \tag{1}$$

where $\sigma^2=\overline{MN}^2=a^2+r^2+2ar\cos\theta$. Writing this equation with $\theta=\theta_1$ and then with $\theta=\theta_2$ and subtracting, we find the equation of the radical axis of the two circles to be

$$[x - a(1 - k^2)] \sin (\theta_1 + \theta_2)/2 - y \cos (\theta_1 + \theta_2)/2 = 0,$$

which passes through the fixed point $[a(1-k^2), 0]$. Eliminating $\theta$ between (1) and its derivative with respect to $\theta$, we find the envelope of the circles $(\Sigma)$ to be an Oval of Descartes with the equation

$$(2) \qquad [x^2 + y^2 - 2ax + (1 - k^2)(a^2 - r^2)]^2 = 4k^2r^2(x^2 + y^2).$$

In case $k=1$, the radical axis of any two circles $(\Sigma)$ passes through the intersection of $xx'$ and $yy'$, and the equation of the envelope of the circles $(\Sigma)$ reduces to

$$[x^2 + y^2 - 2ax]^2 = 4r^2(x^2 + y^2).$$

On writing this in polar coördinates one obtains

$$(3) \qquad \rho = 2(a \cos \theta \pm r)$$

from which the envelope can be readily constructed.

Some interesting properties of the Ovals of Descartes may be found in "*Traitê des Courbes Spêciales Remarquables Planes et Gauches*" by F. Gomes Teixeira, Tome 1, pp. 218–233.

*Editorial Note.* The proposer stated the above results for the radical axis, and that the envelope of the circles $(\Sigma)$ in the general case is an oval of Descartes with $F$, the intersection of the axes $x$, $y$, as one focus: no proof of this last result was given. We shall add some remarks on the solution and consider the other foci. It is simpler to write (2) in the polar form

$$(2) \qquad \rho^2 - 2a\rho \cos \theta + (1 - k^2)(a^2 - r^2) - 2kr\rho = 0.$$

In obtaining this equation we get also another with the last term positive; but the two equations give the same curve and we discard the second. Hence in (3) we may omit the minus sign. Let $F$ and $F'$ be a pair of foci for an oval of Descartes; and set $FF'=d$, $FP=\rho$, $F'P=\rho'$. Then, if $\lambda$ and $\mu$ are constants, the curve is defined by the equation $\lambda\rho+\rho'=\mu$. Taking $F$ as the pole and $FF'$ as the axis, we obtain the equation of the curve in polar coordinates

$$(4) \qquad (\lambda^2 - 1)\rho^2 - 2\rho[\mu\lambda - d \cos \theta] + \mu^2 - d^2 = 0.$$

The identification of (2) and (4) and the solution of the resulting three equations gives two cases for $F'$

$$F_1': \quad d = (1 - k^2)a, \qquad \lambda = \pm k, \qquad \mu = \mp r(1 - k^2),$$
$$F_2': \quad d = (a^2 - r^2)/a, \qquad \lambda = \pm r/a, \qquad \mu = \mp kd.$$

For (3), where $k=1$, the foci $F$ and $F_1'$ coincide, but for $r \neq a$ we have the focus $F_2'$. If also $r=a$ we obtain the cardioid, for which all three foci coincide.

For the case $k=1$ it is quite easy by synthetic geometry to determine the form of the envelope.

3814 [1937, 55]. *Proposed by I. J. Schoenberg, Colby College.*

Let $f(t)$ be a one-valued and complex-valued function which satisfies the functional relation

$$e^{iks}f(t) = f(t+s) - f(s) \tag{1}$$

for all real values of $t$ and $s$, where $k$ is a real constant $\neq 0$. Prove that

$$f(t) = C(e^{ikt} - 1)/ik \qquad (C \text{ a constant}). \tag{2}$$

*Remark.* It should be noticed that for $k=0$ (1) and (2) reduce to (1′) $f(t+s)=f(t)+f(s)$ and (2′) $f(t)=Ct$ respectively. It is known that (1′) implies (2′) only if some additional assumption is made on $f(t)$, for instance boundedness in the neighborhood of some point (see G. Hamel, Mathematische Annalen, vol. 60, 1905, pp. 459–462). No such assumption is needed if $k\neq 0$.

*Solution by Gaines B. Lang, Georgia School of Technology*

Multiplying both sides of (1) by $e^{ikt}$, we obtain

$$e^{ik(t+s)}f(t) = e^{ikt}f(t+s) - e^{ikt}f(s). \tag{3}$$

Interchanging $s$ and $t$ in (1) we obtain

$$e^{ikt}f(s) = f(t+s) - f(t). \tag{4}$$

Combining (3) and (4) we have

$$e^{ik(t+s)}f(t) = e^{ikt}f(t+s) - f(t+s) + f(t) \tag{5}$$

or

$$[e^{ik(t+s)} - 1]f(t) = [e^{ikt} - 1]f(t+s).$$

Hence:

$$\frac{f(t)}{e^{ikt} - 1} = \frac{f(t+s)}{e^{ik(t+s)} - 1}.$$

Since the last equation is an identity, and since one member depends only on $t$ while the other depends only on $(t+s)$, it is clear that each member must be a constant. Hence (2) results.

Solved also by M. T. Bird, B. P. Hoover, O. J. Ramler, R. A. Rosenbaum, C. E. Springer and the proposer.

3815 [1937, 110]. *Proposed by P. Turán, Budapest, Hungary.*

Given the function of a real variable $x$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{4^n};$$

show that there is a positive constant $c_1$, independent of $x$, such that

$$|f(x)| < c_1 \log\log x, \qquad x > e. \tag{1}$$

Show also that there exists a sequence $x_1 < x_2 < \cdots \to \infty$, and a positive constant $c_2$, independent of $x$, such that

$$(2) \qquad |f(x_\nu)| > c_2 \log\log x_\nu, \qquad \nu = 1, 2, \cdots .$$

*Solution by the Proposer.*

Proof of (1). We have

$$f(x) = \sum_{n \leqq [\log x]} + \sum_{n > [\log x]} = S_1 + S_2.$$

$$(a) \qquad S_1 < \sum_{n \leqq [\log x]} \frac{1}{n} < 2 \log\log x.$$

$$(b) \qquad S_2 < \sum_{n > [\log x]} \frac{x}{n4^n} < x \sum_{n > [\log x]} \frac{1}{4^n} < c_3.$$

From (a) and (b) part (1) follows.

Proof of (2). Set $x_\nu = (4^\nu - 1)2\pi/3$, then $x_\nu/2\pi$ is an integer; and, if $\mu \leqq \nu$,

$$(c) \quad (4^\nu - 1)/3 = 1 + 4 + \cdots + 4^{\nu-1} \equiv 1 + 4 + \cdots + 4^{\mu-1} \equiv (4^\mu - 1)/3, \bmod 4^\mu.$$

$$(d) \qquad f(x_\nu) = \sum_{\mu=1}^{\infty} \frac{1}{\mu} \sin \frac{4^\nu - 1}{3} \frac{2\pi}{4^\mu} = \sum_{\mu=1}^{\nu} + \sum_{\mu=\nu+1}^{\infty} .$$

For the first sum in (d) we have by (c)

$$(e) \quad \left| \sum_{\mu=1}^{\nu} \frac{1}{\mu} \sin \frac{4^\mu - 1}{3} \frac{2\pi}{4^\mu} \right| = \left| \sum_{\mu=1}^{\nu} \frac{1}{\mu} \sin \left( \frac{2\pi}{3} - \frac{2\pi}{3 \cdot 4^\mu} \right) \right| > c_4 \log \nu;$$

and for the second sum we have

$$(f) \qquad \left| \sum_{\mu=\nu+1}^{\infty} \frac{1}{\mu} \sin \frac{4^\nu - 1}{3} \frac{2\pi}{4^\mu} \right| < \frac{4^\nu - 1}{3} 2\pi \cdot \frac{1}{4^{\nu+1}} \cdot \frac{4}{3} < \frac{2\pi}{9} \cdot$$

Since $\nu \sim \log x_\nu / \log 4$, if $\nu \to \infty$, we have at once from (e), (f), (d) the desired result for (2).

*Editorial Notes.* Since $f(x)$ is an odd function we may take $x > 0$ in the consideration of $|f(x)|$. For the proof of (a) we have the elementary inequalities

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

where $n \geqq 2$ for the left side inequality. The inequality in (a) is not true unless $n \geqq 3$ and $x \geqq e^3$. If these latter inequalities are true, then $1 \leqq \log n$ and (a) is then true; and we may take $c_1 = 2.1$.

In (f) the terms of the sum are all positive and hence $f(x_\nu) > (\sqrt{3}/2) \log \nu$, $x_\nu < 4^\nu(2\pi/3)$. It is convenient to consider first $\nu \geqq 2$, and then $\log x_\nu < \nu[\log 4 + 2^{-1} \log (2\pi/3)] < 1.76\nu$. We then have

$$\log \nu > \log \log x_\nu / [1 + \log 1.76/\log 2].$$

This gives

$$f(x_\nu) > 0.47 \log \log x_\nu,$$

which is also true for $\nu = 1$.

3816 [1937, 110]. *Proposed by E. Weiszfeld, Budapest, Hungary.*

Given the function of the complex variable $z$

$$f(z) = \sum_{i=1}^{n} \frac{a_i - z}{|a_i - z|},$$

where $a_i$, $(i = 1, 2, \cdots, n)$, denotes any given complex number; show that $|f(z)|$ takes on its maximum value in any domain not containing the points $a_i$ at the boundary of the domain.

*Solution by P. Erdös, Budapest, Hungary.*

The theorem is equivalent to the following vector-theorem.

Let $n$ points $A_1, A_2, \cdots, A_n$ be given in the plane, and $P$ an arbitrary point: we define the vector

$$\overrightarrow{f(P)} = \sum_{i=1}^{n} \frac{\overrightarrow{PA_i}}{\overline{PA_i}}$$

For any domain not containing the points $A_1, A_2, \cdots, A_n$, the function $\overrightarrow{f(P)}$ takes the maximum of its absolute value at the boundary of the domain.

Let us suppose that the theorem is not true. Then there is in the interior of the domain a point $O$ for which $|\overrightarrow{f(O)}|$ is maximum.

Let $R$ be the endpoint of $\overrightarrow{f(O)}$. Take on the straight line $OR$ a point $Q$ lying in the interior of the domain such that $O$ lies between $Q$ and $R$. We prove that in contradiction to our supposition

$$|\overrightarrow{f(Q)}| > |\overrightarrow{f(O)}|. \tag{1}$$

Take with $O$ as origin a rectangular coordinate-system such that the vector $\overrightarrow{f(O)}$ lies in the direction of the positive $x$ axis. The projections of any vector $v$ upon the $x$ and $y$ axis we denote by $v_x$ and $v_y$ respectively. We have evidently

$$\overrightarrow{f(O)} = \sum_{i=1}^{n} \frac{\overrightarrow{OA_i}}{\overline{OA_i}}, \qquad \overrightarrow{f_x(O)} = \sum_{i=1}^{n} \left(\frac{\overrightarrow{OA_i}}{\overline{OA_i}}\right)_x. \tag{2}$$

But

$$\left(\frac{\overrightarrow{QA_i}}{\overline{QA_i}}\right)_x > \left(\frac{\overrightarrow{OA_i}}{\overline{OA_i}}\right)_x, \qquad (i = 1, 2, \cdots, n), \tag{3}$$

since the length of each vector is 1 and the first vector forms with the $x$ axis a smaller angle than the second. Hence by addition

$$f_x(\overrightarrow{Q}) = \sum_{i=1}^{n} \left( \frac{\overrightarrow{QA_i}}{\overline{QA_i}} \right)_x > \sum_{i=1}^{n} \left( \frac{\overrightarrow{OA_i}}{\overline{OA_i}} \right)_x; \tag{4}$$

from which by (2)

$$| \overrightarrow{f_x(Q)} | > | \overrightarrow{f(O)} |. \tag{5}$$

But since the vector cannot be smaller than its projection we obtain

$$| \overrightarrow{f(Q)} | > | \overrightarrow{f(O)} |,$$

the required inequality.

*Remark.* By the same method, we may prove the following general theorem:

Let $\phi_1(x), \phi_2(x), \cdots, \phi_r(x)$ be positive, monotonically increasing continuous functions defined for any real positive numbers. Then the function

$$F(z) = \sum_{i=1}^{n} \frac{a_i - z}{| a_i - z |} \phi_i(| a_i - z |)$$

takes the maximum of its absolute value for any domain not containing the complex numbers $a_1, a_2, \cdots, a_n$ at the boundary of the domain.

*Editorial Note.* In the theorem of the problem, but not in the Remark, the case where all the $n$ points lie on a straight line should be excluded. For, if $A_1$ and $A_n$ are the extreme points, we may select a region cutting the segment $MN$ from the straight line so that all its points lie in the region on the same side of segment $A_1A_n$. Then for each point of $MN$ we have the maximum value $n$. If we make this exclusion, then in (3) there is at least one value of $i$ for which the given sign $>$ must be used; and in no case can $<$ be used as appears in the proof.

## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

Professor B. P. Reinsch of Florida Southern College has been elected president of the Florida Academy of Sciences.

Professor J. A. Shohat of the University of Pennsylvania during the spring of 1939 is giving a series of six lectures on interpolation, approximation, and mechanical quadrature at the Graduate School of the Department of Agriculture in Washington, D. C.

At Brooklyn College, Professor L. T. Moore and Thomas Nicholson have been granted sabbatical leaves of absence for the year 1938–39. A. W. Landers, Jr., has been granted a sabbatical leave for the spring term, 1939. The following appointments have been made in the department of mathematics for the year 1938–39; Dr. Aaron Fialkow, Jeanette Fox Keston, Dr. Moses Richardson, and W. J. Van Stockum.

G. E. Jones, formerly supervisor of schools of Bienville Parish, has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Dr. P. S. Wagner, professor of mathematics at Lebanon Valley College for the past twelve years, died December 5, 1938.

## THE COWLES COMMISSION FOR RESEARCH IN ECONOMICS

The Cowles Commission for Research in Economics will hold its Fifth Annual Research Conference on Economics and Statistics at Colorado College, Colorado Springs, U.S.A., from Monday, July 3 to Friday, July 28, 1939. The purpose of the Conference is to provide research workers an opportunity to present their problems and results before a group qualified to contribute constructive discussion. With the formal sessions limited to the morning hours, the atmosphere is considered more conductive to leisurely and thoughtful discussion than it is at the crowded programs of the regular sessions of learned societies.

The Cowles Commission Research Conferences originated in a series of informal sessions during the summer of 1935 after the meeting of the Econometric Society at Colorado Springs on June 22–24 of that year. At these gatherings various papers were presented and discussed by economists who remained in the vicinity. The meetings were so successful that it was decided to continue them in subsequent years. The scope of the Conferences has grown steadily. In 1935, the first year, there were 8 papers presented by 7 different lecturers, the total number attending being 25 of whom 5 were from out-of-town. Last summer 38 papers were presented by 27 different lecturers, 192 individuals, of whom 93 were from out-of-town, attending part or all of the sessions. Participants have come from all parts of the United States and from 16 foreign countries.

Among the speakers have been R. G. D. Allen, Louis H. Bean, Harold T. Davis, Edward L. Dodd, Griffith C. Evans, Mordecai Ezekiel, Arne Fisher, Irving Fisher, R. A. Fisher, Ragnar Frisch, Thornton C. Fry, Corrado Gini, Frank L. Griffin, Harold Hotelling, Edward V. Huntington, Emil Lederer, Wassily Leontief, A. P. Lerner, Alfred J. Lotka, Jakob Marschak, Karl Menger, James Harvey Rogers, Charles F. Roos, René Roy, Henry Schultz, Walter A. Shewhart, Carl Snyder, Gerhard Tintner, Abraham Wald, Ralph J. Watkins, S. S. Wilks, Elmer J. Working, Holbrook Working, and Theodore O. Yntema.

Each year the Cowles Commission publishes a report of the meetings, occupying about 100 pages, which includes abstracts of all papers presented. These reports are distributed to several thousand leading economists and statisticians throughout the world.

There is no charge for attendance at the Conference. All serious students are welcome. Room and board for those attending, and for their husbands or wives, will be available in dormitories of Colorado College at about $40 for the four weeks or $11 per week. Those interested should notify the Cowles Commission, 301 Mining Exchange Building, Colorado Springs, Colorado, U.S.A., in order that they may receive in the spring a preliminary announcement of the program.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.

## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, May 13.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 13.
MICHIGAN, Ann Arbor, March 18.
MINNESOTA.
MISSOURI, Springfield, April 28.
NEBRASKA, Lincoln, May 5.
OHIO, Columbus, April 8
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 2–3.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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## THE TWENTY-THIRD ANNUAL MEETING OF THE ASSOCIATION

The twenty-third annual meeting of the Mathematical Association of America was held at Richmond and Williamsburg, Virginia, Wednesday to Saturday, December 28–31, 1938, in conjunction with the meetings of the American Association for the Advancement of Science, the American Mathematical Society, and the National Council of Teachers of Mathematics. Three hundred eighty-nine were in attendance at the meetings, including the following one hundred eighty-nine members of the Association:

C. R. Adams, Brown University
V. W. Adkisson, University of Arkansas
R. P. Agnew, Cornell University
Harriet W. Allen, Hollins College
R. C. Archibald, Brown University
T. B. Ashcraft, Colby College
C. S. Atchison, Washington and Jefferson College
Robert Atkinson, The Shipley School

N. H. Ball, U. S. Naval Academy
Helen Barton, Woman's College, University of North Carolina
R. F. Bell, Montgomery, West Virginia, High School
A. A. Bennett, Brown University
G. D. Birkhoff, Harvard University
C. J. Blackall, College of St. Thomas
J. W. Blincoe, University of Tennessee
L. M. Blumenthal, University of Missouri
Paul Boeder, American Optical Company
H. E. Bray, Rice Institute
W. C. Brenke, University of Nebraska
Foster Brooks, Kent State University
Lillian O. Brown, Hood College
E. T. Browne, University of North Carolina
J. Hobart Bushey, Hunter College
Jewell Hughes Bushey, Hunter College
W. E. Byrne, Virginia Military Institute

W. D. Cairns, Oberlin College
Eleanor Calkins, College of William and Mary
Iris Callaway, University of Georgia
D. R. Carpenter, Roanoke College
I. S. Carroll, Syracuse University
W. B. Carver, Cornell University
J. W. Cell, North Carolina State College
H. C. Christofferson, Miami University
Abraham Cohen, Johns Hopkins University
L. W. Cohen, University of Kentucky
Nancy Cole, Sweet Briar College
J. B. Coleman, University of South Carolina
J. L. Coolidge, Harvard University
W. A. Cordrey, Lambuth College
Richard Courant, New York University
Elizabeth B. Cowley, Pittsburgh, Pennsylvania
H. B. Curry, Pennsylvania State College
D. R. Curtiss, Northwestern University
E. H. Cutler, Lehigh University

H. T. Davis, Northwestern University
F. F. Decker, Syracuse University
L. L. Dines, Carnegie Institute of Technology
Sister Mariola Dobbin, Rosary College
Arnold Dresden, Swarthmore College
F. G. Dressel, Duke University
W. L. Duren, Jr., Tulane University
O. L. Dustheimer, Baldwin-Wallace College
L. A. Dye, The Citadel, Charleston, S. C.

E. J. Finan, Catholic University
B. F. Finkel, Drury College
D. A. Flanders, New York University
L. R. Ford, Armour Institute of Technology
Tomlinson Fort, Lehigh University
J. S. Frame, Brown University
T. C. Fry, Bell Telephone Laboratories

R. E. Gaines, University of Richmond
J. J. Gergen, Duke University
F. J. Gerst, Loyola University
J. S. Gold, Bucknell University
W. C. Graustein, Harvard University
C. H. Graves, Pennsylvania State College
C. D. Gregory, College of William and Mary

D. W. Hall, University of Pennsylvania
Harris Hancock, University of Cincinnati
Isabel Harris, Westhampton College
M. L. Hartung, University of Chicago
Nola L. A. Haynes, Columbia, Missouri
Olive C. Hazlett, University of Illinois

E. R. Hedrick, University of California at Los Angeles
E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N. J.
Einar Hille, Yale University
J. H. Hlavaty, Bronx High School of Science
T. R. Hollcroft, Wells College
Charles Hopkins, Tulane University
W. A. Hurwitz, Cornell University
R. O. Hutchinson, Tennessee Polytechnic Institute

J. E. Ikenberry, Franklin and Marshall College

Dunham Jackson, University of Minnesota
C. G. Jaeger, Pomona College
Fritz John, University of Kentucky
F. E. Johnston, George Washington University
B. W. Jones, Cornell University
Olive M. Jones, Queens-Chicora College

A. J. Kempner, University of Colorado
J. R. Kline, University of Pennsylvania
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P. V. Kunkel, Cedar Crest College

W. D. Lambert, U. S. Coast and Geodetic Survey
Lincoln LaPaz, Ohio State University
Gillie A. Larew, Randolph-Macon Woman's College
C. G. Latimer, University of Kentucky
Nathan Lazar, Alexander Hamilton High School, Brooklyn, N. Y.
J. B. Linker, University of North Carolina
S. B. Littauer, U. S. Naval Academy
Mayme I. Logsdon, University of Chicago
W. F. Long, Franklin and Marshall College
R. G. Lubben, University of Texas

C. C. MacDuffee, University of Wisconsin
H. M. MacNeille, Kenyon College
H. F. Mac Neish, Brooklyn College
William Marshall, Purdue University
Carol V. McCamman, McKinley High School, Washington, D. C.
N. H. McCoy, Smith College
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E. J. McShane, University of Virginia
A. K. Mitchell, Trinity College
Virginia Modesitt, Randolph-Macon Woman's College
E. C. Molina, Bell Telephone Laboratories
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CORA STRONG, Woman's College, University of North Carolina
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OTTO SZÁSZ, University of Cincinnati

CARRIE B. TALIAFERRO, State Teachers College, Farmville, Va.
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J. H. TAYLOR, George Washington University
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C. W. WILLIAMS, Washington and Lee University
K. P. WILLIAMS, Indiana University
H. A. WOOD, Connecticut State College
F. L. WREN, George Peabody College

B. F. YANNEY, College of Wooster
MABEL M. YOUNG, Wellesley College

The meetings of the American Association for the Advancement of Science at Richmond were of especial interest to mathematicians because of the fact that Professor G. D. Birkhoff had been serving as president of the American Association the past year and gave his retiring presidential address on Tuesday evening on the subject "Intuition, reason and faith in science." This lecture, which was given in The Mosque, and the reception following it at the Hotel Jefferson were attended by a large number of mathematicians and other scientists. After the two joint sessions Wednesday morning, the mathematicians transferred their activities to Williamsburg for the remainder of their programs. The Association and the Society are indebted to Professor C. H. Wheeler III and his associates for their efficiency in caring for the housing and sight-seeing needs of those who attended the Richmond meetings and for their comfortable transportation to Williamsburg.

At Williamsburg the group was comfortably housed in the dormitories of the College of William and Mary, meals being served in the college dining hall. Pleasant features of the week were the daily afternoon teas served in Barrett Hall by the ladies of the College of William and Mary and other Virginia colleges. Many availed themselves of the delightful opportunity of viewing the Restoration Buildings at Williamsburg and of taking trips by bus to Jamestown and other nearby places of historic interest. This was enhanced by a preparatory lecture on Wednesday evening by Mr. Geddy of the Restoration, followed by a series of beautiful colored slides with accompanying comment by Mrs. J. M.

Stetson. A resolution expressing on behalf of the Association, the Society, and the National Council, our thanks to Professor and Mrs. Stetson and their colleagues for their fine efforts in providing convenient and comfortable arrangements for the mathematicians and their guests was offered by Professor L. R. Ford and adopted unanimously at the annual dinner on Thursday evening.

Three hundred seven attended this dinner, which was held in the larger college dining room. Professor F. D. Murnaghan acted as toastmaster. President J. S. Bryan welcomed the group and recounted the great ebb and flow in the history of the College of William and Mary. Professor G. T. Whyburn spoke of the commendable trait of perseverance in research and paid tribute to the inspiration given by the one or two generations that have preceded the present young mathematicians. Professor J. L. Coolidge gave an interesting account of the mathematical history of the College and of its offspring, the University of Virginia. There were also brief talks by Professor R. L. Moore representing the Society and Professor W. B. Carver representing the Association.

The American Mathematical Society held sessions for the reading of short papers on Wednesday afternoon and Thursday and Friday mornings. On Thursday afternoon Professor R. L. Moore delivered his retiring presidential address on the subject "On certain abstract spaces." This was followed by an address by Professor H. A. Rademacher on "Fourier expansions of modular functions and theorems on partitions." At this same session the Society announced the award of the Bôcher Prize to Professor John von Neumann for his paper "Almost periodic functions and groups" published in the *Transactions of the American Mathematical Society*, vol. 36; following this, Professor von Neumann gave a brief resumé of his paper.

Aside from the joint session Friday afternoon, the National Council of Teachers of Mathematics held two sessions Friday morning in Williamsburg. At a section on arithmetic there were papers by T. G. Foran of Catholic University on "An experimental study of the relation of homework to achievement in arithmetic"; by H. E. Benz of Ohio University on "Some preliminary considerations relating to arithmetic in the high school"; and by B. R. Buckingham of Ginn and Company on "The contribution of arithmetic to liberal education." At a section on secondary mathematics there were papers by Herbert ReBarker of East Carolina Teachers College on "Meaningful mathematics"; by F. G. Lankford, Jr. of the University of Virginia on "An analytical study of high school plane geometry"; by M. L. Hartung of the University of Chicago on "The evaluation of achievement in high school mathematics." These were followed by a brief talk by Professor K. P. Williams of Indiana University "On the Report of the Joint Commission."

The Mathematical Association participated in two joint sessions at Richmond and one at Williamsburg, and held a separate session on Saturday morning preceded by the annual business meeting. The programs follow, together with abstracts of some of the papers:

### Joint Session of the Association with Sections A and E of the American Association and the American Mathematical Society

At this session Professor W. D. Cairns gave his retiring vice-presidential address as Chairman of Section A on "Seismology from a mathematical viewpoint." This paper in condensed form appeared in *Science* for February 10, 1939.

### Joint Session of the Association with Sections A and L of the American Association and the American Mathematical Society

At this session Professor R. C. Archibald gave his retiring vice-presidential address as Chairman of Section L on "Mathematicians, and poetry and drama." This address appeared in *Science* for January 13 and 20, 1939.

### Joint Session of the Association with the National Council of Teachers of Mathematics

1. "A college mathematics teacher views teacher training" by Professor A. A. Bennett, Brown University.
2. "The professional preparation of mathematics teachers" by Professor F. L. Wren, George Peabody College for Teachers.
3. "Mathematics in the training of arithmetic teachers" by Professor R. L. Morton, Ohio University.

The first of these three papers will appear in an early issue of this Monthly, the others in early issues of *The Mathematics Teacher.*

### Separate Session of the Association

1. Annual business meeting and election of officers.
2. "Limits to the characteristic roots of a matrix" by Professor E. T. Browne, University of North Carolina.
3. "Finite deformations of an elastic solid" by Professor F. D. Murnaghan, Johns Hopkins University.
4. "Differential geometry in the large" by Professor S. B. Myers, University of Michigan.

2. Professor Browne's paper will appear in an early issue of the Monthly.

3. The talk by Professor Murnaghan showed how the relations connecting the stress tensor in the theory of elasticity with the gradient of the energy of deformation relative to the strain tensor may be derived without appealing to the simplifying assumptions (such as the infinitesimal nature of the strain) of the established classical theory. In the case of hydrostatic pressure the predictions of Professor Murnaghan's theory have been confirmed in a remarkable manner by experiment; a single constant formula gives, to within experimental error, the relation between pressure and change of volume up to pressures of 45,000 atmospheres (corresponding to a decrease in volume of the amount 2:1). Reference was made to a paper (giving this experimental verification) by Birch, *Journal of Applied Physics*, April, 1938.

4. Professor Myers gave in a non-technical fashion a summary of some recent developments in Riemannian geometry in the large. Among the topics discussed were the papers of H. Hopf on spaces of constant curvature and on the curvature integral in $n$-dimensions, those of Hopf, Rinow, and Myers on analytic Riemannian manifolds, those of Whitney on the imbedding of differentiable manifolds in euclidean space, those of Myers and Whitehead on the "absolute minimum locus," and the joint work of Myers and Steenrod on isometries of Riemannian manifolds.

## Meetings of the Board of Trustees

Eight members of the outgoing and the incoming Board were present at the Williamsburg meetings.

The following forty persons were elected to membership on applications duly certified:

G. E. Albert, Ph.D.(Wisconsin) Instr., Ohio State Univ., Columbus, Ohio

Emil Artin, Ph.D.(Leipzig) Prof., Indiana Univ., Bloomington, Ind.

H. P. Atkins, Jr., M.S.(Brown) Part-time Instr., Brown Univ., Providence, R. I.

Grace E. Bates, M.S.(Brown) Teacher, George School, George School, Pa.

J. W. Beach, M.S.(Iowa State Coll.) Head of dept., Univ. of Dubuque, Dubuque, Iowa

C. A. Bridger, M.S.(Oregon State Coll.) Vital Statistician, Div. of Public Health, Dept. of Public Welfare, Boise, Idaho.

Rev. John Capesius, A.M.(Alabama) Prof., Math. and Physics, St. Bernard Coll., St. Bernard, Ala.

Janet C. Durand, A.M.(Pennsylvania) Prof., Beaver Coll., Jenkintown, Pa.

G. M. Ewing, Ph.D.(Missouri) Instr., Univ. of Missouri, Columbia, Mo.

J. A. Garrett, A.M.(Peabody) Prof., Arkansas A. and M. Coll., Monticello, Ark.

E. F. Gillette, A.B.(Hamilton) Instr., Middlebury Coll., Middlebury, Vt.

Lilah G. Godfrey, M.S.(Washington) Asst. Prof., Walla Walla Coll., College Place, Wash.

Etta Greenberg, A.M.(So. Calif.) Teacher, Washington Irving High School, New York, N. Y.

O. H. Hamilton, Ph.D.(Texas) Asst. Prof., Oklahoma A. and M. Coll., Stillwater, Okla.

H. H. Harman, M.S.(Chicago) Research asst., Univ. of Chicago, Chicago, Ill.

J. H. Hlavaty, B.S.(C.C.N.Y.) Acting chm. of dept., High School of Science, New York, N. Y.

W. E. Johnson, B.S.(Howard Univ.) Registrar and Instr., Leland Coll., Baker, La.

E. S. Kennedy, A.M.(Lehigh) Grad. student, Lehigh Univ., Bethlehem, Pa.

Marie Litzinger, Ph.D.(Chicago) Asso. Prof., Chm. of dept., Mount Holyoke Coll., South Hadley, Mass.

D. M. MacEwen, Ph.D.(New York Univ.) Teacher, The Prep. School, C.C.N.Y., New York, N. Y.

M. S. Macphail, Ph.D.(Oxford) Instr., Acadia Univ., Wolfville, Nova Scotia

Sister Mary Gertrude, A.B.(Marquette) Grad. student, Marquette Univ., Milwaukee, Wis.

Sister Mary of Mercy, A.M.(Catholic Univ.) Instr., Incarnate Word Coll., San Antonio, Texas

K. F. McLaughlin, A.M.(Yale) Grad. asst., Univ. of Oklahoma, Norman, Okla.

Deane Montgomery, Ph.D.(Iowa) Asso. Prof., Smith Coll., Northampton, Mass.

Frances A. Mullen(Mrs. U. J.), A.M.(Chicago) Teacher, Fenger High School, Chicago, Ill.

Jack Neely, A.B.(New River State Coll.) Teacher, Kingston High School, Kingston, W. Va.

J. T. O'Callahan, A.M.(Boston Coll.) Chm. of dept., Holy Cross Coll., Worcester, Mass.

B. J. PETTIS, Ph.D.(Virginia) Sterling fellow, Yale Univ., New Haven, Conn.

Sister MARY H. REILLY, Ph.D.(Catholic Univ.) Head of dept., Teachers Coll. Athenaeum of Ohio, Cincinnati, Ohio

G. DEB. ROBINSON, Ph.D.(Cambridge) Asst. Prof., Univ. of Toronto, Toronto, Ont., Canada

Sister M. DEPAZZI ROCHFORD, M.S.(Iowa) Head of dept., Briar Cliff Coll., Sioux City, Iowa

JACOB RUSH, A.B.(Columbia) Teacher, Brooklyn Tech. High School, Brooklyn, N. Y.

I. L. STRIGHT, A.M.(Allegheny Coll.) Teacher, High School, Freedom, Pa.

ALVIN SUGAR, Ph.D.(California) Prof., Head of dept., Math. and Physics, St. Francis Coll., Loretto, Pa.

R. K. WAKERLING, A.B.(California) Teaching asst., Univ. of California, Berkeley, Calif.

M. S. WEBSTER, Ph.D.(Pennsylvania) Instr., Purdue Univ., Lafayette, Ind.

N. W. WELLS, A.M.(Rice Inst.) Instr., Springfield Jr. Coll., Springfield, Ill.

JOSEPHINE J. WILLIAMS, A.M.(Radcliffe) Teacher, The Baldwin School, Bryn Mawr, Pa.

H. A. WOOD, B.S.(Mass. Inst. of Tech.) Instr., Connecticut State Coll., Storrs, Conn.

The financial report of the Secretary-Treasurer for the year 1938 was presented and accepted.

The Trustees voted to approve the recommendation of the committee on the Chauvenet Prize; the award was announced at the annual business meeting on Saturday.

A letter from Professor G. C. Evans was read, indicating that there is danger that algebra will be taken out of the ninth grade, not merely on the West Coast but over the country, and that this will come to a head at the meeting of the N. E. A. next summer. It was voted to refer this letter to the Joint Commission on the Place of Mathematics for a report and for any action that they might deem advisable in collecting authoritative opinion and forceful arguments, and in establishing contact with members of the National Education Association.

A report was received from the Association's committee on the 1940 Congress according to which it is expected that there will be two symposia in Section VI, the didactic part of the Congress, with suitable speakers and topics, the speakers to be announced by the officers of the Congress.

The following were appointed associate editors of the MONTHLY for the year 1939, as nominated by Professor Moulton:

| | | |
|---|---|---|
| W. F. Cheney, Jr. | E. H. C. Hildebrandt | R. G. Sanger |
| Otto Dunkel | C. A. Hutchinson | D. E. Smith |
| B. F. Finkel | J. R. Musselman | Virgil Snyder |
| T. C. Fry | H. L. Olson | R. J. Walker |

## Annual Business Meeting

The annual business meeting and election of officers was held Saturday morning, December 31, 1938. The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He also reported the deaths of the following members:

E. E. Allen, Professor of mathematics, Occidental College. (May 6, 1938)
R. B. Allen, Professor of mathematics, Kenyon College. (March 4, 1938)
E. W. Brown, Professor emeritus of mathematics, Yale University. (July 22, 1938)
T. W. Edmondson, Professor emeritus of mathematics, New York University. (November 4, 1938)
Carl Gundersen, Professor emeritus of mathematics, Oklahoma A. and M. College. (April 11, 1938)
J. J. Luck, Professor of mathematics, University of Virginia. (September 15, 1938)
J. B. Meyer, Professor of mathematics, Valley City, North Dakota, State Teachers College. (February 25, 1938)
James Pierpont, Professor emeritus of mathematics, Yale University. (December 9, 1938)
Mother Antoinette Spies, Professor of mathematics, Convent of the Sacred Heart, London, England. (November 7, 1938)
H. W. Stager, Retired, Head of mathematics department, Salinas, California, Junior College. (December 20, 1937)
C. M. Titus, Assistant Professor of mathematics, University of California, Davis Branch of the College of Agriculture. (May 6, 1937)
H. W. Tyler, Professor emeritus of mathematics, Massachusetts Institute of Technology; Consultant in science, Library of Congress. (February 2, 1938)
P. S. Wagner, Professor of mathematics, Lebanon Valley College. (December 5, 1938)
W. E. Wilbur, Associate Professor of mathematics, University of New Hampshire. (September 3, 1938)

The amendment concerning life membership rates, which was published in the October issue of the Monthly, was adopted; these new rates are given on page 134 of this issue of the Monthly.

President Kempner announced the fifth award of the Chauvenet Prize of $100 to Professor G. T. Whyburn for his article "On the structure of continua" published in the *Bulletin of the American Mathematical Society*, volume 42 (1936) pp. 49–73. This award covers the years 1935–37 and is given "for a noteworthy expository paper published in English by a member of the Association."

The result of the election of officers was as follows:

President for 1939–40: W. B. Carver, Cornell University.

Vice-Presidents for 1939: W. L. Hart, University of Minnesota; F. D. Murnaghan, Johns Hopkins University.

Additional members of the Board of Trustees, to serve until January 1942: A. A. Bennett, Brown University; R. D. Carmichael, University of Illinois; G. C. Evans, University of California; A. J. Kempner, University of Colorado.

## REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 9, 1938

### RECEIPTS

| | | |
|---|---|---|
| Balance Dec. 10, 1937 | | $6,171.33 |
| 1937 indiv. dues | $ 447.80 | |
| 1937 inst. dues | 45.50 | |
| 1937 subscriptions | 28.40 | |
| 1938 indiv. dues | 7,237.91 | |
| 1938 inst. dues | 630.00 | |
| 1938 subscriptions | 978.09 | |
| Initiation fees | 284.00 | |
| Advertising | 575.53 | |
| Authors' reprints | 301.39 | |
| Sale copies of MONTHLY | 161.31 | |
| Sale First Carus Mon. | 22.50 | |
| Sale Second Carus Mon. | 12.50 | |
| Sale Third Carus Mon. | 18.75 | |
| Sale Fourth Carus Mon. | 8.75 | |
| Sale Fifth Carus Mon. | 11.25 | |
| Sale Archibald's Outline Hist. of Math. | 39.78 | |
| *Annals* subscriptions | 7.50 | |
| *Duke Journal* subscriptions | 9.00 | |
| Sale Rhind Papyrus | 145.00 | |
| Drury Coll. int. Hardy Fund | 120.00 | |
| Int. Peoples Bkg. Co. | 22.01 | |
| Int. Cleveland Trust Co. | 53.35 | |
| Int. Genl. End. Fund | 718.70 | |
| Int. Carus Fund | 131.25 | |
| Int. Chace Fund | 238.99 | |
| Int. Chauvenet Fund | 15.00 | |
| Payment from restricted Carus Fund | 49.70 | |
| Payment from restricted Chace Fund | 2.20 | 12,316.16 |
| Total 1938 receipts to date | | $18,487.49 |
| Total expenditures | | 13,054.86 |
| Balance to end of 1938 business | | $ 5,432.63 |
| Received on 1939 business | | 860.41 |
| Book balance Dec. 9, 1938 | | $ 6,293.04 |

### EXPENDITURES

| | | |
|---|---|---|
| Publisher's bills (Nov.'37–Sept. '38) | | $5,253.85 |
| Reprints | | 297.99 |
| *Register* | | 477.06 |
| President's office | | 10.00 |
| Editor-in-Chief's office | | 670.42 |
| Committee on Membership | | 171.33 |
| Committee on Tests | | 8.13 |
| Secretary-Treasurer's office | | |
| Postage | $ 416.77 | |
| Bond | 11.26 | |
| Safety deposit | 4.40 | |
| Report on examination of accounts | 150.00 | |
| Office supplies | 126.13 | |
| Express, tel., etc. | 76.42 | |
| Clerical work | 2,561.75 | |
| Printing | 207.12 | |
| Bank tax | 38.29 | |
| Typewriter | 87.56 | 3,679.70 |
| *Annals* subvention | | 250.00 |
| *Duke Journal* subvention | | 200.00 |
| Expense of sections from init. fees | | 382.67 |
| Indianapolis meeting | | 120.52 |
| New York meeting | | 120.55 |
| Paid *Annals* subscriptions | | 17.50 |
| Forwarded *Annals* subscriptions | | 7.50 |
| Paid *Duke Journal* subscriptions | | 2.00 |
| Forwarded *Duke Journal* subscriptions | | 7.00 |
| Sust. memb. in Amer. Math. Soc. | | 100.00 |
| Refund subscriptions | | 6.80 |
| Storage back copies MONTHLY | | 30.00 |
| Insurance back copies MONTHLY | | 6.40 |
| Paid back copies MONTHLY | | 303.60 |
| Paid B. F. Finkel int. Hardy Fund | | 120.00 |
| Library expense chiefly binding | | 130.23 |
| Excess cost Youngstown S. & T. bonds | | 8.82 |
| Transfer to Carus Mon. Fund | | 275.64 |
| Transfer to Chace Fund | | 363.15 |
| Expense acct. 1940 Congress | | 34.00 |
| Total expenditures | | $13,054.86 |
| Checking account | | $ 384.29 |
| Oberlin Savgs. Bk. acct. restricted | | 833.00 |
| Peoples Banking Co. savgs. acct. | | 465.11 |
| Cleveland Trust Co. savgs. acct. | | 2,610.64 |
| Youngstown S. & T. bonds | | 2,000.00 |
| Bank balance Dec. 9, 1938 | | $ 6,293.04 |

## Exhibit of the Funds of the Association

### Carus Monograph Fund

| | | |
|---|---|---|
| Balance December 10, 1937 | | $6,685.96 |
| Receipts: Sales | $ 82.16 | |
| Interest | 177.91 | 260.07 |
| | | $6,946 03 |
| Certificate of deposit | $2,091.20 | |
| C. & O. $3\frac{1}{2}\%$ Refunding Mortgage Bonds Series D, 1996 | 2,000.00 | |
| $3\frac{1}{8}\%$ U. S. Treasury Bond of 1946–49 | 1,000.00 | |
| 3% HOLC Bond 1944–52 | 1,000.00 | |
| U. S. Savings Bonds | 150.00 | |
| Cash in bank, restricted, certificate of participation | 497.00 | |
| Cash in bank, unrestricted | 207.83 | |
| Balance December 9, 1938 | | $6,946.03 |

### Arnold Buffum Chace Fund

| | | |
|---|---|---|
| Balance December 10, 1937 | | $7,318.40 |
| Receipts: Sale Papyrus | $ 220.00 | |
| Interest | 245.96 | 465.96 |
| | | $7,784.36 |
| $3\frac{1}{8}\%$ U. S. Treasury Bonds 1946–49 | $2,000.00 | |
| 3% HOLC Bond 1944–52 | 1,300.00 | |
| U. S. Savings Bonds | 1,125.00 | |
| Western United Gas & Electric Co. $5\frac{1}{2}\%$ Bonds Series A, 1955 | 2,500.00 | |
| Certificate of deposit | 577.41 | |
| Cash in bank, restricted, certificate of participation | 22.00 | |
| Cash in bank, unrestricted | 259.95 | |
| Balance December 9, 1938 | | $7,784.36 |

### Chauvenet Prize Fund

| | | |
|---|---|---|
| Balance December 10, 1937 | | $ 672.94 |
| Interest | | 15.00 |
| | | $ 637.94 |
| 3% HOLC Bond 1944–52 | $ 500.00 | |
| Cash in bank, unrestricted | 187.94 | |
| Balance December 9, 1938 | | $ 687.94 |

### Life Membership Fund

| | |
|---|---|
| Liability on life memberships as of January 1, 1938 | $ 804.83 |
| To be transferred to current funds, surplus | 22.06 |
| Liability on life memberships as of January 1, 1939 | $ 782.77 |

GENERAL ENDOWMENT FUND

| | | |
|---|---|---|
| Balance December 10, 1937 | | $18,200.00 |
| $3\frac{1}{4}\%$ U. S. Treasury Bonds 1944–46 | $1,000.00 | |
| $3\frac{1}{4}\%$ U. S. Treasury Bonds 1943–45 | 1,000.00 | |
| 3% HOLC Bonds 1944–52 | 5,500.00 | |
| Land Trust Certificate, Hotel Cleveland Building Site | 700.00 | |
| $3\frac{3}{4}\%$ Montana Power Co. First Mortgage Bonds, 1966 | 2,000.00 | |
| 5% Texas Power and Light Co., First Mortgage Gold Bond, 1956 | 1,000.00 | |
| $3\frac{1}{2}\%$ C. & O. Refunding Mortgage Bond Series C, 1996 | 1,000.00 | |
| $3\frac{3}{4}\%$ Pennsylvania R. R. Co. Bonds Series C, 1970 | 2,000.00 | |
| $3\frac{3}{4}\%$ Bethlehem Steel Co. Consol. Mortgage Bonds, 1966 | 2,000.00 | |
| Oberlin Savings Bank savings account | 2,000.00 | |
| Balance December 9, 1938 | | $18,200.00 |

Of the funds on hand, indicated in the first division of this financial report, $207.83 belongs to the Carus Monograph Fund, $259.95 to the Arnold Buffum Chace Fund, $187.94 to the Chauvenet Prize Fund, while $782.77 is held as Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1939.

When the accounts were closed December 9, 1938, there remained on the total business for 1938 the following items:

| BILLS RECEIVABLE | | BILLS PAYABLE | |
|---|---|---|---|
| 1938 individual dues | $200.00 | Publisher's bills (Oct.–Dec. '38) | $1,850.00 |
| Advertising | 100.00 | Editor-in-Chief's office | 150.00 |
| | | Secretary-Treasurer's office | 400.00 |
| | $300.00 | Subsidy *Duke Journal* | 50.00 |
| | | Carus Monograph Fund | 207.83 |
| | | Chace Fund | 259.95 |
| | | Chauvenet Prize Fund | 187.94 |
| | | Life Membership Fund | 782.77 |
| | | Init. fees due to sections | 880.00 |
| | | | $4,768.49 |

If to the balance on 1938 business shown in this report, $5,432.63, there be added the estimated bills receivable, $300.00, and there be subtracted the estimated bills payable, $4,768.49, there results an estimated final balance on 1938 business of approximately $964.00, somewhat less than the corresponding amount of one year ago. This is a good showing when one considers the necessary increases in the items of printing, the office of the editor-in-chief, and the clerical work in the office of the secretary-treasurer.

W. D. CAIRNS, *Secretary-Treasurer*

## LIFE MEMBERSHIP IN THE MATHEMATICAL ASSOCIATION

In accordance with the action of the Association at its Williamsburg meeting, members may obtain life membership in the Association by the payment, at the first of any calendar year, of an amount indicated in the accompanying table. In estimating one's age the birthday anniversary nearest to the first of January when payment is made should be taken.

| Age | Fee | Age | Fee | Age | Fee |
|---|---|---|---|---|---|
| 20 | $89.49 | 40 | $71.72 | 60 | $46.74 |
| 21 | 88.71 | 41 | 70.62 | 61 | 45.40 |
| 22 | 87.93 | 42 | 69.50 | 62 | 44.06 |
| 23 | 87.16 | 43 | 68.36 | 63 | 42.73 |
| 24 | 86.40 | 44 | 67.21 | 64 | 41.40 |
| 25 | 85.64 | 45 | 66.03 | 65 | 40.08 |
| 26 | 84.86 | 46 | 64.83 | 66 | 38.76 |
| 27 | 84.06 | 47 | 63.62 | 67 | 37.46 |
| 28 | 83.24 | 48 | 62.39 | 68 | 36.17 |
| 29 | 82.40 | 49 | 61.15 | 69 | 34.89 |
| 30 | 81.54 | 50 | 59.89 | 70 | 33.63 |
| 31 | 80.65 | 51 | 58.62 | 71 | 32.38 |
| 32 | 79.75 | 52 | 57.33 | 72 | 31.16 |
| 33 | 78.82 | 53 | 56.03 | 73 | 29.95 |
| 34 | 77.87 | 54 | 54.72 | 74 | 28.76 |
| 35 | 76.90 | 55 | 53.41 | 75 | 27.60 |
| 36 | 75.91 | 56 | 52.08 | 76 | 26.47 |
| 37 | 74.89 | 57 | 50.75 | 77 | 25.36 |
| 38 | 73.86 | 58 | 49.42 | 78 | 24.28 |
| 39 | 72.80 | 59 | 48.08 | 79 | 23.22 |

W. D. CAIRNS, *Secretary-Treasurer*

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## THE THIRTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The thirteenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Ursinus College, Collegeville, Pennsylvania, on Saturday, November 26, 1938, Professor L. L. Smail presiding.

The attendance was thirty, including the following seventeen members of the Association: J. A. Benner, P. A. Caris, J. W. Clawson, J. E. Davis, F. L. Dennis, D. W. Hall, V. V. Latshaw, F. L. Manning, C. A. Nelson, L. L. Smail, W. M. Smith, E. P. Starke, A. W. Tucker, R. M. Walter, Anna Pell Wheeler, A. H. Wilson, R. C. Yates.

At the business meeting the following officers were elected for next year: Chairman, J. B. Reynolds, Lehigh University; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Tomlinson Fort, H. W. Brinkmann,

J. R. Kline. It was agreed to hold the next meeting at Lehigh University, Bethlehem, Pa., on Saturday, December 2, 1939.

The following papers were presented:

1. "Functions and sequences" by Professor Anna Pell Wheeler, Bryn Mawr College.

2. "Undergraduate courses in topology and other phases of geometry" by Professor A. W. Tucker, Princeton University.

3. "Meeting the challenge to secondary mathematics" by W. D. Carpenter, Germantown Academy, introduced by Professor Hedlund.

4. "Linkages" by Professor R. C. Yates, University of Maryland.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Wheeler presented a study of the correspondence between certain classes of functions analytic in the interior of the unit circle, and of sequences of complex numbers, with special emphasis on such properties as linearity of the classes.

2. Professor Tucker gave a description of the content of two one-term (40-hour) courses in geometry taught by him at Princeton University: *Introduction to Modern Geometry*, for sophomores; *Elementary Topology*, for juniors and seniors. The first of these deals with geometries of Klein, *i.e.*, geometries characterized by their groups of transformations. The intuitive material of elementary synthetic geometry is used to construct "working models" of Euclidean, conformal, projective, affine, elliptic, and hyperbolic geometries. From these "working models" the principal theorems and invariants can be read off very easily. The second course deals with topological properties in two and three dimensions which may readily be visualized by students having reasonable spatial perceptions: the classification of 2-dimensional manifolds, examples of 3-dimensional manifolds, the concepts of topological space and continuous mapping, fixed points and critical points, graphs and complexes.

3. In speaking of the problems of the teachers of secondary mathematics, Mr. Carpenter, while approving of the general aims of the new type of college board examinations, felt that the inclusion of elementary calculus and analytic geometry would cause a lack of thoroughness in the other subjects taught. The criticism of secondary mathematics must be met, he said, by the colleges using improved methods in the training of future mathematics teachers; by the consolidation of small high schools and the employment of capable guidance directors; by the writing of new text-books along the lines laid down in the preliminary report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics; and by a general offensive program in which all mathematics teachers take part.

4. Professor Yates presented an account of the historical development of linkages—beginning with the common pantograph of Scheiner in 1631. The demands of James Watt at the close of the 18th century brought renewed interest in the conversion of circular into rectilinear motion, a problem which

many confused with squaring the circle. The solution was first given by Sarrut in 1853, then by Peaucellier in 1864, both of which lay unnoticed until Lipkin, a student of Tschebyscheff, independently recreated Peaucellier's mechanism. Fanned by Sylvester's enthusiasm, interest in general linkwork immediately flamed high to attract the attention of men like Cayley, Kempe, Hart, Darboux, Clifford, Koenigs, Sir William Thompson, Darwin, Mannheim, and a host of lesser minds. The epidemic was so fierce and so universal that the subject was drained almost completely dry in the short span of five or six years. The drop in interest followed Sylvester's departure for America and Kempe's proof of the remarkable theorem that any algebraic curve, no matter how complex, can be described by a linkage. Two dozen or more working models of linkages designed for line motion, trisection, and the description of conics and many higher plane curves were placed on exhibition.

P. A. CARIS, *Secretary*

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## LONG CYCLES AS A RESULT OF REPEATED INTEGRATION

A. WALD, Cowles Commission and Columbia University

**1. Introduction.** Herbert E. Jones pointed out in his paper, "The theory of runs applied to time series," presented before the Third Annual Research Conference of the Cowles Commission, that the repeated summations (in the continuous case integrations) of a random series, each referred to its mean, approach a cosine curve with a period equal to the length of the series. This fact was stated as an empirical law, without mathematical proof.

This remarkable fact was also mentioned in H. T. Davis's paper, "Mathematical adventures in social science," published in this MONTHLY in the February 1938 number.

The mathematical proof for this statement was given by E. J. Moulton in his article, "The periodic function obtained by repeated accumulation of a statistical series," which appeared in the November 1938 number of the MONTHLY. For simplicity, Moulton dealt with the case of a continuous function, so that the theory of Fourier series could be applied. He showed that the iterative process leads to a multiple of the first non-vanishing term of the Fourier series development of the function.

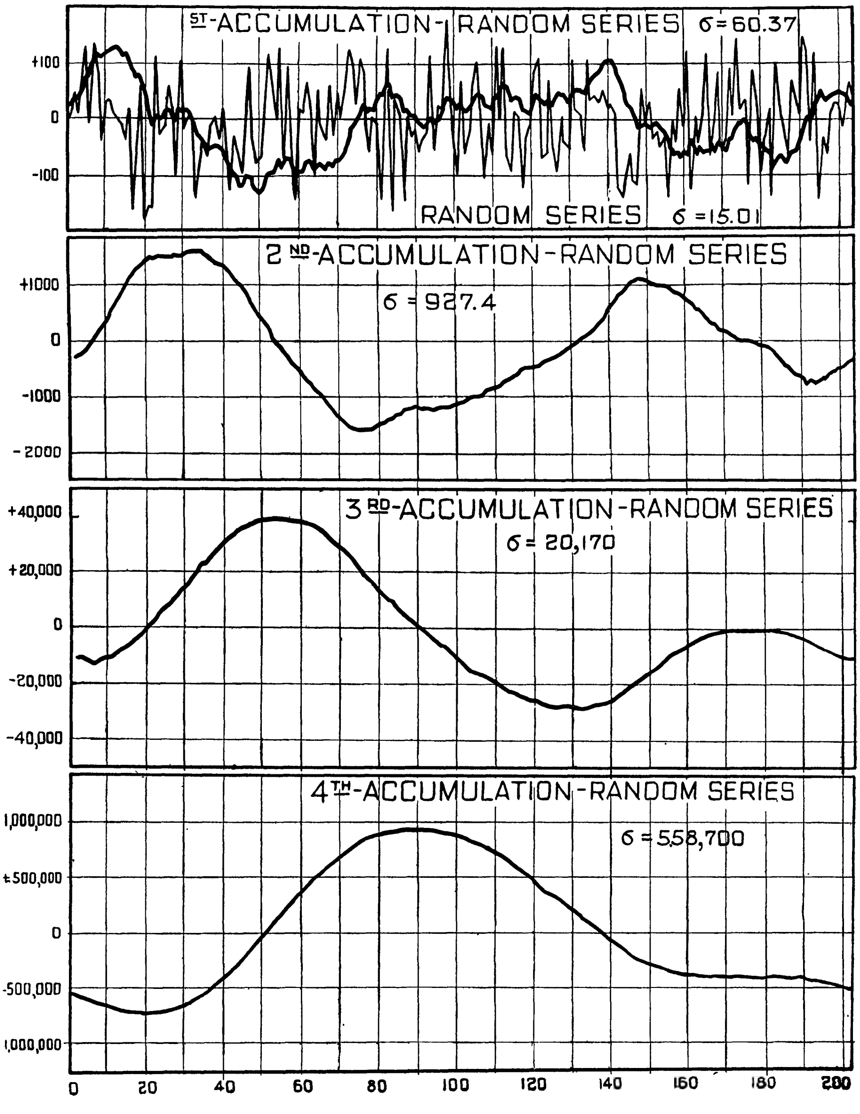
As Moulton pointed out in his paper, this result seems to have no special significance for random series, since it holds for any function which can be developed in Fourier series. I should like to remark that there may still be a significant difference between random series and other functions as to how fast the cosine function will be approached by the iterative process. Hence the following statistical problem arises: *To determine, in terms of probability, how fast the repeated integrations of a random series approach a cosine function.* The solution of this problem might be of interest also in economics, where we sometimes have to deal with accumulated (repeatedly summed or integrated) random series, and are interested in knowing how fast they approach a cosine function. We shall deal here with this statistical problem.

**2. The iterative process.** Denote by $x(t)$ a function defined in the interval $0 \leqq t \leqq L$. The iteration process is given by the following recursion formula:

$$(1) \quad S_{n+1}(t) = \frac{2\pi}{L}\left[\int_0^t S_n(s)ds - \frac{1}{L}\int_0^L (L-s)S_n(s)ds\right], \qquad (n = 0, 1, 2, \cdots),$$

where $S_0(t)$ is the function $x(t)$ referred to its mean; that is to say,

$$(2) \qquad S_0(t) = x(t) - \frac{1}{L}\int_0^L x(t)dt.$$



The recursion formula given here differs by the constant factor $2\pi/L$ from that given by Davis and by Moulton. This factor has been introduced in order to keep constant (*i.e.*, independent of $n$) the amplitude of the fundamental period in the Fourier series development of $S_n(t)$. If $x(t)$ is representable in a Fourier series in the interval $0 \leqq t \leqq L$, and if $A \cos (\alpha + 2\pi t/L)$ is the first non-vanishing term in the Fourier development, then, according to Moulton's theorem,

$$(3) \qquad S_n(t) - A \cos \left( \frac{2\pi t}{L} + \alpha - \frac{n\pi}{2} \right)$$

converges uniformly towards zero with increasing $n$.

**3. Exact formulation of the statistical problem.** Let us consider $N$ observations giving numbers $a_1, a_2, \cdots, a_N$, where $a_i$ is the observed value of the random variable $z_i$ $(i=1, \cdots, N)$. We make the following assumptions:

I. The expected value of $z_i$ is equal to zero, $(i=1, \cdots, N)$.

II. The expected value of the product $z_i z_j$ is equal to zero if $i \neq j$.

III. The variance $\sigma_i^2$ of the random variable $z_i$ is finite, $(i=1, \cdots, N)$.

These assumptions are very weak. We do not assume that the random variables are normally distributed, and we do not even assume that they have the same distribution.

As is well known, there exists exactly one trigonometric polynomial

$$(4) \qquad P(t) = A_0 + A_1 \cos qt + B_1 \sin qt + \cdots + A_m \cos mqt + B_m \sin mqt$$

such that $P(j) = a_j$, $(j=1, \cdots, N)$, where $m = \frac{1}{2}N$ or $\frac{1}{2}(N-1)$ according as $N$ is even or odd, and where $q = 2\pi/N$.

Instead of the discrete series $a_1, \cdots, a_N$, we shall for simplicity consider the continuous function $P(t)$ in the interval $0 \leqq t \leqq N$. We shall apply the iteration process to the function $P(t)$. Hence

$$S_0(t) = P(t) - \frac{1}{N}\int_0^N P(t)dt,$$

and

$$S_{n+1}(t) = q\left[\int_0^t S_n(s)ds - \frac{1}{N}\int_0^N (N-s)S_n(s)ds\right], \qquad (n = 0, 1, 2, \cdots),$$

where $q = 2\pi/N$. For brevity let

$$D_n(t) = S_n(t) - [A_1 \cos (qt - n\pi/2) + B_1 \sin (qt - n\pi/2)].$$

Since $A_1 \cos qt + B_1 \sin qt$ is the fundamental term of $P(t)$, we have, according to Moulton's theorem,

$$\lim_{n\to\infty} D_n(t) = 0.$$

As a measure for the distance between $S_n(t)$ and the periodic function $A_1 \cos (qt - n\pi/2) + B_1 \sin (qt - n\pi/2)$, the expression

$$\delta_n = \left\{\frac{1}{N}\int_0^N [D_n(t)]^2 dt\right\}^{1/2} \tag{5}$$

can be introduced. Then the statistical problem to be solved can be formulated as follows: For any positive number $\delta$ and for any positive integer $n$ we have to determine (or at least give a lower limit for) the probability that $\delta_n \leqq \delta$.

**4. The solution of the problem.** As is well known, the coefficients of the trigonometric polynomial $P(t)$ defined in (4) are given by the following equations:

$$\begin{aligned} A_0 &= \frac{1}{N}\sum_{j=1}^{N} a_j; \\ A_k &= \frac{2}{N}\sum_{j=1}^{N} a_j \cos kjq, \qquad (k = 1, 2, \cdots, m); \\ B_k &= \frac{2}{N}\sum_{j=1}^{N} a_j \sin kjq, \qquad (k = 1, 2, \cdots, m). \end{aligned} \tag{6}$$

For $k=m$ the above equation holds only if $N$ is odd. For simplicity let us assume that $N$ is odd, and therefore $m=\frac{1}{2}(N-1)$.

It is easy to show that

$$S_0(t) = P(t) - A_0,$$

and

$$S_n(t) = \sum_{k=1}^{(N-1)/2} \frac{1}{k^n}\left[A_k \cos (kqt - n\pi/2) + B_k \sin (kqt - n\pi/2)\right].$$

Hence

$$D_n(t) = \sum_{k=2}^{(N-1)/2} \frac{1}{k^n}\left[A_k \cos (kqt - n\pi/2) + B_k \sin (kqt - n\pi/2)\right],$$

and therefore

$$\delta_n = \left\{\sum_{k=2}^{(N-1)/2} \frac{1}{2k^{2n}}(A_k^2 + B_k^2)\right\}^{1/2}.$$

We shall now calculate the second moment of $\delta_n$ about the origin; that is to say, the expected value $E(\delta_n^2)$ of $\delta_n^2$. It is obvious that

$$E(\delta_n^2) = \sum_{k=2}^{(N-1)/2} \frac{1}{2k^{2n}}\left[E(A_k^2) + E(B_k^2)\right]. \tag{7}$$

From (6) and assumption II it follows that

$$E(A_k^2) + E(B_k^2) = \frac{4}{N^2} \sum_{j=1}^{N} \sigma_j^2, \tag{8}$$

where $\sigma_j^2 = E(z_j^2)$. Writing $\bar{\sigma}^2$ for the arithmetic mean of the variances $\sigma_1^2, \cdots, \sigma_N^2$, we get from (7) and (8)

$$E(\delta_n^2) = \alpha_n^2 \bar{\sigma}^2,$$

where

$$\alpha_n^2 = \frac{2}{N} \sum_{k=2}^{(n-1)/2} \frac{1}{k^{2n}}, \qquad \alpha_n > 0. \tag{9}$$

Applying the well known inequality of Tschebyscheff we obtain: The probability that $\delta_n$ will not exceed $\lambda\alpha_n\bar{\sigma}$ is greater than or equal to $1-\lambda^{-2}$ where $\lambda$ is an arbitrary positive number.

We can summarize our results in the following:

THEOREM: *Let $a_1, \cdots, a_N$ be a series of observed values, where $a_i$ is the observed value of the random variable $z_i$ $(i=1, \cdots, N)$ and $N$ is an odd number. The random variables satisfy the assumptions* I–III. *Let $P(t)$ denote the trigonometric polynomial of the order $\frac{1}{2}(N-1)$ defined in the interval $0 \leqq t \leqq N$ which goes through the points $a_1, \cdots, a_N$. Let $S_n(t)$ denote the nth accumulation of $P(t)$. Then the probability that the distance between $S_n(t)$ and the fundamental term of $P(t)$ with the phase reduced by $n\pi/2$ will be less than $\lambda\alpha_n\bar{\sigma}$ is greater than or equal to $1-\lambda^{-2}$, where $\lambda$ is an arbitrary positive number, $\alpha_n$ is given by* (9), *and $\bar{\sigma}^2$ denotes the arithmetic mean of the variances $\sigma_1^2, \cdots, \sigma_N^2$, where*

$$\sigma_1^2 = E(z_1^2), \cdots, \sigma_N^2 = E(z_N^2).$$

Denote by $A^2$ the expected value of the squared amplitude of the fundamental term of $S_n(t)$. Since this squared amplitude is equal to $A_1^2 + B_1^2$, we have

$$A^2 = E(A_1^2) + E(B_1^2) = \frac{4}{N} \bar{\sigma}^2.$$

Hence $\alpha_n\bar{\sigma} = A\beta_n$, where

$$\beta_n^2 = \sum_{k=2}^{(N-1)/2} \frac{1}{2k^{2n}}, \qquad \beta_n > 0.$$

The result in our theorem can obviously be expressed also as follows:

*The probability that the ratio $\delta_n/A$ will not exceed $\lambda\beta_n$ is greater than or equal to $1-\lambda^{-2}$, where $\lambda$ is an arbitrary positive number.*

This result remains obviously valid also if we substitute for the recursion formula (1) the recursion formula used by Davis and by Moulton, in which the factor $2\pi/L$ is omitted in (1).

In fact, the $n$th accumulation $S_n(t)$ calculated by means of their formula is $(N/2\pi)^n$ times greater than $S_n(t)$ calculated by means of the recursion for-

mula (1). Hence also $\delta_n$ and $A$ become $(N/2\pi)^n$ times greater, and therefore the ratio remains unchanged.

Herbert Jones formed as an example a random series by mixing the deviations from the trend of the Dow-Jones industrial averages over the period 1897 to 1913. The Figure shows the first four accumulations of the random series. For the distance $\delta_3$ between the third accumulation and the fundamental term of the random series with the phase reduced by $3\pi/2$ we have obtained the value 0.3516. In order to apply the results of the theorem, we have also calculated $\alpha_3$, when $N=205$, the value turning out to be 0.013. The estimated value of $\bar{\sigma}$ was 15.01. Hence, by applying our theorem, we get the following result:

The probability that $\delta_3$ is less than 0.3516, is greater than or equal to $1-\lambda^{-2}$ which is found to be 0.693. This is a very reasonable result, and one which shows that the accumulations of the random series converge as fast as we might have expected on the basis of our theorem.

*Editorial Note.* My note which appeared in the November 1938 number of the MONTHLY was worked out in March 1938, but publication was postponed for other material. In August a paper was received from Dr. Wald which was written independently of mine, and which contained much of the material of my paper and in addition most of the material of his present paper. I am happy to record Dr. Wald's independent solution of the problem which I treated.—E. J. M.

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## A POSTULATIONAL BASIS FOR PROBABILITY*

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In a postulational system, some terms are taken as undefined. The possibility of interpreting these undefined terms as familiar notions motivates the construction of the system. The system then serves to exhibit logical interrelationships among certain properties of these notions, independently of details irrelevant to those properties. The system which follows is thought to do this for the elementary theory of probability in a particularly concise manner.

The conciseness is obtained by using the two-valued calculus of propositions informally. The theory of real numbers, and a few principles from the calculus of propositional functions, are also presupposed.†

A second feature is that probability is associated with a propositional function $A(x)$, rather than simply an event $A$.‡ This makes the system a more ac-

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† The system can be completely formalized by incorporating it into a formal system for these three subjects. The continuity of the real numbers, and the functional calculus, are required only for Parts III and IV of our system.

‡ This point of view is adopted in H. Reichenbach, Wahrscheinlichkeitsrechnung, Leiden, 1935.

curate portrayal of the notions represented. However, by writing $A$, $P(A)$, $\cdots$ as abbreviations for $A(x)$, $P_x(A(x))$, $\cdots$, the theorems may be read in the language of events.

**1. Introduction and interpretation.** On a trial of an event governed by certain conditions, the event occurs or fails to occur. The result is a fact to which no probability attaches. Probability attaches rather to an *a priori* description of the event and conditions, in which there is indeterminacy. This idea is clarified by referring to a totality of possible trials of the event under the conditions. The probability measure is relative to this totality.

In this connection, it may be of interest to quote a paradox from Lewis Carroll (C. L. Dodgson), *Pillow Problems*, London, Macmillan, 1894, p. 18:

PROBLEM 72. Aug. 9, 1887. (Transcendental probabilities!) A bag contains 2 counters as to which nothing is known except that each is either black or white. Ascertain their colors without taking them out of the bag.

*Solution*: We know that if a bag contained 3 counters, 2 being black and one white, the chance of drawing a black one would be 2/3; and that any *other* state of things would *not* give this chance.

Now the chances that the given bag contains $(\alpha)BB$, $(\beta)BW$, $(\gamma)WW$, are respectively 1/4, 1/2, 1/4.

Add a black counter.

Then the chances that it contains $(\alpha)BBB$, $(\beta)BWB$, $(\gamma)WWB$, are, as before, 1/4, 1/2, 1/4.

Hence the chance of now drawing a black one is

$$\tfrac{1}{4}\cdot 1 + \tfrac{1}{2}\cdot\tfrac{2}{3} + \tfrac{1}{4}\cdot\tfrac{1}{3} \quad \text{or} \quad \tfrac{2}{3}.$$

Hence the bag now contains $BBW$ (since any *other* state of things would *not* give this chance).

Hence, before the black counter was added, it contained $BW$, *i.e.*, one black counter and one white.

At first, the chance of drawing a black counter refers to all trials of the experiment, to draw one counter from 3 fixed counters. Later, it refers to all trials of the experiment, to select 2 counters, add a black one, and draw one from the resulting 3.

We use a letter $X$ to designate the totality of possible trials, or a variable $x$ ranging over $X$. When the description of event and conditions is translated into propositional terminology, we have a propositional function $A(x)$. The function $A(x)$ takes the value *true* or *false*, according as the event occurs or fails to occur on the trial $x$. The probability $P_x(A(x))$ is a number resulting from an operation $P_x$ over the range $X$. EXAMPLE: If $A(x, y)$ is a propositional function of $x$ and $y$, $P_x(A(x, y))$ is a numerical function of $y$, and $P_y(P_x(A(x, y))=1/2)$ is a fixed number.

In the postulational system, the probability $P_x(A(x))$ is introduced as an undefined real number. For this purpose it is unnecessary to analyze $X$ in detail. It suffices to regard $X$, $X_B$, $S$, $\cdots$ with variables $x$, $x_B$, $s$, $\cdots$ as formal

devices for distinguishing different types of indeterminacy. However, a few examples will indicate the possibilities of further interpretation.

EXAMPLE 1. $X$ is the set of all possible tosses of a coin. The $x$'s are the individual tosses. $A(x) \equiv$ (a head is shown on toss $x$). $P_x(A(x)) = 1/2$.

EXAMPLE 2. $X$ is the set of all possible ordered pairs of tosses of a coin. $A(x) \equiv$ (a head is shown on the second toss of the pair $x$). $B(x) \equiv$ (a tail $\cdots$ on the first $\cdots$). $C(x) \equiv$ (a tail $\cdots$ on the first, and a head $\cdots$ on the second $\cdots$). $X_B$ is the subset of $X$ for which $B(x)$ is *true*. $P_x(A(x)) = 1/2$. $P_x(B(x)) = 1/2$. $P_x(C(x)) = 1/4$. $P_{x_B}(A(x_B)) = 1/2 = P_x(C(x))/P_x(B(x))$.

EXAMPLE 3. $S$ is a set of infinite sequences of tosses of a coin. $A(s) \equiv$ (heads and tails alternate in sequence $s$). $P_s(A(s)) = 0$.

The phrase "all possible trials" refers to a totality which may be described in theoretic or empirical terms. A theoretic description might consist in specifying the totality of sets of values of those parameters which would determine the outcome of the trial. An empirical description might consist in specifying more or less definite conditions for experimental trials.

Suppose that $\{A, B, \cdots\}$ is a class of events subject to the same set of conditions—a trial of one is simultaneously a trial of the others. In propositional terminology, it is a class of propositional functions $\{A(x), B(x), \cdots\}$ of the same variable $x$. There may be constructed within the class compound events, such as "$A$ occurs and $B$ fails." These are formed by the operations of the propositional calculus, of which we use three. $A \cdot B$ ("$A$ and $B$"): *true* when both $A$ and $B$ are *true*, otherwise *false*. $A \vee B$ ("$A$ or $B$"): *true* when at least one of $A$ and $B$ is *true*, otherwise *false*. $\overline{A}$ ("not $A$"): *true* when $A$ is *false*, otherwise *false*.

Two expressions which agree in truth value, for every determination of values for the component letters, are equivalent in the propositional calculus. For example: $\overline{A \cdot B} \equiv \overline{A} \vee \overline{B}$. Since each letter admits only two values, an equivalence of the propositional calculus can be verified by a finite series of tests. The variable $x$ does not play a rôle in these operations. However, in situations in which $A(x)$ and $B(x)$ can be shown to agree in truth value for each value of $x$, by other methods, we likewise write $A \equiv B$.

Part I of the postulational system deals with the probabilities of the events of the fundamental class $\{A, B, \cdots\}$. If $A$ and $B$ are identified with point sets, then Postulate 2 asserts that the probability of $A$ is an additive function of point sets, since $B$ is arbitrary. Definition 1 expresses the ordinary meaning of the words "mutually exclusive" in symbolic form. Two events are mutually exclusive, if their simultaneous occurrence is contradictory.

In Part II, an additional condition is imposed on the event $A$, the additional condition to consist in the occurrence of $B$. The probability of $A$ under the augmented conditions ("$A$ if $B$") is denoted by $P_B(A)$. In terms of propositional functions, $P_B(A) = P_{x_B}(A(x_B))$ where $x_B$ is a variable ranging over those values of $x$ for which $B(x)$ is *true*.* Postulate 4 relates probabilities with respect to $x$

* The definition of $A(x)$ outside the range of $x_B$ may be left unspecified.

and $x_B$.* Then Theorems 5, 11 show that it is unnecessary to introduce the other probability postulates for $x_B$.

In Part III, the Laplace definition of probability, as the number of favorable cases divided by the total number of cases, when there is a finite number of equally likely cases, is deduced from results of Part I. The concept of equal likelihood is here transferred from the foundations of the theory to its applications.† The result extends readily to the case of a continuous variable. If Lebesgue measure is used in the formulation, the extension of Theorem 3 to a denumerable infinity of events is required, since Lebesgue measurement employs denumerably infinite coverings.

In Part IV, we introduce another range of variation $S$, representing a collection of infinite sequences $s=(x_{s,1}, x_{s,2}, \cdots, x_{s,n}, \cdots)$ of trials of the fundamental events. The function $r_{s,n,A}$ represents the number of successes of $A$ in the first $n$ trials of the sequence $s$. In the usual relative frequency definition of probability, it is assumed that $r_{s,n,A}/n$ has a limit $l$, and $P(A)$ is defined as $l$. It suffices for our purpose to take Bernoulli's theorem as a postulate, with $S$ and $r_{s,n,A}$ as undefined terms. Then if $S$ is assumed to be a collection for which $r_{s,n,A}/n$ has a unique limit $l$ (as in the scheme of von Mises‡), it becomes a theorem, that $P(A)=l$.

**2. The postulational system.** PART I. FUNDAMENTAL PROBABILITY. The undefined terms are $X$ and $P$; $X$ is the range of a variable $x$; $A(x)$, $B(x)$, $\cdots$ are any propositional functions of $x$; $A(x)$, $P_x(A(x))$, $\cdots$ are abbreviated to $A$, $P(A)$, $\cdots$.

POSTULATE 1. *With $A$ is associated a non-negative real number $P(A)$. If $A\equiv B$, then $P(A)=P(B)$.*

POSTULATE 2. $P(A\cdot B)+P(A\cdot \overline{B})=P(A)$.

THEOREM 1. $P(A \vee B)=P(A)+P(B)-P(A\cdot B)$.

*Proof.* By the propositional calculus,

$$[(A\vee B)\cdot A]\cdot B \equiv A\cdot B, \qquad (A\vee B)\cdot \overline{A} \equiv B\cdot \overline{A},$$
$$[(A\vee B)\cdot A]\cdot \overline{B} \equiv A\cdot \overline{B}, \qquad B\cdot A \equiv A\cdot B.$$

Using these relations, we have

$$P(A\vee B) = P[(A\vee B)\cdot A] + P[(A\vee B)\cdot \overline{A}] \qquad \text{(P. 2)§}$$

* The reader may prefer to regard the equation $P(A\cdot B)=P(B)P_B(A)$ as the definition of $P_B(A)$, as it is in A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933, p. 7.

† For an interesting justification and discussion of Laplace's definition of probability, together with a discussion of other definitions of probability, see T. C. Fry, Fundamental Concepts in the Theory of Probability, this MONTHLY, vol. 41, 1934, pp. 206–217.

‡ R. von Mises, Wahrscheinlichkeitsrechnung, Leipzig and Vienna, 1931.

§ (P. 2) indicates "by Postulate 2"; (p.c., P. 1) indicates "by propositional calculus and Postulate 1"; (Th. 1) indicates "by Theorem 1"; (hyp.) indicates "by hypothesis"; *etc.*

$$
\begin{aligned}
&= P\{[(A \vee B)\cdot A]\cdot B\} + P\{[(A \vee B)\cdot A]\cdot \overline{B}\} + P[(A \vee B)\cdot \overline{A}] && \text{(P. 2)}\\
&= P\{A\cdot B\} + P\{A\cdot \overline{B}\} + P[B\cdot \overline{A}] && \text{(p.c., P. 1)}\\
&= P\{A\cdot B\} + P\{A\cdot \overline{B}\} + P[B\cdot \overline{A}] + P(B\cdot A) - P(A\cdot B) && \text{(p.c., P. 1)}\\
&= P(A) + P(B) - P(A\cdot B) && \text{(P. 2).}
\end{aligned}
$$

THEOREM 2. $P(A \vee B) \geqq P(B)$. $P(A\cdot B) \leqq P(B)$.

*Proof.* We have

$$
\begin{aligned}
P(A \vee B) &= P(A) + P(B) - P(A\cdot B) && \text{(Th. 1)}\\
&= P(A\cdot B) + P(A\cdot \overline{B}) + P(B) - P(A\cdot B) && \text{(P. 2)}\\
&= P(A\cdot \overline{B}) + P(B) \geqq P(B) && \text{(P. 1).}
\end{aligned}
$$

Similarly,

$$P(A \vee B) = P(B \vee A) \geqq P(A),$$

whence by Theorem 1 and Postulate 1,

$$P(A\cdot B) \leqq P(B).$$

DEFINITION 1. *A and B are mutually exclusive if* $A\cdot B \equiv A\cdot \overline{A}$.

THEOREM 3. *If A and B are mutually exclusive, then* $P(A\cdot B)=0$ *and* $P(A \vee B)=P(A)+P(B)$.

*Proof.* We have

$$
\begin{aligned}
P(A \vee \overline{A}) &= P[(A \vee \overline{A})\cdot(A\cdot \overline{A})] + P[(A \vee \overline{A})\cdot(\overline{A\cdot \overline{A}})] && \text{(P. 2)}\\
&= P(A\cdot \overline{A}) + P(A \vee \overline{A}) && \text{(p.c., P. 1).}
\end{aligned}
$$

Hence $P(A\cdot \overline{A})=0$. But by Definition 1, $A\cdot B \equiv A\cdot \overline{A}$. Hence by Postulate 1, $P(A\cdot B)=0$. By Theorem 1, $P(A \vee B)=P(A)+P(B)$.

POSTULATE 3. $P(A \vee \overline{A})=1$.

THEOREM 4. $P(A \vee \overline{A})=P(A)+P(\overline{A})=1$.

COROLLARY. $0 \leqq P(A) \leqq 1$.

Theorem 4 results from Theorem 3, and the corollary follows by Postulate 1.

PART II. CONDITIONAL PROBABILITY. Corresponding to the suggested interpretation, we regard $P_B(A)$ as an abbreviation for $P_{x_B}(A(x_B))$, where $x_B$ is a variable over the set $X_B$ defined thus: $[x \in X_B] \equiv [(x \in X)\cdot B(x)]$.*

POSTULATE 4. *With A and B* ($P(B) \neq 0$) *is associated a real number* $P_B(A)$ *such that* $P(A\cdot B)=P(B)P_B(A)$.

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* It would suffice in the formal system that $P$ be admitted now as a binary operator on a pair of propositional functions $A(x)$, $B(x)$, the result of the operation being abbreviated $P_B(A)$.

THEOREM 5. *For a fixed* $B$, *Postulates* 1–3 *hold with* $P$ *replaced by* $P_B$ $(P(B)\neq 0)$.

*Proof.* By Postulate 4, $P_B(A)=P(A\cdot B)/P(B)$. This ratio has the properties of $P(A)$ in Postulate 1. For Postulates 2–3, let $A$ and $C$ be the variable events, and write

$$\begin{aligned} P_B(A\cdot C)+P_B(A\cdot\overline{C}) &= \{P[(A\cdot C)\cdot B]/P(B)\}+\{P[(A\cdot\overline{C})\cdot B]/P(B)\} && \text{(P. 4)}\\ &= \{P[(A\cdot B)\cdot C]+P\,[(A\cdot B)\cdot\overline{C}]\}/P(B) && \text{(p.c., P. 1)}\\ &= P(A\cdot B)/P(B) = P_B(A) && \text{(P. 2, P. 4)}\\ P_B(A\vee\overline{A}) &= P[(A\vee\overline{A})\cdot B]/P(B) && \text{(P. 4)}\\ &= P(B)/P(B) = 1 && \text{(p.c., P. 1).} \end{aligned}$$

DEFINITION 2. If $P(A)=P_B(A)$, then $A$ and $B$ are *independent* (we suppose $P(A)$, $P(B)$, $P(\overline{A})$, $P(\overline{B})\neq 0$).

THEOREM 6. *If* $A$ *and* $B$ *are independent*, $P(A\cdot B)=P(A)P(B)$ $(P(A)$, $P(B)$, $P(\overline{A})$, $P(\overline{B})\neq 0)$.

THEOREM 7. *Each of the six equations which follow is necessary and sufficient for the independence of* $A$ *and* $B$ $(P(A)$, $P(B)$, $P(\overline{A})$, $P(\overline{B})\neq 0)$:

$$P(A)=P_B(A)=P_{\overline{B}}(A),\ P(B)=P_A(B)=P_{\overline{A}}(B).^*$$

*Proofs.* Theorem 6 follows from Postulate 4 and Definition 2. To establish necessity under Theorem 7, assume $P(A)=P_B(A)$. Then

$$\begin{aligned} P_A(B) &= P(B\cdot A)/P(A) && \text{(P. 4)}\\ &= P(A\cdot B)/P(A) = P(B)P_B(A)/P(A) && \text{(P. 4)}\\ &= P(B)P(A)/P(A) && \text{(hyp.)}\\ &= P(B), \end{aligned}$$

and

$$\begin{aligned} P_{\overline{B}}(A) &= P(A\cdot\overline{B})/P(\overline{B}) && \text{(P. 4)}\\ &= \{P(A)-P(A\cdot B)\}/\{1-P(B)\} && \text{(P. 2, Th. 4)}\\ &= \{P(A)-P(B)P_B(A)\}/\{1-P(B)\} && \text{(P. 4)}\\ &= \{P(A)-P(B)P(A)\}/\{1-P(B)\} && \text{(hyp.)}\\ &= P(A). \end{aligned}$$

Similarly, $P_A(B)=P(B)$ implies $P_{\overline{A}}(B)=P(B)$. The arguments may be reversed to establish sufficiency.

PART III. THE LAPLACE DEFINITION OF PROBABILITY. THEOREM 8. *If* $y$ *is a*

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* For a discussion of these equations based on the Laplace definition of probability, see H. Poincaré, Calcul des Probabilités, Paris, 1912, p. 39.

*function of $x$, if $y$ admits $n$ values $y_1, \cdots, y_n$, if $P(y=y_1)=\cdots=P(y=y_n)$, if $E$ is a subset consisting of $r$ of the $n$ values, and if $A\equiv(y\in E)$, then $P(A)=r/n$.*

*Proof.* Set $p=P(y=y_1)=\cdots=P(y=y_n)$. By Theorem 3 and Postulate 3, $np=1$ or $p=1/n$. Then by Theorem 3, $P(A)=rp=r/n$.

POSTULATE 5. *If $A_1, A_2, \cdots$ are mutually exclusive in pairs, and if $A\equiv A_1\vee A_2\vee\cdots$, then $P(A)=P(A_1)+P(A_2)+\cdots$.*

*Remark.* When Postulate 5 is assumed, Postulate 2 becomes redundant.

THEOREM 9. *If $y$ is a function of $x$, if the range of $y$ is the real interval $(a, b)$, if $P(y\in(a_1, b_1))=P(y\in(a_2, b_2))$ for each two subintervals $(a_1, b_1)$ and $(a_2, b_2)$ of equal length, if $E$ is a set of points in $(a, b)$ with Lebesgue measure $m(E)$, and if $A\equiv(y\in E)$, then $P(A)=m(E)/(b-a)$.*

Proof may be carried out by a straightforward extension of the method of proof of Theorem 8 in conjunction with the approximation process of Lebesgue measurement.

PART IV. THE RELATIVE FREQUENCY DEFINITION OF PROBABILITY. New undefined terms are $S$ and $r$; $S$ is the range of a variable $s$; $\mathfrak{A}(s), \mathfrak{B}(s), \cdots$ are any propositional functions of $s$; $\mathfrak{A}(s), P_s(\mathfrak{A}(s)), \cdots$ are abbreviated $\mathfrak{A}, P(\mathfrak{A}), \cdots$.

POSTULATE 6. *With each $A$, $s$, and positive integer $n$, is associated a positive integer $r_{s,n,A}$. If $\delta>0$, then*

$$\lim_{n\to\infty} P_s\left(\left|\frac{r_{s,n,A}}{n}-P_x(A(x))\right|<\delta\right)=1.$$

POSTULATES 1s–5s. *Postulates 1–5 with $A, B, \cdots$ replaced by $\mathfrak{A}, \mathfrak{B}, \cdots$.*

THEOREM 10. *If for every $s$, $r_{s,n,A}/n\to l$ as $n\to\infty$, then $P(A)=l$.*

*Proof* (abbreviating $r_{s,n,A}$ to $r$, $P(A)$ to $p$, $P_s$ to $P$). Suppose instead that $p\neq l$. Let $\delta=|l-p|/2$. Using the hypothesis $r/n\to l$, divide $S$ into subsets, $S_1=\{$those $s$ for which $|r/n-l|<\delta$ holds for $n\geqq 1\}$, $S_2=\{$those $s$ remaining for which $|r/n-l|<\delta$ holds for $n\geqq 2\}, \cdots$. Using Postulates 5s and 3s, we have

$$\begin{aligned}P\left\{\left|\frac{r}{n}-p\right|<\delta\right\} &= P\left\{\left[\left|\frac{r}{n}-p\right|<\delta\right]\cdot[s\in S_1]\right\}\\ &\quad+P\left\{\left[\left|\frac{r}{n}-p\right|<\delta\right]\cdot[s\in S_2]\right\}+\cdots,\end{aligned}\tag{1}$$

$$1=P(s\in S)=P(s\in S_1)+P(s\in S_2)+\cdots.\tag{2}$$

By Postulate 1s, the terms of both series are positive. By the convergence, given any $\zeta>0$, there is a $k$ such that the remainder of (2) after the $k$th term is $<\zeta$. By Theorem 2s, the terms of (2) are $\geqq$ the corresponding terms of (1). Hence

the corresponding remainder of (1) is also $<\zeta$. Now let $n$ take any value $\geqq k$. Then for $s$ in $S_1, \cdots, S_k$, we have $|r/n-l|<\delta$. Hence

$$\left|\frac{r}{n}-p\right| = \left|\left\{\frac{r}{n}-l\right\}+\{l-p\}\right| = \left|\left\{\frac{r}{n}-l\right\} \pm 2\delta\right| > \delta.$$

It follows by Theorem 3s that the first $k$ terms of (1) vanish, so that the entire series is $<\zeta$. Thus $\lim_{n\to\infty} P\{|r/n-p|<\delta\}=0$ for $\delta=|l-p|/2$, contradicting Postulate 6.

Conditional probability. THEOREM 11. *Postulate* 5 *holds with $P$ replaced by $P_B$, and Postulate* 6 *with $P(A)$ replaced by $P_B(A)$ and $r_{s,n,A}/n$ by $r_{s,n,A\cdot B}/r_{s,n,B}$* $(P(B)\neq 0)$.

The proof is straightforward, using particularly Postulates 4 and 6.*

## NOTE ON AN ABSOLUTE CONSTANT OF KHINTCHINE

D. H. LEHMER, Lehigh University

Let $x$ be a real positive number and let $x=[x]+1/q_1+1/q_2+\cdots$, where $q_i$ are integers $\geqq 1$, be the development of $x$ in a regular continued fraction. The question naturally arises: Can anything be said in advance about the partial quotients $q_i$ of an arbitrarily given number? Strictly speaking the answer is no. In fact a finite or infinite set of $q$'s determine a unique $x$ so that there are $x$'s whose $q$'s satisfy any possible condition one may care to specify. However, if we alter the question so that it reads: Can anything be said about the partial quotients of almost all numbers? then it is possible to say a great deal. P. Levy, A. Khintchine and others† have recently considered this problem. One result may be stated as follows:

THEOREM. *Let*

$$M_n(x) = \sqrt[n]{q_1q_2\cdots q_n}.$$

*Then the set $E$ of all those $x$'s for which*

$$\lim_{n\to\infty} M_n(x) = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\log k/\log 2} = K \tag{1}$$

*does not hold, is of measure zero.*

In other words the geometric mean of the first $n$ partial quotients of almost all numbers tends to an absolute constant $K$ as $n\to\infty$.

* We have considered only asymptotic properties of the number $r_{s,n,A}$. In a fuller treatment of relative frequency, other properties of $r_{s,n,A}$ would be postulated, and some of the earlier postulates for probabilities with respect to $x$ would become theorems.

† Lévy: Bulletin Sociète Mathématique de France, vol. 57, 1929, pp. 190 ff. Compositio Mathématica, vol. 3, 1936, pp. 286–303. Khintchine: Ibid., vol. 1, 1935, pp. 361–382; vol. 3, 1936, pp. 276–285.

The infinite product in (1) converges very slowly. Khintchine gives for $K$ the value 2.6. The purpose of this note is not to point out that this value contains a last figure error, but rather to show how this constant can be easily determined with considerable accuracy, and to call attention to the fact that the seemingly lawless partial quotients of the number $\pi$ appear to behave in a manner demanded by (1).

Taking the natural logarithm of $K$ and setting

$$\begin{aligned} f(x) &= \log x \log\left(1 + \frac{1}{x(x+2)}\right) \\ &= 2 \log x \log (x+1) - \log^2 x - \log x \log (x+2), \end{aligned} \tag{2}$$

we consider, instead of the product, the series

$$S = \log 2 \log K = \sum_{r=1}^{\infty} f(r). \tag{3}$$

Applying the Euler-MacLauren sum formula we have

$$\begin{aligned} \sum_{r=\nu}^{N} f(r) &= \int_{\nu}^{N} f(x)dx + \frac{1}{2} f(\nu) - \frac{1}{2} f(N) + \frac{1}{12} (f'(N) - f'(\nu)) \\ &\quad - \frac{1}{720}(f^{(3)}(N) - f^{(3)}(\nu)) + \frac{1}{30240}(f^{(5)}(N) - f^{(5)}(\nu)) - \cdots . \end{aligned} \tag{4}$$

The integral may be evaluated by applying the following reduction formula to equation (2). We obtain

$$\begin{aligned} \int \log x \log (x+a)dx &= (x+a) \log x \log (x+a) - x \log x \\ &\quad - (x+a) \log (x+a) + 2x - a \int \frac{1}{x} \log (x+a)dx \end{aligned} \tag{5}$$

with $a=0, 1, 2$. Except for $a=0$, we are lead to the non-elementary integral

$$\int \frac{1}{x} \log (x+a)dx.$$

Postponing its evaluation for a moment, we obtain from (2) and (5) after simplification

$$\begin{aligned} \int_{\nu}^{N} f(x)dx &= \{\log N - 1\}\{2(N+1)\log(N+1) - N\log N - (N+2)\log(N+2)\} \\ &\quad + \{\log \nu - 1\}\{\nu \log \nu + (\nu+2)\log(\nu+2) - 2(\nu+1)\log(\nu+1)\} \\ &\quad + 2\int_{\nu}^{N} \frac{1}{x} \log\left(1 + \frac{1}{x+1}\right) dx. \end{aligned} \tag{6}$$

If now we let $N \to \infty$ and note that

$$\log(N+a) - \log N \sim a/N,$$

we see that the entire first term of the right hand side of (6) vanishes. As for the integral

$$I = \int_{\nu}^{\infty} \frac{1}{x} \log\left(1 + \frac{1}{x+1}\right) dx,$$

by expanding the logarithm and writing

$$V_k = \int_{\nu}^{\infty} \frac{dx}{x(x+1)^k},$$

we have

$$I = V_1 - \tfrac{1}{2}V_2 + \tfrac{1}{3}V_3 - \tfrac{1}{4}V_4 + \cdots, \tag{7}$$

where the $V$'s tend rapidly to zero, and may be easily computed by means of the recurrence formula

$$V_{k+1} = V_k - k^{-1}(1+\nu)^{-k} \tag{8}$$

and the initial condition

$$V_1 = \log(1 + 1/\nu). \tag{9}$$

We have then

$$\int_{\nu}^{\infty} f(x)dx = (\log \nu - 1) \log \frac{\nu^{\nu}(\nu+2)^{\nu+2}}{(\nu+1)^{2\nu+2}} + 2I. \tag{10}$$

Returning to (4) we note first that $f(x)$ and all its derivatives vanish as $x \to \infty$. In fact if we set

$$g_a(x) = \log x \log(x+a), \tag{11}$$

so that

$$f(x) = 2g_1(x) - g_0(x) - g_2(x), \tag{12}$$

we find that

$$g_a'(x) = \frac{\log(x+a)}{x} + \frac{\log x}{x+a}, \tag{13}$$

$$g_a^{(3)}(x) = -\frac{3(2x+a)}{x^2(x+a)^2} + \frac{2\log(x+a)}{x^3} + \frac{2\log x}{(x+a)^3}, \tag{14}$$

$$g_a^{(5)}(x) = -10\frac{(2x+a)(5x^2+5xa+3a^2)}{x^4(x+a)^4} + \frac{24\log(x+a)}{x^5} + \frac{24\log x}{(x+a)^5}, \tag{15}$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

Hence we observe that

$$g_a^{(n)}(x) = 0(x^{-n}).$$

Hence not only do $g(x)$ and its derivatives, and hence $f(x)$ and its derivatives, vanish at infinity, but also when $x=\nu$, the sequence $f(\nu), f'(\nu), f^{(3)}(\nu), f^{(5)}(\nu)$, *etc.*, tends to zero rapidly for $\nu$ only fairly large.

Finally, combining (4), (7), and (10) we get

$$S = \sum_{r=1}^{\nu-1} f(r) + (\log \nu - 1) \log \frac{\nu^\nu(\nu+2)^{\nu+2}}{(\nu+1)^{2\nu+2}} + 2\{V_1 - \tfrac{1}{2}V_2 + \tfrac{1}{3}V_3 - \cdots\}$$
$$+ \frac{1}{2}f(\nu) - \frac{1}{12}f'(\nu) + \frac{1}{720}f^{(3)}(\nu) - \frac{1}{30240}f^{(5)}(\nu) + \cdots,$$

where the $V$'s are computed from (8) and (9), and the other terms from (11)–(15).

Choosing $\nu=10$, we find that $S=.68472475$, while if $\nu=15$, $S=.68472480$. Hence it follows from (3) that

$$K = 2.685550.$$

The regular continued fraction for $\pi$ was developed as far as $q_{34}$ (the first 33 $q$'s being correct) by Wallis* in 1685. The first 90 $q$'s have been given recently.† The values of $q_{91} \cdots q_{100}$ are as follows:

1, 2, 1, 3, 1, 2, 1, 1, 10, 2.

The first 100 partial quotients occur with the following frequencies:

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 15 | 10 | 12 | 13 | 14 | 16 | 22 | 24 | 45 | 84 | 99 | 161 | 292 |
| 41 | 22 | 7 | 4 | 2 | 5 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

That is to say, 41 of the partial quotients are 1's, 22 are 2's, and so on. The geometric mean of the first 100 $q$'s is found to be

$$M_{100}(\pi) = 2.6831468,$$

which differs from $K$ by less than 9 parts in ten thousand.

Levy‡ has proved that as $n$ tends to infinity the $n$th root of the denominator $B_n$ of the $n$th convergent tends to

$$e^{\pi^2/12\log 2} = 3.275823$$

for almost all numbers. For $\pi$ we find that

$$\sqrt[100]{B_{100}} = 3.269202,$$

which differs from the limit by 2 parts in a thousand.

* A Treatise of Algebra, London, 1685, pp. 46–55.

† This MONTHLY, vol. 45, 1938, pp. 231–233.

‡ Compositio Mathematica, vol. 3, 1936, p. 289.

There is every reason to believe, then, that $\pi$ (like almost all numbers) does not belong to the set $E$ mentioned in the Theorem. The Naperian base $e$, however, does belong to this set.* In fact, $M_n(e)$ does not even tend to a limit but is of order $\sqrt[3]{n}$. Perhaps the reader will find it of interest to verify that

$$\lim_{n\to\infty} \frac{M_n(e)}{\sqrt[3]{n}} = \sqrt[3]{\frac{2}{3e}} = .625949.$$

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## MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This Department affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

### POSTCARDS ON APPLIED MATHEMATICS

J. L. SYNGE, University of Toronto

Authors of mathematical textbooks should receive from their publishers a copy, surmounted by a skull and cross-bones, of Andrew Marvell's poem "To His Coy Mistress":

. . . My vegetable love should grow
Vaster than empires, and more slow;
An hundred years should go to praise
Thine eyes and on thy forehead gaze;
Two hundred to adore each breast,
But thirty thousand to the rest;
An age at least to every part,
And the last age should show your heart.
For, Lady, you deserve this state,
Nor would I love at lower rate.
But at my back I always hear
Time's wingèd chariot hurrying near;
And yonder all before us lie
Deserts of vast eternity. . . .

That is the true way to court the Queen of the Sciences—if only we had the time. Unfortunately no one has. And as the charms of mathematics proliferate, the courtship becomes more and more perfunctory. Even now one devotes all his adoration to her ear (and swears that its beauties far transcend those of her nose), while another is a nasal specialist and will have nothing to do with ears. A few, a sadly decreased and decreasing few, still possess the enormous intellectual vitality required to adore the whole. A greater number (in which the pres-

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* This is true, in fact, of the $k$-th root of $e^2$, where $k$ is any positive integer.

ent writer includes himself) adore a limited area of the Queen of the Sciences with a genuine adoration based on hard work and understanding, but the remoter parts are lost in a mist of ignorance. When we advance timidly through the mist to explore, we encounter a forbidding sign "Keep out: specialists at work," and if we pursue our quest, a dozen books are hurled at us. These books suggest that the experts believe in courtship through a microscope for they offer a detailed description of the pores, acne, and hair follicles of the Queen rather than an aesthetic appreciation of a whole feature.

Is this inevitable? Are those general ideas which the specialists possess in high degree (and may let slip in conversation, but all too rarely on the printed page)—are those general ideas to be attained only by tedious grubbing through a wilderness of detail? Are short-cuts to knowledge impossible?

The situation as regards mathematics is peculiarly difficult—much more so than for the sciences which rely on observation and experiment. The biologist observes the patterns woven by nature; but the mathematician weaves his own patterns—more, he spins out of his own imagination the thread for the cloth which bears the patterns. To most mathematicians aesthetic appreciation of the pattern is begotten only through tedious weaving; in sudden spurts the realization of beauty comes to them. They wade through exhausting pages of detail to find in some hidden corner the formula or idea which reveals the secret of the pattern.

Mathematics consists of axioms, reasoning and results. Mathematical education puts far the greatest emphasis on the reasoning. It does not seem to matter very much what the axioms are or what the results are, but woe betide the unfortunate student who slips on the reasoning. Put into the language of analogy, it does not matter much what the thread is or what the final pattern is, but shame on the slipshod weaver! A healthy spirit of craftsmanship is shown in this attitude, it is true, but is it not an attitude a little lacking in imagination? Those who subscribe to this view can find in the history of mathematics only food for laughter at the expense of the incompetent craftsmen of the past who left so many loose threads hanging, and would assess higher than Euler a freshman who had learned some rigorous processes without the ghost of an idea whither they were leading him.

Has not mathematics by this time so expanded that a change of emphasis is necessary for very self-preservation? Is it not time to get off the Grecian high-horse, and admit that mathematics is something more than a string of syllogisms? Is not mathematics big enough and important enough in itself for us to drop the silly propaganda that the study of mathematics trains the mind for work in fields remote from mathematical symbols? For a century society has swallowed the dope that the child who could not solve a quadratic equation would burn his hand in the fire because he would lack the reasoning power to withdraw it. Mass education having failed (apparently) to produce the millennium, it looks as if society were getting restive. Bricks will be thrown at education, and the most brittle pane is mathematics-as-a-training-for-the-mind. Shall we make up another one, or shall we tell them the truth—namely that if mathe-

matics were to drop out of education it would make a bigger hole in society than any other subject, advertising and business methods not excepted?

But the mathematics which is a reality does not consist of chains of reasoning, beginning nowhere in particular and ending nowhere in particular. The brain of a mathematician is as full of results (whose precise proofs are temporarily and mercifully forgotten, except when he has to lecture on them) as a swamp is full of mosquitoes. These are facts, as real or more so than a table of physical constants, and if society were to sweep mathematics out of education it would sweep out a storehouse as full of aesthetics as an art gallery, as full of power as a fleet of battleships and as full of riches as a bank; all of which may be submitted as an almost criminal understatement.

A Bessel function, a permutation group, an $n$-space of constant curvature or a linear differential equation of the second order are just as real as an elephant or a hot-dog. They exist and have certain describable properties, and endearing habits which make them in many ways pleasanter life-companions than elephants or hot-dogs. The trouble begins when we start to bring them into society. If an Eskimo asks "What is an elephant?" you say "It is like a very big walrus with a long nose and legs." That gets him somewhere at least. If you are wise you do not start off: "All matter is composed of protons and electrons. . . ." By the time you reach the elephant, the Eskimo has long since remembered an engagement elsewhere.

Yet in mathematical education we still insist in starting off with the proton and electron, so to speak. What was good enough for the Greeks is good enough for us. By the time we have mastered the intricacies of the elephant, we are wearing on towards middle age, and a similar intensive study of hot-dogs does not appeal to us. Hot-dog specialists may turn up their noses at such an unworthy treatment of their favorite theme, but is it not permissible to tell the man who has wasted the flower of his youth on elephants that a hot-dog is a very small elephant with no trunk or legs? In brief and in general, the manner in which anything is to be explained depends as much on the background of the learner as on the thing itself.

Any attempt to explain contact transformations in terms of elephants and hot-dogs is foredoomed to failure. We can be playful up to a point but there are limits. And this is the fate of all too many laudable attempts to bring mathematics to the man in the street. His world of ideas is altogether too remote from the world of mathematics, and although high-school mathematics helps him somewhat, its insistence on rigid discipline (training-the-mind-in-accurate-habits-of-the-thought) cuts him off from the essentially intuitive mode of thought of creative mathematics.

But even if we regretfully write off our debt to the man in the street as a bad one, there are still some problems in exposition much nearer home. Perhaps the most important problem for us is that of communication between mathematicians. How much can we accept as a basis for communication? In fact, what are the mathematical words in which general communications are to be framed?

Time was not so long ago when one might casually mention dynamics, for example, as a basis of explanation (the walrus to the Eskimo) but that is no longer true today. Although the whole language of mathematics becomes richer daily, the vocabulary of our *lingua franca* shrinks, until one begins to wonder whether there will soon be any at all.

The main trouble is that an expert in a field is so familiar with the really fundamental things in that field that he has almost forgotten their existence. He has forgotten that in his early years he had to find the gate in the fence before he could get into the field, and that others might relish some clear indication of the whereabouts of the gate more than a detailed discourse on the latest flower (or weed?) that he has discovered.

One of the facts which the historian of the future will not fail to note regarding our present epoch is the way in which mathematicians have turned from applied mathematics. Mathematicians may be divided into three classes in respect of their attitude towards applied mathematics: (a) those who have nothing to do with applied mathematics and do not want to, regarding it as an inferior type of intellectual exercise; (b) those who would like to be better acquainted with applied mathematics, but cannot find time for prolonged study of what is not their major interest; (c) those primarily interested in applied mathematics, studying the pure almost solely for its repercussions on the applied. There are, it is true, a few fortunate individuals who cannot be so rudely classified, but their number is so small and their capacity so great that the present writer would not presume to write for or about them; they are the sort that seem able to master the contents of a book by glancing at the cover.

The eighteenth century was the age of class (c); the twentieth century is the age of class (a). The nineteenth was the age of transition. (Some pure mathematicians may be surprised to learn that Weierstrass made some fundamental contributions to the theory of dynamical stability.) The transition was doubtless greatly helped by Kelvin and Tait's monumental "Natural Philosophy," succulent Irish stew to the physicist, but a bellyache to the mathematician.

The difficulty which besets every writer on applied mathematics is inevitable; he has to write in terms of physical imagery for his physical readers and in terms of mathematical imagery for his mathematical readers, and it requires something more than an ordinary man to serve two so different ends at the same time. Dynamics and potential theory, established early in the mathematical tradition, are written mathematically. Newer subjects in comparison, electromagnetism, hydrodynamics and elasticity, are written much more for physicists. Relativity is for mathematicians (so much so that physicists will allow for change of mass with velocity without understanding that simultaneity is not absolute, while astronomers will grin at you if you say that space is not Euclidean). Quantum mechanics has an unpleasant way of falling with a bang into the lap of the physicist or into that of the mathematician, with a disconcerting hiatus between.

Class (b) aforesaid deserves a straighter deal from applied mathematicians. This class wants to know what applied mathematics is about and they want to be told in mathematical language, and briefly above all. To do this sort of thing successfully, a mathematician must get over his excessive love of truth (there should be a Greek word for this, with a sting in it). He must be as unqualified in his statements and as unblushing about it as a seventeenth century lover praising his mistress.

As has been remarked above, mathematics is made up of axioms, reasoning and results. Its verbosity is testimony to man's stupidity, for the deductions which can be made logically from any set of axioms should be self-evident to any sufficiently acute mind. Logic is the railway track along which the mind glides easily. It is the axioms that determine our destination by setting us on this track or the other, and it is in the matter of choice of axioms that applied mathematics differs most fundamentally from pure. Pure mathematics is controlled (or should we say "uncontrolled"?) by a principle of ideological isotropy: any line of thought is as good as another, provided that it is logically smooth. Applied mathematics on the other hand follows only those tracks which offer a view of natural scenery; if sometimes the track dives into a tunnel it is because there is prospect of scenery at the far end. When the applied mathematician describes his journeys to a physicist, he devotes himself to the natural scenery he has seen; but when he describes them to the pure mathematician he must stress the axioms from which he started and the high spots of his mathematical argument.

No one can possibly carry about in his head the detail of a textbook of 400 pages. At least, not in his conscious mind; there may be a good deal of it lying down in subconscious regions. When a problem comes up for discussion there is a hurried dive into the subconscious to hunt for the necessary material to deal with it. If the subconscious is an untidy mess, it is unlikely that this material will be found. Then the only plan is to let the problem sink down into the subconscious in the hope that sometime a solution will float up to the surface. Putting such a solution into words for circulation is tedious, almost painful, a fact which explains the great amount of obscure writing in a subject which ought to be crystal clear.

We cannot avoid this difficulty by keeping our knowledge all floating on the surface of the conscious mind. If we try to, we have no room for the everyday consciousness, and we begin to leave our umbrellas behind us and generally to become public nuisances. A better plan is to buoy our subconscious by a few conspicuous buoys floating on the surface of the conscious. When a problem presents itself, we see that it belongs to the region of one buoy rather than another, and it may require only comparatively little diving into the subconscious in its neighborhood to bring up the knowledge necessary to solve the problem. Moreover it will be far easier to put the solution into words because the question has been discussed nearer to the level of consciousness.

These buoys are surely familiar to us all—certain formulas or theorems

which stand out (for us at least) as so fundamental that we would as soon forget our own names as them. But these buoys may serve a double purpose, for they may serve as a guide to the stranger. We may say frankly: "If you cannot understand what these formulas mean, you know nothing about the subject; but if you do understand them, you have a sufficient general grasp of the subject to fill in a thousand and one details in their proper perspective."

What are these buoys? Do writers of textbooks indicate in a preface the few buoys which mark the course through their intricate channels of reasoning? Very rarely. It is no easy task, but apart from its intrinsic difficulty, there seems to be a traditional inhibition—a reluctance to reveal to the public too easily the secrets of a cult. How many students of Euclid have wandered apathetically through logical mazes without knowing that the last buoy was a bundle of regular solids? Today we should sink that buoy, but we must have others. In the study of the Greek geometry of point, line and circle, the first buoy should carry a brief legend giving the rules of the game—the constructions allowed; thus and only thus can we ever get rid of that skeleton at the educational feast—the eternal circle-squarer. After this essential warning buoy, we might have half a dozen others carrying the figures of half a dozen salient propositions. Understand these, you may say, and have a general idea of the links between them, and Euclidean geometry is yours. Here are hard facts suited to a realistic generation, hard physical facts for the physicist, hard mathematical facts for the mathematician. Away with the doctrine of mathematics-as-a-training-for-the-mind, and substitute over the doors of high-schools and universities the realistic Spanish legend: "What Nature has not given, Salamanca cannot supply"!

What are the buoys in applied mathematics? That depends on who is to use them. The applied mathematician will have his own particular branch of applied mathematics dotted with a number of buoys, a smaller number over the other parts of applied mathematics, and (it is to be hoped) a few outlying buoys marking the richest regions of pure mathematics. But let us consider the case of mathematicians of class (b), mentioned above. Their buoys will for the most part mark regions of pure mathematics. Cannot the applied mathematician help with some advice as to a few buoys in applied mathematics, marked not with physical legends (which the pure mathematician would read with distaste), but with mathematical symbols?

The legends on these buoys should be compact enough to go with ease on an ordinary postcard. In fact it should be possible to give on a dozen postcards key-formulas for the whole of applied mathematics. Since size has always been an object of human admiration, from the pyramids and fat women at fairs down to the Empire State and the Queen Mary (and is not a reprint of a hundred pages much more impressive than one of ten?), the above suggestion may seem grossly disrespectful to one's favorite subject. If so, one can best defend oneself by quoting Robert Burns's admirable comparison of two ladies:

Why did God make the gem so small,
And why so huge the granite?
Perhaps it was that men might put
The greater value on it.

Here, then, are postcards on some branches of applied mathematics—not *all* branches, because the present writer does not feel competent to do them adequately. That he dares to make the present attempt at all with any hope of helping anyone thereby is due to the sudden illumination which he received some eighteen years ago when, in the course of a lecture, Dr. L. Silberstein wrote on the blackboard as an epitome of relativity a few formulas equivalent to those given below.

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## DYNAMICS

$$L(\dot{q}, q) = T - V, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \ (i = 1, \cdots, n), \qquad \delta \int L dt = 0;$$

$$p_i = \partial L/\partial \dot{q}_i, \qquad H(p, q) = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = T + V,$$

$$\dot{q}_i = \partial H/\partial p_i, \qquad \dot{p}_i = -\partial H/\partial q_i;$$

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0.$$

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## ELECTROSTATICS

$$\iint E_n dS = 4\pi e, \qquad \Delta V = -4\pi\rho, \qquad \partial V/\partial n = -4\pi\sigma,$$

$$W = \tfrac{1}{2}\sum eV = \frac{1}{8\pi}\iiint E^2 d\tau.$$

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## HYDRODYNAMICS

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i, \qquad \frac{\partial u_j}{\partial x_j} = 0.$$

$$u_i = -\partial\phi/\partial x_i, \qquad \Delta\phi = 0, \qquad \partial\phi/\partial n = -u_n,$$

$$-\partial\phi/\partial t + p/\rho + \tfrac{1}{2}q^2 + \Omega = F(t).$$

$$Y = \rho\kappa U.$$

ELASTICITY

$$E_{ij,j} + \rho X_i = \rho \partial^2 u_i/\partial t^2, \qquad T_i = E_{ij}n_j, \qquad (,j = \partial/\partial x_j);$$
$$\tfrac{1}{2}(u_{j,i} + u_{i,j}) = e_{ij} = E^{-1}\{(1+\sigma)E_{ij} - \sigma\delta_{ij}E_{kk}\},$$

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$$
$$E_{11} = \Phi_{,22}, \qquad E_{12} = -\Phi_{,12}, \qquad E_{22} = \Phi_{,11}, \qquad \Delta\Delta\Phi = 0.$$

GENERAL RELATIVITY

$$ds^2 = g_{ij}dx^i dx^j.$$

Particle: $\delta \int ds = 0.$ Light: $ds = 0.$

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -\kappa T_{ij},$$
$$T_{ij} = (p+\rho)\lambda_i\lambda_j + pg_{ij}.$$

## QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, New York

*The Department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

### ON A GENERALIZATION OF THE "PROBLÈME DES RENCONTRES"

I. KAPLANSKY, University of Toronto

In the classical *problème des rencontres* we are required to find the number of arrangements of the integers 1, 2, $\cdots$, $n$ in which every integer appears out of its natural order. The familiar answer is

$$n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \cdots + (-)^n \frac{1}{n!}\right],$$

which can also be written $\Delta^n 0!$.*

The following generalization will now be proposed. Suppose we have $a_1 + a_2 + \cdots + a_n$ cards of which $a_r$ are marked "$r$" ($r = 1 \cdots n$). The cards marked "$r$" (or $r$-cards) are not to appear in any of $p_r$ specified places. No place, however, is to be forbidden simultaneously to both $r$-cards and $s$-cards. Under these restrictions the number of arrangements of the cards is:

$$F(a_1, p_1)F(a_2, p_2) \cdots F(a_n, p_n)0!,$$

where $F(a, p)$ is the operator

$$E^a - apE^{a-1} + \frac{a(a-1)p(p-1)}{1\cdot 2}E^{a-2} - \cdots.$$

* $E$ and $\Delta$ are finite difference operators, defined by the equations $Eu_n = u_{n+1}$, $\Delta u_n = u_{n+1} - u_n$.

If two $r$-cards are not regarded as distinct the answer should be divided by $a_1!a_2! \cdots a_n!$.

If all the $p$'s are zero the formula gives $(a_1+a_2+\cdots+a_n)!$, which is correct, since there is then no restriction upon the cards. We get a proof by induction upon observing the truth of the relation

$$F(a, p) = F(a, p-1) - aF(a-1, p-1).$$

It is perhaps of interest to examine some particular cases in the light of the general result.

Netto (*Lehrbuch der Combinatorik*, pp. 80–82) proposes the problem of finding the number of terms, in the expansion of a determinant, which contain no element from either diagonal. In the case of even order $2n$, it is clear that we can apply the above result with $a_r = p_r = 2$, obtaining

$$(E^2 - 4E + 2)^n 0!.$$

The first few terms of this series can readily be checked with those given by Netto.

Macmahon (*Combinatory Analysis*, vol. I, pp. 99–114) treats the case $a_r = p_r = \xi_r$, observing that the desired result is the coefficient of $x_1^{\xi_1}x_2^{\xi_2} \cdots x_n^{\xi_n}$ in the expansion of

$$(x_2 + x_3 + \cdots + x_n)^{\xi_1}(x_1 + x_3 + \cdots + x_n)^{\xi_2} \cdots (x_1 + x_2 + \cdots + x_{n-1})^{\xi_n}.$$

The results actually worked out on pp. 108–109 can rapidly be checked. For example, on page 109, we find

$$4\{0; 2^2 1^s\} = P_{s+4} - 4P_{s+3} + 2P_{s+2} + 4P_{s+1} + P_s.$$

Here $P_n$ denotes $\Delta^n 0!$, and the symbol on the left refers to the case when $s$ of the $\xi$'s are equal to 1, while the remaining two are equal to 2. Then the statement is:

$$(E^2 - 4E + 2)^2(E - 1)^s 0! = (\Delta^{s+4} - 4\Delta^{s+3} + 2\Delta^{s+2} + 4\Delta^{s+1} + \Delta^s)0!,$$

which is easily verified by putting $E = 1 + \Delta$.

It is possible, by similar methods, to obtain other general results in combinatory analysis. We can for example write down the number of permutations of $r$ different things taken $n$ at a time, under the condition that

(i) repetitions are allowed, but

(ii) the first element may not be taken 0, 2, 5, or 9 times,
the second element may not be taken 1, 2, or 6 times,
and so on.

The result is

$$G_1(E_r, E_n)G_2(E_r, E_n) \cdots G_r(E_r, E_n)0^n,$$

where $E_n$ acts on $n$, $E_r$ acts on the 0 in $0^n$, and in our example

$$G_1 = E_r - 1 - \binom{n}{2} E_n^{-2} - \binom{n}{5} E_n^{-5} - \binom{n}{9} E_n^{-9},$$

$$G_2 = E_r - \binom{n}{1} E_n^{-1} - \binom{n}{2} E_n^{-2} - \binom{n}{6} E_n^{-6}.$$

Here $E^{-s}0^n$ is to be taken as 0 for $s > n$. Of course the result is intelligible only if the imposed conditions are consistent.

It will be seen that the algebraic expressions used in this note would be extremely awkward to write down without the use of the operator $E$. Its use amounts to an algebraic shorthand; and, like all such, it not only enables formulas to be written compactly, but also stimulates new investigations.

## ANOTHER NOTE ON LINEAR OPERATORS

C. A. HUTCHINSON, University of Colorado

In the operational treatment of linear ordinary differential equations with constant coefficients, the following two formulas for particular integrals occur:

Let

$$F(D) = c_0 D^n + c_1 D^{n-1} + \cdots + c_{n-1} D + c_n,$$

where $c_0, c_1, \cdots, c_n$ are constants and $n$ is a positive integer. If $F(a) \neq 0$, then

$$\frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{F(a)} \cdot \tag{1}$$

If $F(a) = 0$, then there is a positive integer $r$ such that $F(D) = (D-a)^r \cdot \phi(D)$, where $\phi(a) \neq 0$, and

$$\frac{1}{F(D)} e^{ax} = \frac{x^r \cdot e^{ax}}{r! \cdot \phi(a)} \cdot \tag{2}$$

The object of this note is to show that both formulas are comprised in

$$\frac{1}{F(D)} e^{ax} = \frac{x^r \cdot e^{ax}}{F^{(r)}(a)}, \tag{3}$$

where $r$ is the smallest positive integer for which $F^{(r)}(a) \neq 0$.

If $F(a) \neq 0$, then $r = 0$, and (3) is true, by (1). If $F(D) = (D-a)^r \cdot \phi(D)$, where $\phi(a) \neq 0$, then we readily find that $F^{(k)}(a) = 0$, for $k < r$, and since

$$F^{(r)}(D) = (D - a)^r \cdot \phi^{(r)}(D) + \binom{r}{1} \cdot r \cdot (D - a)^{r-1} \cdot \phi^{(r-1)}(D)$$
$$+ \cdots + \binom{r}{r-1} \cdot r! \cdot (D - a) \cdot \phi'(D) + r! \cdot \phi(D),$$

we observe that

$$F^{(r)}(a) = r! \cdot \phi(a).$$

Then (3) follows at once, by (2).

## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

### REVIEWS

*International Encyclopedia of Unified Science.* Encyclopedia and Unified Science, vol. 1, no. 1. By Otto Neurath, Niels Bohr, John Dewey, Bertrand Russell, Rudolf Carnap and Charles W. Morris. 8+75 pages. $1.00. Foundations of the Theory of Signs, vol. 1, no. 2. By Charles W. Morris. 7+59 pages. $1.00. Procedures of Empirical Science, vol. 1, no. 5. By Victor F. Lenzen. 7+59 pages. $1.00. Chicago, The University of Chicago Press, 1938.

These three pamphlets are part of the ambitious undertaking which is outlined in the first of them. The plan for the *Encyclopedia* grew out of the International Congresses for the Unity of Science, which represent a movement for the cooperation of scientists in all fields. The work as projected will begin with two volumes, entitled *Foundations of the Unity of Science,* of ten numbers each, to be followed by other volumes in which it is hoped to systematize the special sciences both separately and in relation to one another.

Such a project may well take more than a generation to its completion, if it is to be so thoroughgoing as Professor Neurath indicates in his article *Unified Science as Encyclopedic Integration*; it is obviously impossible to estimate its ultimate value on the basis of these three numbers from Volume I. In so far as they may be taken to indicate the nature of the whole, it appears likely to have more appeal to students of philosophy than to scientists active in either research or teaching. The articles in Number 1 are more or less interesting and fairly readable, from the point of view of the general reader, but have nothing specifically to say to the mathematician. Dewey's appeal for the extension of the scientific attitude to matters of daily life is timely, if not new. Bohr and Russell briefly bestow their blessings on the enterprise. Professors Carnap and Morris in their articles give perhaps the best indication of the direction in which the effort to unify science is likely to lead. Carnap, seeking a common basis for the empirical sciences, finds it in the "physical language," which, he says, is reducible, by the process of definition, to "observable thing-predicates,"—*i.e.*, the sensory qualities of objects. One naturally wonders how this applies to the Bohr atom and other concepts of modern physics. Morris would have scientists generalize the methods of empirical science and apply them to science itself as a subject. Whether a practicing scientist could undertake this "metascience" without losing touch with the realities of his own science is a doubtful question.

This difficulty is well illustrated in Number 2, in which Morris gives an

introduction to semiotic, the study of signs, in which he says, "logic, mathematics, and linguistics can be absorbed in their entirety." The mathematician who would therefore master semiotic, finds, unfortunately, that he must first master a formidable special vocabulary, and at the end of the pamphlet he seems to have attained no nearer to any application to mathematics.

Professor Lenzen's monograph is extremely readable. The parallels he draws between the biological and the physical sciences are illuminating. While not beyond the grasp of any educated layman, this pamphlet will be found stimulating by the scientist, and should be required reading for graduate students in scientific subjects. But it is hard to see in this factual discussion and comparison, by a scientist, of method in the various sciences any hint of that mystical "metascience" which Morris demands of scientists. In fact, Lenzen specifically concludes that the unity of science lies at present only in unity of procedure and that "it may well be doubted that a limited set of principles will systematize both physical and biological laws." It will be interesting to see whether these conclusions can be upset in the completed *Encyclopedia*, but to the reviewer this seems extremely unlikely.

CONSTANCE BALLANTINE

*Introduction to Economic Statistics.* By William Leonard Crum, Alson Currie Patton, and Arthur Rothwell Tebbutt. New York and London, McGraw-Hill Book Co., 1938, 11+423 pages. $4.00.

This book is divided into the following three parts: Part I, Statistical Data; Part II, General Analytical Methods; Part III, The Analysis of Time Series.

The first two chapters define, in much elaboration, the tasks of the economic statistician, statistical items, statistical series, economic variables, and homogeneity.

The two following chapters present in many details ideas concerning primary and secondary sources, quality of sources, and the collection of data.

Chapters V and VI treat of methods for constructing general statistical tables, working tables, summary tables, and tables for publication.

Part I closes with three chapters on charting categorical series, time series, and frequency series, together with several illustrations.

A student beginning the study of statistics will find this part of the book rather abstract and heavy reading.

The first chapter of the second part lays down six requisites for an average, defines the mean, median, geometric mean, harmonic mean, weighted average, and discusses rates and ratios. Examples illustrating the geometric and harmonic means do not show the need of using these means. The reader concludes the chapter wondering why so many means were found pertaining to adjusted indexes (adjusted for seasonal variations), of which he knows very little.

Chapter XI treats of the properties of averages for frequency series and

methods for computing these averages. The student is not informed why so many measures of central tendency are obtained for data pertaining to the capital of National Banks in Michigan and weekly earnings of machine bookkeepers. The meaning and use of the exponential and inverse exponential averages for a series are poorly explained. The student reading this chapter is bewildered, since he is not instructed as to when to use the different means; he gets the idea that all of the means should be computed for each series. Comparisons of the various averages are inadequate. The authors have shown clearly, by examples, how to calculate the various means, but interpretations of the results of these calculations are rather meager.

The following chapter introduces dispersion through interquartile range, the average deviation, standard deviation, and coefficients of dispersion, together with methods of computing them. More stress is placed on calculating dispersion than on interpretation.

The next two chapters contain information relating to the properties of the normal curve, graduating by use of the normal curve, the probable error concept, characteristics of a frequency distribution, and their probable errors, skewness, and curve fitting. The part devoted to sampling and the probable error of the mean is entirely too brief, for this is the most important part of statistics.

Part II concludes with three chapters on correlation tables, scatter diagrams, lines of regression, the linear correlation coefficient, standard error of estimate, correlation ratio, spurious correlation and partial correlation.

The student of economics will certainly not be bothered with many mathematical derivations of formulas; these have been reduced nearly to a minimum.

Part III opens with two chapters devoted to price relatives, index numbers, weighted index numbers, ways of computing index numbers, and tests for good index numbers.

Chapter XX treats of secular trend based on various time units, the least squares method being used for determining the trend lines. A method is given for removing trend from the raw data.

The chapter following presents clearly the meaning of seasonal variations by use of graphs, by various averages, link relative and chain indexes; it also contains methods well illustrated for removing seasonal variation in time series for various time units.

The last two chapters in Part III define cycles in a clear and concise manner, contain methods for comparing cyclical fluctuations pertaining to different time series, point out vividly the different usages of the word cycles, and give a splendid treatment of lags concerning time series. They set forth tersely ways of finding the amount of lag by use of correlation coefficients and the ways of forecasting. These last chapters contain a great deal of useful information.

Economics students will find Part III to be the best part of the book.

Appendices A, B, C, and D contain numerical data for certain charts of

Parts I and III, laboratory procedure, ordinates, and areas pertaining to the normal curve and a table of common logarithms of numbers.

This book in the hands of a skillful teacher should prove to be a good classroom text.

W. D. Baten

*The physical treatises of Pascal: The Equilibrium of Liquids and The Weight of the Mass of the Air.* Translated by I. H. B. Spiers and A. G. H. Spiers. (With notes and introduction by Frederick Barry.) New York, Columbia University Press, 1937. 26+181 pages.

This book is Number XXVIII of the *Records of Civilization, Sources and Studies*, edited under the auspices of the Department of History, Columbia University. This particular work is published under the supervision of Professor Frederick Barry, associate professor of history at Columbia, who wrote the foreword and bibliographical notes. We find here, in an English translation, Pascal's "Traitez de l'équilibre des liqueurs et de la pesanteur de la masse de l'air," which were probably written in 1654 and were published posthumously at Paris in 1663. Added are translations of parts of Stevin's book on Statics, of Galileo's remarks on Nature's abhorrence of a vacuum, and of Torricelli's letters on the pressure of the atmosphere, to which reference has been made in Pascal's text.

It is always good to have such writings together in an English translation, though it is not difficult, in this case, to have access to Pascal's original text, which is also a classic of scientific style and has been published in the collected works of Pascal, in the edition of Brunschvicg and Boutroux (1904–14), as well as in the later one of Strowski (1923–31). The forty pages of appendices on Stevin, Galileo and Torricelli, together with the notes of Professor Barry, make the work more nearly complete as a historical introduction to the theory of fluid equilibrium.

Pascal's interest in these problems dates back at least to the time that he had heard of Torricelli's experiments on the suspension of mercury in a barometric tube, which Torricelli described in 1644 in a letter to M. Ricci, reprinted in translation in the appendix of the present volume. Pascal performed the experiment repeatedly and saw the relation between the suppositional atmospheric pressure and the hydrostatic pressure explained by Stevin. Pascal's work shows a remarkable interplay of deductive and experimental procedure, and it ends with what seems a complete verification of all his contentions.

The papers of Pascal, reprinted in the present volume, are not all that Pascal wrote on the subject. They represent, however, a most comprehensive and concise account of his experiments and thoughts. For a complete study of Pascal's relations to this subject we must turn to the other places, quoted by Professor Barry.

The quaint pictures of the old edition are nicely reproduced.

D. J. Struik

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.*

## UNDERGRADUATE PUBLICATIONS

From time to time we have noted publications or bulletins issued by mathematics clubs. These publications appear in varied form, ranging from a few mimeographed pages to bound volumes.

One of the more elaborate club publications comes from Brooklyn College, where it has been published for six years under the title, *The Math Mirror*. The 1938 edition was noteworthy for the following articles: Pascal-Brianchon theorem, by Jacob Halpern and Leon Ginsberg; Brooklyn College student-teachers, by Professor W. L. Schaaf; An extension of Fermat's general theorem, by I. H. Rose; To bisect an angle with compasses only, by Benjamin Liebowitz; Brooklyn College student proves Fermat's last theorem! (or does he?), by M. Cohen; The deltoid, by Israel Rose; A problem in the theory of numbers, by Max Famely; Two interesting cubics, by Emanual Mehr; The distributive law in vector analysis, by M. S. Cohen. *The Math Mirror* also contains departments devoted to mathematical contests, and to activities of Pi Mu Epsilon and the Mathematics Club.

Club bulletins were also received this year from Kappa Mu Epsilon of Kansas State Teachers College at Emporia, Mississippi Beta chapter of Kappa Mu Epsilon, and Kappa Mu Epsilon of Illinois State Normal University.

## CLUB REPORTS, 1937–38

### *Pi Mu Epsilon, Brooklyn College*

A report of the year's program included discussions on Bell's *Men of Mathematics*, and *Continuous Geometries* as developed by von Neumann. The chapter sponsored the second annual Metropolitan Intercollegiate Contest in which Cooper Union placed first, with Brooklyn College and Columbia University close behind. In the spring, Abraham Hillman, Maurice Schuklin and Bernard Sherman were chosen to represent Brooklyn College in the William Lowell Putnam Intercollegiate Competition. The chapter also sponsored a joint meeting of the New York chapters at which Professor MacNeish spoke on The place of mathematics in the curriculum.

### *Kappa Mu Epsilon, Mississippi State College*

Under programs for the year we find that some of the meetings included discussion on the following topics: Impossibility of making a planet trip; Construction of a decagon inscribed in a circle; Humorous solution of the tortoise and hare problem; Design of future automobile bodies; Factoring of numbers by inspection; Solution of problems on magic squares; Extraction of roots by use of the binomial theorem; A four-element number system; Special proof applied to the Pythagorean theorem. The outstanding paper was presented by a special guest, Eckford Cohen, Starkville high school boy, who gave his original solutions of the cubic and quartic equations. On March 17, when Mississippi State College had its first Engineer's Day, the local chapter of K.M.E. placed on exhibit certain mathematical oddities and geometrical problems, including explanations of the mathematical principles involved in the various instruments used by the engineer.

### *Pi Mu Epsilon, University of Nebraska*

The year's activities included one meeting at the college observatory, an initiation banquet in February at which twenty-four new members were admitted, an initiation picnic in May with seventeen members added, and prize examinations in analytic geometry and calculus sponsored by the chapter. The ten dollar awards in each were won by Roland Fricke and Ted Nelson, respectively.

### *Kappa Mu Epsilon, Alabama College*

One of the outstanding features of the year's program was the open meeting in November at which all students in the department of mathematics were guests. Topics discussed at the other monthly meetings were: Flatland; Mathematical games; Galois and the theory of groups; Mathematical recreations; The valuable nothing; Mathematical magic; New names in mathematics; Mathematical curiosities.

### *Pi Mu Epsilon, University of Alabama*

The report of this chapter suggests a particularly active program of discussions on mathematical topics at the regular meetings. These included: Difference operators; On volumes bounded by cylindrical surfaces; René Descartes; Sign language; The solution of a certain linear partial differential equation of the first order; An application of congruences to secret codes; Some properties of the differential equation $Mdx+Ndy=0$, where $M$ and $N$ are polynomials; The determination of square roots of a given matrix; Some properties of the Galois field of order 9; Conformal representation of the function $W=\log (z-1)(z-w)(z-w^2)$; A locus problem associated with the polar of a point with respect to a conic. A bridge party in December and a picnic in May furnished the social side of the club's calendar.

Director, Mary E. Forman; Vice-Director, F. H. Mitchell; Secretary, W. F. Adams; Treasurer, H. S. Thurston; Librarian, Dr. F. A. Lewis.

### *Kappa Mu Epsilon, Kansas State Teachers College at Pittsburg*

During the year four open meetings were held, with discussions by students and members of the staff on such topics as: Decimalization of industry; Mathematics as a tool subject or as a system of thought; Mathematical numbers and operations considered as having directions; Photography; Let's check the hypothesis; Attempts at solving industrial differences; Milestones in the evolution of number concepts; Difference between algebraic and transcendental numbers, Graphic solution of a linear and of a quadratic equation. A Christmas party, an initiation banquet, and a spring picnic rounded out the year. A summer chapter meeting was held in July, with seventy-one members in attendance. Short demonstrations were given on the use of the angle mirror, hypsometer and clinometer, the transit, and the plane table.

President, Willetta German; Vice-President, Lysle Mason; Secretary, Aileene Kingsbury; Treasurer, Joseph Campbell; Corresponding Secretary, W. H. Hill.

### *Mathematics Clubs, St. Lawrence University*

Pi Mu Epsilon and Alpha Mu Gamma, local mathematics clubs founded in 1917, hold joint meetings bi-monthly throughout the school year. The program included student discussions on: Short cuts in mathematics; Logarithms; Theory and use of special purpose slide rules; Uses of the carpenter's square; Non-Euclidean Geometry; The sextant and its use; Opportunities in actuarial mathematics; Theory and use of the planimeter, differentiator, and integraph. The past year these organizations jointly sponsored a mathematics puzzle contest open to all undergraduate students, which aroused a great deal of interest on the campus and in the town. The prizes awarded were two polyphase duplex slide rules. The clubs have in mind sponsoring an interscholastic mathematical contest for neighboring high schools.

President, Thomas Branagan; Vice-President, Helen Gilbert; Secretary-Treasurer, John Burgess; Director, Dr. O. Kenneth Bates.

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Connecticut State College, Storrs, Connecticut.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

## PROBLEMS FOR SOLUTION

E 371. *Proposed by H. T. R. Aude, Colgate University.*

A box contains a sufficient number of coins, which are pennies, dimes and half-dollars. Find the smallest sum, also the largest sum, which can be counted out in two and only two ways, by using one hundred coins including at least one of each kind.

E 372. *Proposed by Virgil Claudian, Bucharest, Roumania.*

The variable point $Q$ moves on a circle through the fixed point $A$, and $B$ is another fixed point in the same plane. The points $R$ and $S$ are the feet of the perpendiculars from $A$ and $Q$ on $BQ$ and $AB$ respectively. The line through $B$, parallel to $RS$, meets $AQ$ at $P$. Find the locus of $P$.

E 373. *Proposed by David Segal, Kosow Huculski, Poland.*

Show that, for every odd positive integer $n$, there exists a denumerable set of number pairs, $(a, b)$, such that $a^2+b^2$ is the $n$th power of an integer.

E 374. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

What relationship exists between the sides of triangle $ABC$ if the bisector of angle $A$, the median from vertex $B$ and the altitude from vertex $C$ are concurrent? Can the three sides be commensurable if the triangle is not equilateral?

## SOLUTIONS

E 310 [1938, 479]. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form $aabb$ which is the square of $bb$. Show that the numbers $b$ and $a^2+4(a-1)^2$ are perfect squares. Determine the base of such a system and the values of $a$ and $b$, knowing that $a$ is also a perfect square.

*Solution by E. P. Starke, Rutgers University.*

Put $R$ for the base. By hypothesis we have $(aR^2+b)(R+1)=b^2(R+1)^2$, or $aR^2+b=b^2(R+1)$. Hence $b^2-b$ is divisible by $R$, and we may put $b^2=b+kR$, and thus have $aR^2+b=(b+kR)(R+1)$. This gives $R=(k+b)/(a-k)$. Since $b<R$ and $k<R$ by definition, $k+b<2R$. Consequently, $k+b=R$ and $a-k=1$.

Then $R=a+b-1$ and $k=a-1$, so that the equation $b^2=b+kR$ becomes

$$b(b-a)=(a-1)^2. \tag{1}$$

Since this shows that every common factor of $a$ and $b$ must divide 1, it follows that $a$ and $b$ are relatively prime, and so are $b$ and $b-a$. Thus (1) implies that $b$ and $b-a$ are separately squares. Hence we write

$$b=u^2, \qquad b-a=v^2, \qquad a-1=uv. \tag{2}$$

From this it follows that $a^2+4(a-1)^2=(u^2-v^2)^2+4u^2v^2=(u^2+v^2)^2$, which proves the first part of the theorem.

The result of eliminating $a$ and $b$ from (2) is $u^2-uv-1=v^2$, or (replacing $u+2v$ by $w$), $w^2+4=5u^2$. Such a Pell equation arises frequently and yields an easy solution by means of reduction formulae. In this case we obtain

$$u_{n+1}=3u_n-u_{n-1}, \qquad w_{n+1}=3w_n-w_{n-1} \tag{3}$$

with $u_0=1$, $w_0=1$, $u_1=2$, $w_1=4$. Other values for $(u_n, w_n)$ are (5, 11), (13, 29), (34, 76), (89, 199), etc. Hence the first few values of $(a, b, R)$ are given by (3, 4, 6), (16, 25, 40), (105, 169, 273).

The requirement that $a$ be also a square, say $a=t^2$, demands, according to (2), that $v^2+t^2=u^2$, with $t$, $u$ and $v$ relatively prime. Hence either $v=2rs$, $t=r^2-s^2$ and $u=r^2+s^2$, or else $v=r^2-s^2$, $t=2rs$ and $u=r^2+s^2$. But $a=uv+1$ then becomes $(r^2-rs+s^2)^2=5r^2s^2+1$, or $(s^2+2r^2)^2=5r^4+1$, respectively, for which the only apparent solution in positive integers is $r=2$, $s=1$; resulting in $a=16$, $b=25$, $R=40$, given above.

Also solved by C. W. Trigg and the proposer.

E 327 [1938, 248]. *Proposed by Philip Franklin, Massachusetts Institute of Technology.*

It was shown by Fermat that, for any number $k$, it is possible to find $k$ distinct right triangles, each with integral sides, having the same area. Find sets for $k=1, 2, 3, 4, 5$, having as common areas 6, 210, 840, $341880=2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 37$ and $3738746400=2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13^2\cdot 19$ respectively. Are these the smallest possible values?

*Solution by D. L. MacKay, Evander Childs High School, New York.*

If $x$ and $y$ are the generators, then the sides of a right triangle may be represented by $a=x^2-y^2$, $b=2xy$, $c=x^2+y^2$, and the area $s=xy(x^2-y^2)$. Then by substitution the following right triangles are obtained:

| $k$ | $x$ | $y$ | $a=x^2-y^2$ | $b=2xy$ | $c=x^2+y^2$ | $s=xy(x^2-y^2)$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 3 | 4 | 5 | 6 |
| 2 | 5 | 2 | 21 | 20 | 29 | 210 |
| | 6 | 1 | 35 | 12 | 37 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 7 | 3 | 40 | 42 | 58 | 840 |
| | 7 | 5 | 24 | 70 | 74 | |
| | 8 | 7 | 15 | 112 | 113 | |
| 4 | $22\sqrt{2}$ | $15\sqrt{2}$ | 518 | 1320 | 1418 | 341,880 |
| | 37 | 33 | 280 | 2442 | 2458 | |
| | 40 | 37 | 231 | 2960 | 2969 | |
| | 56 | 55 | 111 | 6160 | 6161 | |
| 5 | $11\sqrt{m}$ | $8\sqrt{m}$ | 155,610 | 480,480 | 505,050 | 37,383,746,400 |
| | $19\sqrt{n}$ | $8\sqrt{n}$ | 270,270 | 276,640 | 386,750 | |
| | $22\sqrt{p}$ | $3\sqrt{p}$ | 518,700 | 144,144 | 538,356 | |
| | $38\sqrt{q}$ | $11\sqrt{q}$ | 343,980 | 217,360 | 406,900 | |
| | $507\sqrt{3}$ | $32\sqrt{3}$ | 768,073 | 97,344 | 774,219 | |

where $m=2\cdot3\cdot5\cdot7\cdot13$, $n=2\cdot5\cdot7\cdot13$, $p=2^2\cdot3\cdot7\cdot13$, and $q=2^2\cdot5\cdot13$.

However, the following generators give a smaller value of $s$ for $k=5$

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | $96\sqrt{3}$ | $91\sqrt{3}$ | 2,805 | 52,416 | 52,491 | 73,513,440 |
| | $91\sqrt{3}$ | $85\sqrt{3}$ | 3,168 | 46,410 | 46,518 | |
| | 130 | 108 | 5,236 | 28,080 | 28,564 | |
| | $51\sqrt{6}$ | $40\sqrt{6}$ | 6,006 | 24,480 | 25,206 | |
| | $68\sqrt{3}$ | $42\sqrt{3}$ | 8,580 | 17,136 | 19,164 | |

E 328 [1938, 249]. *Proposed by C. A. Murray, West Texas Teachers College.*

In the equation, $y=x^3+px^2+qx$ show how to determine all pairs of integral values of $p$ and $q$, for which the equation $y=0$ will have distinct integral roots, and the two bend points integral coördinates.

*Solution by E. P. Starke, Rutgers University.*

The abscissae of the bend points are roots of $3x^2+2px+q=0$. If these are integers, $p$ and $q$ must be multiples of 3. Let $p=3P$ and $q=3Q$. Then $x^2+2Px+Q=0$, must have integer roots, and so must $x^2+3Px+3Q=0$. If $d$ is any integer divisor of $P$ greater than unity, and if $d^2$ divides $Q$, we may set $x=zd$ and get $z^2+3(P/d)z+3Q/d^2=0$ and $z^2+2(P/d)z+Q/d^2=0$ as two new equations whose roots are all integers. Hence we need only investigate those cases in which $q$ is not divisible by the square of any divisor of $p$. [For we may multiply any $p$ and $q$ which comply with these conditions by $d$ and $d^2$ respectively, and thus get the remaining cases.]

Since $x^2+2Px+Q=0$ and $x^2+3Px+3Q=0$ both have integer roots, their discriminants must be perfect squares. That is, we may set $9P^2-12Q=(3a)^2$ and $P^2-Q=b^2$. Then $P^2=4b^2-3a^2$. By the preceding paragraph, $a$, $b$ and $P$ are relatively prime. Thus we may, in $3a^2=(2b+P)(2b-P)$, put $(2b+P)=3au/v$ and $(2b-P)=av/u$, where $u$ and $v$ are relatively prime integers. Then $b/a=(3u^2+v^2)/4uv$, and $P/a=(3u^2-v^2)2uv$. Since $a$, $b$ and $P$ are relatively prime, also $u$ and $v$, we must have either $a=2uv$, $b=(3u^2+v^2)/2$, $P=3u^2-v^2$, $Q=3(u^2-v^2)(9u^2-v^2)/4$; or else $a=4uv$, $b=3u^2+v^2$, $P=6u^2-2v^2$, $Q=3(u^2-v^2)$

$(9u^2-v^2)$, according as $u$ and $v$ are both odd, or one even and one odd. [Since only the squares of $u$ and $v$ appear in $p$ and $q$, we need only consider their positive values. If either $u$ or $v$ were zero, the original cubic equation would have a double root, contrary to hypothesis.] These conditions compel the bend points to have also integer ordinates. Hence the complete solution sought may be expressed in the form, $p=6d(3u^2-v^2)$, and $q=9d^2(u^2-v^2)(9u^2-v^2)$, where $u$ and $v$ are relatively prime integers, and $d$ is an integer or half an integer according as one or both of $u$ and $v$ are odd.

Conversely, with these values of $p$ and $q$, our cubic equation becomes

$$y = x^3 + 6d(3u^2 - v^2)x^2 + 9d^2(9u^2 - v^2)(u^2 - v^2)$$

and the equation, $y=0$, has the roots, $x=0$, $3d(v+3u)(v-u)$, $3d(v-3u)(v+u)$. The derived equation, $3x^2+4d(3u^2-v^2)x+3d^2(9u^2-v^2)(u^2-v^2)=0$, has the integer roots, $x=3d(v+u)(v-u)$, $d(v-3u)(v+3u)$, as abscissae of the bend points.

Also solved by H. L. Lee.

E 329 [1938, 249]. *Proposed by V. Thêbault, Le Mans, France.*

How may we locate two points on the sides of a triangle so that the segment joining them may divide the area in a given ratio and have a maximum or minimum length?

*Solution by V. W. Graham, Harcourt Street High School, Dublin, Ireland.*

Let $ABC$ be the given triangle, with $AC<AB$. Take $AB$ as unit length and mark off on it $AP=k$, the given ratio. Find a mean proportional between $AB$ and $AP$ and lay it off on $AB$ as $AD$. Then $AD^2=AB\cdot AP=k$. Through $D$ draw $DE$ parallel to $BC$, cutting $AC$ at $E$. Now the areas of triangles $ADE$ and $ABC$ are in the ratio, $AD^2/AB^2=k$. Then triangle $ADE$ has $k$ times the area of triangle $ABC$. Also, $AE<AD$.

Find a mean proportional between $AD$ and $AE$, and lay it off on $AB$ as $AX$. Lay off $AY$ on $AC$, equal to $AX$. Then $X$ and $Y$ are the required points with reference to vertex $A$. [Similar constructions will yield lines corresponding to the other two vertices of triangle $ABC$, and the shortest of the three is the desired line.] $XY<DE$ because $DE^2=AD^2+AE^2-2AD\cdot AE\cdot\cos A$, and $XY^2=AX^2+AX^2-2\ AX\cdot AX\cdot\cos A$, so that the difference, $DE^2-XY^2=AD^2+AE^2-2\ AX\cdot AX=(AD-AE)^2$, which is necessarily positive. $XY$ is thus not merely less than $DE$, but also less than any other line, $FG$, with $F$ on $AB$ and $G$ on $AC$, such that the area of triangle $AFG$ equals that of triangle $AXY$, by an identical argument. (The areas may best be compared by the formula, $S=\frac{1}{2}AD\cdot AE\cdot\sin A$).

Also solved by W. B. Clarke, D. L. MacKay, E. P. Starke and the proposer.

E 330 [1938, 249]. *Proposed by Fred Discepoli, New York.*

Find a triangular number whose digit pattern in the decimal system is *aabb*, and determine whether or not the solution is unique.

*Solution by C. W. Trigg, Los Angeles City College.*

$N=\frac{1}{2}n(n+1)=aabb=11(100a+b)$. Hence $n$ is of the form, $11k$, or $11k-1$. An examination of the eighteen values of $N$ corresponding to $44<n<143$, (the limits imposed on $n$ because $1000<N<10000$) reveals the unique solution, $n=66$, $N=2211$. In the course of this examination there appeared three triangular numbers of the form, $abba$, namely: 3003, 5995, and 8778.

Also solved by Walter Penney and the proposer.

E 331 [1938, 249]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Determine a triangle whose six angle bisectors, three altitudes, five radii, and area are rational numbers.

*Solution by E. P. Starke, Rutgers University.*

If the altitudes and area are rational, then of course the sides are rational, and so is $s$, the semi-perimeter. Then also the inradius is rational. The circumradius is $a/2 \sin A=a/4 \sin \frac{1}{2}A \cos \frac{1}{2}A$, and the radii of the escribed circles are $s \tan \frac{1}{2}A$, $s \tan \frac{1}{2}B$, and $s \tan \frac{1}{2}C$.

It is not difficult to show that the internal and external bisectors of angle $A$ have lengths $2bc\cdot\cos \frac{1}{2}A/(b+c)$ and $2bc\cdot\sin \frac{1}{2}A/|b-c|$, respectively, with analogous formulas for the other bisectors.

These will all be rational if the sides are rational and the sines and cosines of the half angles are rational. To effect this, choose side $a$ rational, and choose angles $A$ and $B$ such that $\sin \frac{1}{2}A$, $\sin \frac{1}{2}B$, $\cos \frac{1}{2}A$ and $\cos \frac{1}{2}B$ are rational. Then $b=a\cdot\sin \frac{1}{2}B \cos \frac{1}{2}B/(\sin \frac{1}{2}A \cos \frac{1}{2}A)$ and $\sin \frac{1}{2}C=\cos \frac{1}{2}(A+B)$ are rational, and similarly $\cos \frac{1}{2}C$ and $c$ are rational. Now if $\cos\theta$ and $\sin\theta$ are rational, we have, upon clearing $\sin^2\theta+\cos^2\theta=1$ of fractions, an integral relation of the form, $x^2+y^2=z^2$, whose solution in integers is well known. Hence we put $\cos \frac{1}{2}A =2pq/(p^2+q^2)$, $\sin \frac{1}{2}A=|p^2-q^2|/(p^2+q^2)$, $\cos \frac{1}{2}B=2mn/(m^2+n^2)$, and $\sin \frac{1}{2}B =|m^2-n^2|/(m^2+n^2)$, with any convenient rational value for $a$.

The isosceles triangle with sides 25, 25 and 14 meets all requirements except that one external bisector is infinite. The simplest triangle with integer sides which fully meets the requirements has sides 125, 154 and 169. These values result from $\sin \frac{1}{2}A=3/5$, $\cos \frac{1}{2}A=4/5$, $\sin \frac{1}{2}B=5/13$, $\cos \frac{1}{2}B=12/13$, and $a=169$. Then the area is $\frac{1}{2}ab\cdot\sin C=9240$; the three altitudes are 18480/169, 3696/25 and 120; the inradius is 165/4, the circumradius is 4225/48; the radii of the escribed circles are 168, 280/3 and 132; the internal bisectors are 23100/29, 4004/3 and 975/2.

Also solved by the proposer.

E 332 [1938, 249]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Prove that $1^p+2^p-3^p+4^p-5^p-6^p+7^p+8^p- \cdots +(2^n-1)^p=0$ for every positive integer $p<n$, where the sign of a term of the form $m^p$ is negative or positive according as $m$ is or is not one of the $2^k$ integers following $2^{k+1}$, where $k$ is any integer.

*Solution by V. W. Graham, Harcourt Street High School, Dublin, Ireland.*

$$1-(1-e^{x})(1-e^{2x})(1-e^{2^2x})\cdots(1-e^{2^{n-1}x})=e^{x}+e^{2x}-e^{3x}\cdots e^{2^{n-1}x}.$$

Equating the coefficients of $x^p/(\underline{|p})$, where $0<p<n$, we get

$$0=1^p+2^p-3^p\cdots(2^n-1)^p.$$

The rule for finding the sign of a term seems to be inaccurate, as it gives

$$1^p+2^p-3^p+4^p-5^p-6^p+7^p+8^p-9^p-10^p-11^p-12^p+13^p+14^p+15^p\cdots,$$

and we should get

$$1^p+2^p-3^p+4^p-5^p-6^p+7^p+8^p-9^p-10^p+11^p-12^p+13^p+14^p-15^p\cdots.$$

Also solved by E. P. Starke.

E 333 [1938, 319]. *Proposed by O. E. Henry, Pittsburgh Public Schools.*

The equations $SLED+SNOW=RIDE$ and $SLED-SNOW=BOB$ are the results of coding the digits of numbers by assigning to each digit a different letter which replaces it throughout the equations. Identify the numerals and determine whether or not the solution is unique.

*Solution by Mary L. Constable, Philadelphia High School for Girls.*

In the addition, $ED+OW=DE$, or $DE+100$. Therefore $OW=9(D-E)$ or $100-9(E-D)$. Thus $OW$ is a two-place multiple of 9 or 100 minus such a multiple, and lies among the seventeen values, 09, 18, 19, 27, 28, 36, 37, 45, 46, 54, 63, 64, 72, 73, 81, 82 and 91. [$W\neq 0$.] For any value of the letter $O$, the subtraction determines two values possible for $E$, but one of which can determine a corresponding value of $D$ consistent with the addition. Most of the above values of $OW$ thus determine values of $E$, $D$ and $B$ which must be ruled out because of the duplication of some digit. There remain the following six sets of subtraction columns (two right-hand columns only) $12-09=03$, $35-18=17$, $93-46=47$, $06-54=52$, $52-73=79$, $87-91=96$.

We may now use the first $B$ in $BOB$ to get accompanying values of $L$ and $N$. Noting that $S<5$, we next obtain corresponding values of $S$ and $R$. Duplicate digits appear in every case excepting that for which $SLED=2893$ and $SNOW=2146$, so that the solution is unique.

Also solved by W. E. Buker, Fred Discepoli, Wm. Douglas, L. S. Johnston, J. S. Leech, Helen T. Raudenbush, W. B. Rufus, M. A. Scheier, E. P. Starke, Harriet A. Welch and the proposer.

E 334 [1938, 319]. *Proposed by J. R. Musselman, Western Reserve University.*

The corners of two rectangles, named counterclockwise, are $A$, $B$, $C$ and $D$, and $D$, $E$, $F$ and $G$. (They share the vertex $D$.) Prove or disprove that the midpoints of the segments $AG$ and $CE$, together with the centers of the given rectangles, constitute the corners of another rectangle.

*Solution by L. S. Johnston, University of Detroit.*

Let $P$ be the center of $ABCD$, $Q$ the center of $DEFG$, $R$ the midpoint of $AG$, $S$ the midpoint of $CE$, $M$ the midpoint of $AD$ and $N$ the midpoint of $DC$. The quadrilateral $PSQR$ is a parallelogram, since $PR$ and $SQ$ are each parallel to, and equal to half of, $CG$. Now if also $PR$ is perpendicular to $PS$, the triangles $PMR$ and $PNS$ are similar, since $PM$, $RM$, and $PR$ are respectively perpendicular to $PN$, $NS$ and $PS$. Then $PM/PN=MR/NS$, or $CD/AD=DG/DE$. Thus the given rectangles are similar, with corresponding sides appearing in counterclockwise sequence from $A$ and $D$, respectively. Conversely, if the two given rectangles are similar and similarly placed with respect to $A$ and $D$ respectively, it is obvious that the triangles $PMR$ and $PNS$ are similar, and, since $PM$ and $MR$ are respectively perpendicular to $PN$ and $NS$, the sides $PR$ and $PS$ are perpendicular, and $PSQR$ is a rectangle.

It may be remarked that if $PSQR$ is a rectangle, then $AE$ and $CG$ intersect at one of the two intersections of the circumscribing circles of the two given rectangles, the other intersection being of course the vertex $D$.

In his solution, H. T. R. Aude points out that $PSQR$ will also be a rectangle, independent of the similarity of the given rectangles, if $DE$ and $DG$ chance to lie along $DA$ and $DC$ respectively.

One member of the Association "proved" that $PSQR$ would always be a rectangle.

Also solved by W. E. Buker, V. W. Graham, Jack Lorell, D. L. MacKay, E. P. Starke, C. W. Trigg and Harriet A. Welch.

E 335 [1938, 319]. *Proposed by J. H. Edmonston, Washington, D. C.*

In the proof of the binomial theorem for positive, integral exponents, use is made of the identity, ${}_{n+1}C_r={}_nC_r+{}_nC_{r-1}$. Prove the generalized proposition,

$$ {}_{n+t}C_r = \sum_{k=0}^{t} {}_tC_k \cdot {}_nC_{r-k}, \quad \text{for} \quad t \leqq r. $$

*I. Solution by E. P. Starke, Rutgers University.*

Consider ${}_{n+t}C_r$ from the point of view of the theory of combinations, letting $n+t$ elements be $t$ of one sort and $n$ of another. The various combinations of $n+t$ elements taken $r$ at a time may be separated into those containing 1 of the first sort, 2 of the first sort, $\cdots$, $t$ of the first sort. The number of combinations containing $k$ of the first sort, and hence $r-k$ of the second sort, is ${}_tC_k \cdot {}_nC_{r-k}$. Thus the sum with respect to $k$, of these products, is the total, ${}_{n+t}C_r$.

*II. Solution by V. W. Graham, Harcourt Street High School, Dublin, Ireland.*

$$ (1+x)^t = {}_tC_0 + {}_tC_1x + {}_tC_2x^2 + \cdots + {}_tC_tx^t. $$

$$ (1+x)^n = {}_nC_0 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n. $$

If we multiply out and collect terms, the coefficient of $x^r$ will be

$$_{n+t}C_r = {}_tC_0 \cdot {}_nC_r + {}_tC_1 \cdot {}_nC_{r-1} + {}_tC_2 \cdot {}_nC_{r-2} + \cdots + {}_tC_t \cdot {}_nC_{r-t},$$

if $r \geqq t$. If $r \leqq t$, we obtain a similar result,

$$_{n+t}C_r = {}_tC_0 \cdot {}_nC_r + {}_tC_1 \cdot {}_nC_{r-1} + {}_tC_2 \cdot {}_nC_{r-2} + \cdots + {}_tC_r \cdot {}_nC_0,$$

which is known as Vandermonde's Theorem.

Also solved by G. W. Petrie, Maude Willey, Samuel Zuckerman and the proposer.

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3906. *Proposed by W. B. Campbell, Drexel Institute.*

A model for a spool-type hyperboloid consists of two disks of radius $R$, free to rotate and slide on a rod of radius $r$, and a set of connecting thin strings of length $2h$. With the parts initially set up as a right circular cylinder, the strings serving as elements, the disks are rotated through equal angles $\theta$, in opposite senses. Discuss the resulting positions of the strings, especially after they come in contact with the rod, if the latter is (a) smooth, (b) rough.

3907. *Proposed by V. W. Graham, The High School, Dublin, Ireland.*

Find all the roots of the equation

$$\frac{(x^2 - x + 1)^3}{x^2(x-1)^2} = \frac{(a^2 - a + 1)^3}{a^2(a-1)^2}.$$

3908. *Proposed by Otto Dunkel, Washington University.*

Given a simplex $S$ in $n$ dimensions with the vertices $A_i$, $i = 1, 2, \cdots, n+1$, let $p_1, p_2, \cdots, p_{n+1}$ denote the normal homogeneous coördinates of a point $P$ with respect to the basis figure $S$, where the $p_i$'s are equal to or proportional to the respective $n+1$ distances of $P$ from the faces of $S$. Prove that the equation of the sphere circumscribing $S$ is

$$\sum_{i \neq j} \frac{p_i p_j}{h_i h_j} e_{ij}^2 = 0, \qquad (n+1)n/2 \text{ terms},$$

where $e_{ij} = \overline{A_i A_j}$ and $h_i$ is the length of the altitude of $S$ from $A_i$.

3889 [1938, 554]. *Correction.* In the second line from the bottom replace "Euler line" by "Euler circle, that is, the nine-point circle."

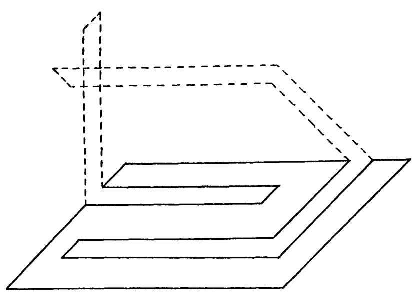
## SOLUTIONS

3763 [1935, 627]. *Proposed by Paul Erdös, The University, Manchester, England.*

Given any simple polygon $P$ which is not convex, draw the smallest convex polygon $P'$ which contains $P$. This convex polygon $P'$ will contain the area $P$ and certain additional areas. Reflect each of these additional areas with respect to the corresponding added side, thus obtaining a new polygon $P_1$. If $P_1$ is not convex, repeat the process, obtaining a polygon $P_2$. Prove that after a finite number of such steps a polygon $P_n$ will be obtained which will be convex.

*Solution by Bêla de Sz. Nagy, Szeged, Hungary.*

The process described in the above problem, *i.e.*, the reflection of *all* additional areas, does not always lead from a simple polygon to a simple one, as shown in the following example:



This means that the repeating of this process is not always possible.

In order to avoid this difficulty we modify the process in the following way. Instead of reflecting *all* additional areas mentioned in the problem we reflect *only one* of them, so obtaining obviously always a simple polygon again. We agree to define the process also for convex polygons as the process of leaving them invariant.

Let $A_1^0, A_2^0, \cdots, A_\sigma^0$ be the vertices of the given simple polygon $P^0$. Applying the process $n$ times leads to a polygon $P^n$, the points $A_\nu^0$ $(\nu=1, 2, \cdots, \sigma)$ being carried thereby into the points $A_\nu^n$. Let us denote by $C^n$ the least convex polygon containing $P^n$ in its interior. Each polygon in the sequence $P^0$, $C^0$, $P^1$, $C^1$, $P^2$, $C^2$, $\cdots$ contains obviously the foregoing ones in its interior. The lengths of all polygons $P^n$ being plainly the same, there is a circle containing all $P^n$'s in its interior. This implies that the sequence of the points $A_\nu^n$ $(n=0, 1, 2, \cdots)$ has at least one point of accumulation.

It follows readily from the nature of the above process that if $B$ is a point on, or inside of, $P^m$, then dist $(B, A_\nu^n) \leqq$ dist $(B, A_\nu^{n+1})$ for $n \geqq m$. Especially we

have: dist $(A_\nu^m, A_\nu^n) \leqq$ dist $(A_\nu^m, A_\nu^{n+1})$ for $n \geqq m$. From this it follows that the sequence of the points $A_\nu^n$ $(n=0, 1, 2, \cdots)$ may have only a single point of accumulation. It is thus convergent: $A_\nu^n \to A_\nu$ for $n \to \infty$.

The polygon $P=(\overline{A_1A_2}, \overline{A_2A_3}, \cdots, \overline{A_{\sigma-1}A_\sigma}, \overline{A_\sigma A_1})$, being the limit of the sequence $P^n$, is also the limit of the sequence $C^n$ and is therefore convex.

Denote by $c_\nu(r)$ the interior of the circle of radius $r$ drawn around $A_\nu$ as center.

Let $A_\mu$ be a convexity-point of $P$ (i.e., such that $A_{\mu-1}, A_\mu, A_{\mu+1}$ do not lie on the same straight line; $A_\sigma$ being denoted also as $A_0$, $A_1$ as $A_{\sigma+1}$). We may find then obviously a straight line $L$ and a positive number $\rho$ such that $c_\mu(\rho)$ lies wholly on one side of $L$ while all $c_\lambda(\rho)$ $(\lambda \neq \mu)$ lie on the other side. For $n \geqq n_0(\mu)$ we shall certainly have: $A_\nu^n \in c_\nu(\rho)$ for $\nu=1, 2, \cdots, \sigma$. $L$ separates thus $A_\mu^n$ from the other points $A_\lambda^n$ $(\lambda \neq \mu)$. Hence $A_\mu^n$ is a convexity-point of $P^n$. It must be therefore invariant: $A_\mu^{n+1}=A_\mu^n$. This implies that for $n \geqq n_0(\mu)$: $A_\mu n_0(\mu)=A_\mu^n$. So is $A_\mu^n=A_\mu$ for $n \geqq n_0(\mu)$.

Let now $A_{\mu_1}, A_{\mu_2}, \cdots, A_{\mu_s}$ be all the convexity-points of $P$. We have then $A_{\mu_r}^N=A_{\mu_r}$ $(r=1, 2, \cdots, s)$ for $N=\max(n_0(\mu_1), n_0(\mu_2), \cdots, n_0(\mu_s))$.

This involves that $C^N=P$ and therefore also that $P^n=P$ for $n \geqq N$. We thus obtain after a finite number of steps a convex polygon indeed.

3817 [1937, 111]. *Proposed by V. Thébault, Le Mans, France.*

Parallels, with arbitrary direction, drawn through the vertices $A$, $B$, $C$ of a triangle cut any given transversal $\Delta$ of the plane in $\alpha$, $\beta$, $\gamma$. The parallels to $BC$, $CA$, $AB$ through $\alpha$, $\beta$, $\gamma$ determine, by intersections of suitable pairs of the nine lines, a triangle $A_1B_1C_1$ symmetrically equal to triangle $ABC$ (see J. Neuberg, *Wiskundij Tydschrift*, t.X. p. 80). Show that the center of symmetry of $ABC$ and $A_1B_1C_1$ describes the Newton line of the quadrilateral $(ABC, \Delta)$ when the direction of $A\alpha$, $B\beta$, $C\gamma$ varies.

*Solution by J. H. Butchart, Phillips University, Enid, Okla.*

Let the position vectors of $A$, $B$, $C$ be $\mathbf{a}$, $\mathbf{b}$, and zero respectively. Let the transversal meet $CA$ and $CB$ in points whose position vectors are $k\mathbf{a}$ and $l\mathbf{b}$. Use $\mathbf{a}+m\mathbf{b}$ as the vector determining the direction of the parallels through the vertices. Then the position vectors of $\alpha$, $\beta$, $\gamma$ are respectively:

$$[kl(\mathbf{a} + m\mathbf{b}) + m(k\mathbf{a} - l\mathbf{b})]/(l + km),$$
$$[kl(\mathbf{a} + m\mathbf{b}) - (k\mathbf{a} - l\mathbf{b})]/(l + km),$$
$$kl(\mathbf{a} + m\mathbf{b})/(l + km).$$

The vectors to $A_1$, $B_1$, $C_1$ are accordingly:

$$[kl(\mathbf{a} + m\mathbf{b}) - l\mathbf{a} + l\mathbf{b}]/(l + km),$$
$$[kl(\mathbf{a} + m\mathbf{b}) + km\mathbf{a} - km\mathbf{b}]/(l + km),$$
$$[kl(\mathbf{a} + m\mathbf{b}) + km\mathbf{a} + l\mathbf{b}]/(l + km).$$

We see at once that the vectors $B_1C_1$ and $A_1C_1$ are $\mathbf{b}$ and $\mathbf{a}$. The midpoint of the line $CC_1$ has the position vector $[km(\mathbf{a}+l\mathbf{b})+l(k\mathbf{a}+\mathbf{b})]/2(l+km)$. Hence the center of symmetry moves along the line announced.

Solved also by O. J. Ramler, C. E. Springer and the proposer.

*Editorial Note.* The solutions by Ramler and Springer used areal coördinates; and Ramler showed that the locus of the center of symmetry is parallel to the isotomic conjugate of $\Delta$. A synthetic proof is simple. Let $\Delta$ cut $BC$, $CA$, $AB$ in $L$, $M$, $N$; and let $(P)$ be the locus of $P$, the center of similitude of the triangles $ABC$ and $A_1B_1C_1$. When the direction of the parallels is that of $BC$, the points $A_1$, $\beta$, $\gamma$ coincide in $L$; $A$, $\alpha$, $B_1$, $C_1$ lie on a straight line parallel to $BC$; and $P$ is the midpoint of $AL$, $BB_1$, $CC_1$. There are two other similar cases; when the direction of the parallels is that of $CA$, and that of $AB$. It follows that $(P)$ passes through the midpoints of $AL$, $BM$, $CN$, the diagonals of the quadrilateral $(ABC, \Delta)$. Also we have three cases for which the triangles $A_1B_1C_1$ are congruent and symmetric to $ABC$. We shall prove that $(A_1)$, the locus of $A_1$, is a straight line, and similarly for $(B_1)$ and $(C_1)$; it will then follow from the above that these three loci are parallel, and that $A_1B_1C_1$ is always the symmetric of $ABC$, and that $(P)$ is a straight line $d$ parallel to $(A_1)$, $(B_1)$, $(C_1)$, and finally that the midpoints of the diagonals of $(ABC, \Delta)$ lie on the straight line $d$.

The points $\beta$ and $\gamma$ form a projective range on $\Delta$ with $L$ and the point at infinity $\infty_\Delta$ on $\Delta$ as self-corresponding points. Hence $C_1A_1$ and $A_1B_1$ are corresponding elements of two projective pencils with centers $\infty_{CA}$ and $\infty_{AB}$, the points at infinity on $CA$ and $AB$. The line at infinity $\infty_\Delta \infty_{CA} \infty_{AB}$ is a self corresponding element, and hence the locus $(A_1)$ is a straight line. Similarly, $(B_1)$ and $(C_1)$ are straight lines, and the proof is complete.

The last part of the proof may be put in a different form. From what has been said about the ranges of points $\beta$ and $\gamma$, it follows that $L\beta/L\gamma$ is a constant, and hence the similar triangles $A_1\beta\gamma$ have $L$ as the center of similitude. Thus $(A_1)$ is a straight line through $L$.

3818 [1937, 111]. *Proposed by V. Thébault, Le Mans, France.*

Consider a triangle $ABC$, a transversal $\Delta$, passing through the orthocenter $H$, which cuts $BC$, $CA$, $AB$ in $\alpha$, $\beta$, $\gamma$, and the line $\Delta'$ which joints the orthocenters $H$, $H_a$, $H_b$, $H_c$ of triangles $ABC$, $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$. Show that: (1) The line $\Delta$ and the Newton lines of the quadrilaterals $(ABC, \Delta)$ and $(ABC, \Delta')$ meet in a point $P$. (2) The sides of the triangles symmetrically equal to $ABC$, $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ with respect to $P$ pass respectively through the orthocenters $(H_a, H_b, H_c)$, $(H, H_b, H_c)$, $(H, H_c, H_a)$, $(H, H_a, H_b)$.

*Solution by Otto J. Ramler, Catholic University of America.*

Consider the sides of the triangle $ABC$ and the transversal $\Delta$ as tangents to a parabola $y^2=4ax$. Then the equations of the lines $BC$, $CA$, $AB$ and $\Delta$ may be expressed respectively as $x-m_iy+am_i^2=0$, $(i=1, 2, 3, 4)$. The orthocenter of $ABC$ has coördinates $(-a, as_1+as_3)$ where $s_1=m_1+m_2+m_3$, $s_3=m_1m_2m_3$. Since $\Delta$ passes through the orthocenter we have

$$(1)\qquad m_4^2 - m_4(s_1 + s_3) - 1 = 0.$$

Since the abscissa of the orthocenter is independent of $m_i$, we have a verification of the well known theorem that the orthocenters of all triangles circumscribed to a parabola lie on the directrix. Hence the line $\Delta'$ is the line $x=-a$. The Newton line of the quadrilateral $(ABC, \Delta)$ is parallel to the axis of the parabola at a distance $d$ from it; its equation is

$$(2)\qquad y = d = a(m_1 + m_2 + m_3 + m_4)/2$$

The equation of the Newton line of $(ABC, \Delta')$ is readily computed to be

$$(3)\qquad x + s_3 y = a(s_1 s_3 - 1)/2.$$

The equation of $\Delta$ is $x-m_4y=-am_4^2$. Applying condition (1), we can readily show that lines (2), (3), and $\Delta$ meet in the point $P$ whose coördinates are $-a(1+\sigma_4)/2$, $a\sigma_1/2$, where $\sigma_1=m_1+m_2+m_3+m_4$, $\sigma_4=m_1m_2m_3m_4$.

The points $(x, y)$ and $(x', y')$ are symmetrically situated with respect to $P$ if

$$(4)\qquad x = -x' - a(1+\sigma_4), \qquad y = -y' + a\sigma_1.$$

Substituting in the equations $x-m_iy+am_i^2=0$ of the lines $BC$, $CA$, $AB$, and $\Delta$ we obtain,

$$(5)\qquad x' + a + a\sigma_4 + m_i a\sigma_1 - m_i y' - am_i^2 = 0.$$

The coördinates of the orthocenters $H_a$, $H_b$, $H_c$, $H$ are respectively $[-a, a(\sigma_1-m_i+\sigma_4m_i^{-1})]$, $(i=1, 2, 3, 4)$, and they satisfy equation (5), thus proving the second part of the problem.

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## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

During his visit to Harvard University on an exchange professorship, Professor A. Denjoy of the Sorbonne gave several lectures at Princeton, Cornell, and Columbia Universities and the Institute for Advanced Study.

Professor Solomon Lefschetz of Princeton University delivered a series of lectures in Brussels during February on an appointment by the Belgian Educational Foundation, Inc.

Professor Marston Morse of the Institute for Advanced Study has been elected Vice-President of the A. A. A. S. and Chairman of Section A. Professor W. M. Whyburn of the University of California at Los Angeles is a member of the sectional committee.

Assistant Professor Jewell Hughes Bushey of Hunter College has been promoted to an associate professorship.

Assistant Professor Teresa Cohen of Pennsylvania State College has been promoted to an associate professorship.

W. E. Cox of Mississippi State College has been promoted to an assistant professorship.

Mary L. Elveback, a graduate student and research statistician at the University of Minnesota, has received an appointment to the Dorothy Bridgman Atkinson Fellowship for 1939–40 given by the American Association of University Women. This fellowship is to permit Miss Elveback to study mathematical statistics at the University of London and at other universities in England.

Professor D. R. Curtiss of Northwestern University is on leave of absence for the second semester, 1938–1939.

Professor W. B. Ford and Associate Professor V. C. Poor of the University of Michigan are on leave of absence for the second semester, 1938–39.

Associate professor G. A. Hedlund of Bryn Mawr College has been appointed to a professorship at the University of Virginia.

Professor A. W. Hobbs of the University of North Carolina was on leave of absence for the winter quarter.

Dr. H. L. Krall of Pennsylvania State College has been promoted to an assistant professorship.

Professor H. G. Lieber of Long Island University and Dr. L. R. Lieber, director of the Galois Institute of Mathematics, have been appointed visiting professors of modern mathematics in the graduate school of Duquesne University.

Assistant Professor G. M. Merriman of the University of Cincinnati has been promoted to an associate professorship.

Professor Otto Neugebauer of Copenhagen, formerly of Göttingen, the well-known mathematical historian and former editor of the Zentralblatt für Mathematik, has accepted a call from Brown University. He will take residence in the autumn of 1939.

Dr. Abba V. Newton of the American International College has been appointed to a professorship at Hartwick College.

Dr. Arthur Ollivier of Mississippi State College has been promoted to an assistant professorship.

Associate Professor W. V. Parker of Louisiana State University has been promoted to a professorship.

Associate Professor N. E. Rutt of Louisiana State University has been promoted to a professorship.

Dr. G. E. Schweigert of the University of Virginia has accepted a position at Wright Junior College, Chicago.

Professor Frederick Wood, head of the department of mathematics at the University of Nevada, has been made dean of the College of Arts and Sciences at that institution.

The following appointments to instructorships are announced:

University of Alabama: Warren Ambrose
Our Lady of Cincinnati College: Dr. Haim Reingold
Georgetown University: Dr. R. L. Mooney
University of Maryland: Mrs. Martha H. Plass
University of Missouri: D. L. Waidelich
Monticello College: R. F. Jackson
University of Nevada: Everett Harris
University of Oregon: Dr. T. S. Peterson
Pennsylvania State College: Dr. Leonidas Alaoglu
Queens College: Dr. Mary B. Haberzetle
Ursinus College: Dr. F. L. Dennis

Professor Ellen L. Burrell, head of the department of pure mathematics at Wellesley College until her retirement in 1916, died on December 3, 1938, at the age of eighty-eight years.

Dr. Fabian Franklin, professor of mathematics at Johns Hopkins University from 1879 to 1895 and subsequent editor of the Baltimore News and associate editor of the New York Evening Post, died on January 8, 1939, at the age of eighty-six years.

J. E. Ostrander, from 1897 to 1928 professor of mathematics and civil engineering at the Massachusetts State College at Amherst, died on October 19, 1938, at the age of seventy-three years.

Commander Milton Updegraff, professor of mathematics of the United States Naval Academy, died on September 12, 1938, at the age of seventy-seven years.

## A CHICAGO CONFERENCE ON THE CALCULUS OF VARIATIONS

During recent years the Department of Mathematics of the University of Chicago has placed emphasis on a particular field each Summer Quarter. Algebra was stressed in 1938, and analysis will be emphasized in the summer of 1939. A feature of this summer's program will be a seminar on the calculus of variations conducted by G. A. Bliss, and participated in by Max Coral, L. M. Graves, M. R. Hestenes, E. J. McShane, W. T. Reid, and M. F. Smiley, members of the summer staff.

In the period from Tuesday, June 27 to Friday, June 30 there will be a special Conference on the Calculus of Variations, including a number of addresses by members of the staff and others, as follows: *Length and area*, and *Geometrical approach to the Plateau problem*, by Tibor Rádo, Ohio State University; *The problem of Plateau-Riemann* and *Minimal surfaces of higher topological structure*, by Jesse Douglas; *Existence theorems for multiple integrals*, by E. J. McShane, University of

Virginia. *The equations of Haar and differentiability of their solutions*, by Max Coral, Wayne University; *The Jacobi condition for multiple integrals*, by L. M. Graves; *The field-theory for multiple integrals*, by G. A. Bliss; *Sufficiency proofs by expansion methods*, by W. T. Reid; *Functional topology and analysis in the large*, and *Variational theory in the large*, by Marston Morse, Institute for Advanced Study; *The problem of Bolza*, by M. R. Hestenes; *Logical analysis of the semi-continuity properties of line integrals*, by Karl Menger, University of Notre Dame.

A series of lectures preparatory to the addresses of the Conference will be given in the seminar in the preceding week, June 21–24. All mathematicians who may be interested are cordially invited to attend the Conference and the preparatory lectures in the seminar. A limited number of rooms will be available in Judson Court, a modern residence hall facing the Midway, for the week of the conference at $2.75 per day for room and board. Other housing accommodations near the University are also available. Reservations and requests for information should be addressed to M. R. Hestenes, Eckhart Hall, University of Chicago, Chicago, Illinois.

## THE INTERNATIONAL MATHEMATICAL CONGRESS IN 1940

On invitation by the American Mathematical Society, an International Congress of Mathematicians will be held in Cambridge, Massachusetts, September 4–12, 1940.

The first International Congress of Mathematicians was held in Chicago in 1893 in connection with the World's Columbia Exposition. Since that time congresses have been held at about four year intervals, all of which have been in Europe except one which was held in Toronto in 1924; the most recent one was held in Oslo in 1936. At recent congresses the number of members has been about 600, representing about 40 countries, and the number of papers submitted has been about 250.

Following precedent, there will be a score of invited addresses, each an hour long, and sectional meetings for shorter papers. An innovation will be a number of conferences on topics in which research is at present particularly vigorous.

That social features and entertainment will be adequately provided need not be mentioned to the large group of mathematicians who were so delightfully entertained at Cambridge in connection with the Harvard Tercentenary in 1936.

A more detailed preliminary announcement of the congress was published in the *Bulletin of the American Mathematical Society* last September. Further information concerning the congress may be obtained by writing to the American Mathematical Society, 531 West 116th Street, New York, N. Y.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.


## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY, Murray, April 28–29.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Grounds, Md., May 13.
MICHIGAN, Ann Arbor, March 18.
MINNESOTA, Northfield, May 13.
MISSOURI, Springfield, April 28.
NEBRASKA, Lincoln, May 5.
OHIO, Columbus, April 8
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 2–3.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.


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## THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on December 10, 1938. The chairman, Michael Goldberg, Navy Department, presided over both sessions, morning and afternoon. Six papers were read at the morning session and two at the afternoon session. Professor Monroe Martin of the University of Maryland was the invited speaker, giving as his address "The problem of three bodies."

The Section gave a rising vote of thanks to the University of Maryland and its Department of Mathematics for their generous hospitality.

There were sixty-four in attendance including the following thirty-four members of the Association: O. S. Adams, N. H. Ball, T. E. Berry, G. A. Bingley, Archie Blake, C. C. Bramble, Randolph Church, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, Michael Goldberg, L. M. Kells, W. D. Lambert, A. E. Landry, Florence P. Lewis, S. B. Littauer, Carol V. McCamman, E. J. McShane, Florence M. Mears, T. W. Moore, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, R. E. Root, J. L. Stearn, T. H. Taliaferro, John Tyler, C. E. Van Orstrand, G. C. Vedova, C. H. Wheeler III, G. T. Whyburn, C. W. Williams, John Williamson, R. C. Yates.

The following papers were read:

1. "On a class of distributions approaching normality" by G. B. Dantzig, Bureau of Labor Statistics, Washington, D. C., introduced by the chairman.
2. "A method of obtaining a conjugate function" by John Beek, Jr., National Bureau of Standards, introduced by the chairman.
3. "An algorithm for polynomial approximations" by Dr. L. S. Dederick, Aberdeen Proving Ground.
4. "The dual quartic" by C. W. Williams, Washington and Lee University.
5. "On the tenth book of Euclid's *Elements*" by Professor G. C. Vedova, St. John's College.
6. "On a sixteenth century algebraist" by Professor Jacob Klein, St. John's College, introduced by the secretary.
7. "Three-dimensional models of Dedekind structures" by Professor Randolph Church, Postgraduate School, U. S. Naval Academy.
8. "The problem of three bodies" by Professor Monroe Martin, University of Maryland.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Given an initial discrete distribution $f(x)$, a variable distribution $f_{n+1}(x)$ is generated from $f_n(x)$ by the following recurrence formula: $f_{n+1}(x)=f_n(x)+f_n(x-1)+\cdots+f_n(x-a_n)$, where $x$ takes on all integer values extending from $-\infty$ to $+\infty$, and $a_n$ is any set of positive integers. This recurrence formula

may be considered as a generalization of the "Pascal Triangle" relationship for the binomial coefficients, namely:

$$\binom{n+1}{x} = \binom{n}{x} + \binom{n}{x-1},$$

where we should note that in this example $a_n = 1$ for all $n$. Setting

$$\Gamma = \lim_{n\to\infty} \sum_{i=1}^{n} c_i^2 \Big/ \left( \sum_{i=1}^{n} c_i \right)^2$$

where $c_i = a_i^2 + 2a_i$, Mr. Dantzig demonstrated the following theorem: *A condition necessary and sufficient that $f_n(x)$ (normalized) approach the normal curve of error with growing $n$ is $\Gamma = 0$.*

2. Mr. Beek's procedure involved constructing a function $w$ of $z$ $(=x+iy)$ such that the real part of $w$ is the given function $u(x, y)$. This is accomplished by substitution, $w$ being given by the equation $w = 2u(z/2+c/2, z/2i-c/2i)$, where $c$ is so chosen that the part of $w$ in which $c$ appears becomes a pure imaginary number. The desired conjugate function is given immediately as the imaginary part of $w$. This method may be used only if the given function is defined for complex values of $x$ and $y$.

3. The determination of a polynomial of specified degree which is the best fit for a series of values of a function where the values of the argument are equally spaced is a special problem under the method of least squares. Dr. Dederick reduced the solution of this problem to a convenient algorithm which takes advantage of the two special characteristics of the problem and also of the device of replacing powers of the argument by the corresponding binomial coefficients.

4. A plane curve is said to be dual if its order and class are equal, the number of its point and line singularities are, respectively, equal to each other, and the curve is unicursal. Mr. Williams considered a dual quartic, *i.e.*, one with two cusps and one node. He exhibited a conic with respect to which the line and point singularities of this curve stand in a pole-polar relation.

5. Among the host of those who have in the past concerned themselves, in one way or another, with Euclid's tenth book, many have commented on its difficulty, some on the inverted logical order used in the exposition of its subject, and a few have hinted at the mythical "secret method of analysis of the Greeks" as a possible explanation of the phenomenon. Professor Vedova showed what logically prior considerations could have led to Euclid's seemingly arbitrary definitions, and what logically prior plan could have suggested the seemingly arbitrary development of his theory. He derived, incidentally, a simple way of solving the general quadratic for its positive real roots by ruler and compasses, and pointed out wherein, in his opinion, Euclid displayed fine artistry and a keen sense of the logically rigorous.

6. Professor Klein discussed the work of Simon Stevin (1548–1620) who was the real discoverer of decimal fractions, although they had been used sporadi-

cally before. That discovery was based on a fundamental change in the concept of number, which influenced the algebra of Stevin. Reference was made to Stevin's general philosophical outlook and his decisive influence on Descartes.

7. Elementary properties of finite Dedekind structures were illustrated by Professor Church by several plane graphs and wire models of structures, among which were those associated with the abelian groups of types $(2^2, 2^2)$, $(2^3, 2^2)$, $(2, 2, 2)$, $(2^2, 2^2, 2)$ and $(3, 3, 3)$.

8. Professor Martin presented a brief survey of the work of Newton, Clairaut, Euler, and Lagrange devoted to establishing simple periodic solutions and algebraic integrals of the problem of three bodies, followed by a discussion of the result of Bruns which states that there are no algebraic integrals of the problem of three bodies independent of the ten classical ones known to Clairaut and Euler. The work of Painlevé and Sundman on the nature of the motion in the neighborhood of collision was outlined, and Sundman's theorem on the existence of power series adequate to follow the motion of the three bodies throughout the entire course of their motion was discussed. The concluding part of the paper was devoted to the restricted problem of three bodies as introduced by Jacobi. A new elementary derivation of the differential equations of motion was obtained, and a description was given of the work of Poincaré, Birkhoff, Perron, Wintner, and Strömgren on periodic solutions.

S. B. LITTAUER, *Secretary*

## ON THE GRAEFFE METHOD OF SOLUTION OF EQUATIONS

L. L. CRONVICH, Tulane University

**1. Introduction.** One of the classical methods of solving algebraic equations was developed by Graeffe, and is explained in many treatises.* Graeffe's method has been extended so that in the course of the computation one can gain information as to the character of the roots. In a paper by Hutchinson, published in this MONTHLY in 1935,† there is an extended account of the Graeffe method and of the characterization of roots by means of a table of numbers connected with Graeffe's successive equations. Hutchinson gave a list of 11 rules to be used in such a characterization. In the present paper it is my intention to give a new list of rules, which shall be more specific and which shall include all the cases given by Hutchinson. A detailed comparison of my rules with Hutchinson's will be given later. I shall not include proofs, since these can be inferred from Hutchinson's paper.

* See, for example, Bauer, Vorlesungen über Algebra, Leipzig, 1903; Berg, Heaviside's Operational Calculus, New York, 1929; Doherty and Keller, Mathematics of Modern Engineering, Book I, New York, 1936; Runge, Praxis der Gleichungen, Berlin and Leipzig, 1921; Runge und König, Numerisches Rechnen, Berlin, 1924; Whittaker and Watson, The Calculus of Observations, London, 1924.

† On Graeffe's method for the numerical solution of algebraic equations, this MONTHLY, vol. 42, 1935, pp. 149–161.

**2. An example.** For purposes of illustration I shall consider the equation

$$x^5 - x^4 - 9x^3 + 129x^2 + 20x - 500 = 0.$$

Graeffe's method consists of the creation of a set of equations, all of the same degree, whose Encke roots (roots with the signs reversed) are, respectively, the squares of the original Encke roots, the fourth powers of the original Encke roots, the eighth powers, etc. The roots of these equations are progressively separated more and more widely so that eventually it is possible to use the relations between the roots and coefficients of an equation for a closely approximate solution. In the table below I present in tabular form the coefficients of the original equation and the five succeeding equations for this example. These are to be found in the rows from the third to the eighth of the table. The value of $m$ in the table designates the power to which the roots of the original equation have been raised. The characterization of the nature of the roots of the original equation is determined from the nature of the columns in this table.

Note. In this Table $1.29^2 = 1.29 \times 10^2$, $1.294^5 = 1.294 \times 10^5$, etc.

| Col. No. | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x$ | $c$ |
| $m=1$ | 1 | $-1$ | $-9$ | $1.29^2$ | 20 | $-5.00^2$ |
| $m=2$ | 1 | 19 | $3.79^2$ | $1.80^4$ | $1.294^5$ | $2.50^5$ |
| $m=4$ | 1 | $-3.97^2$ | $-2.816^5$ | $2.354^8$ | $7.74^9$ | $6.25^{10}$ |
| $m=8$ | 1 | $7.21^5$ | $2.82^{11}$ | $5.97^{16}$ | $3.05^{19}$ | $3.906^{21}$ |
| $m=16$ | 1 | $-4.42^{10}$ | $-6.50^{21}$ | $3.55^{33}$ | $4.64^{38}$ | $1.526^{43}$ |
| $m=32$ | 1 | $1.395^{22}$ | $3.56^{44}$ | $1.26^{67}$ | $1.07^{77}$ | $2.33^{86}$ |

**3. Types of columns.** A column is said to be *regular* if eventually each coefficient is very nearly the square of the coefficient standing above it. In our example, columns 1, 4, and 6 are regular.

A column is said to be *oscillating* if the signs of its terms are in some places positive and in some places negative. This oscillation in sign is due to the presence of imaginary roots. Two columns are said to have *exactly equal rates of oscillation* when the changes in sign occur at the same rows throughout the table, or to have *approximately equal rates of oscillation* if some corresponding coefficients differ in sign near the beginning of the table but eventually all corresponding coefficients agree in sign. The second and third columns in my table are oscillating columns, and appear to have exactly equal rates of oscillation.

All columns which are not regular or oscillating are called *irregular*. As Hutchinson showed, in irregular columns the coefficients may be a fraction of what they would be if the column were regular. Column 5 is irregular, each coefficient being nearly half the square of the coefficient standing above. Irregular columns may also have indefinite rates which seem to follow no fixed pattern. They may have negative signs near the top of the table, but eventually only positive signs appear in these columns; in this respect irregular columns of indefinite rates differ from oscillating columns.

**4. Rules of identification.** Since it has been found that the best guide to the proper choice of roots is the determination of the number and location of oscillating columns present, the rules have been developed according to the number of such oscillating columns. Further, since the peculiarities due to certain combinations of roots are unchanged by additional roots which differ entirely from those causing the peculiarities, we need have only special rules for identifying all such combinations. This fact makes it possible to reduce considerably the number of rules of identification. In the following summary it is understood that the peculiarly behaving columns mentioned always occur in a group of columns which have a regular column adjacent on each side of the group, thus separating it from any other groups.

Summary of Rules of Identification

1. No oscillating columns, and all coefficients positive after the given equation.—All real roots.
   *a*. All columns regular.—Distinct real roots.
   *b*. Adjacent irregular columns, $(n-1)$ in number, increasing at rates proportional to the reciprocals of the binomial coefficients.—$n$ multiple roots of equal absolute value.
2. One oscillating column.
   *a*. One oscillating column only.—A pair of imaginary roots.
   *b*. An oscillating column with an irregular column adjacent on each side having an indefinite rate and no possibility of having negative signs.—A pair of imaginary roots whose modulus is equal to the absolute value of two real roots of equal absolute value.
3. Two oscillating columns.
   *a*. Two adjacent oscillating columns with approximately equal rates.—A pair of imaginary roots whose modulus is equal to the absolute value of a real root.
   *b*. Two adjacent oscillating columns with exactly equal rates of oscillation and having one irregular column adjacent on each side with an indefinite rate and no possibility of having negative signs.—A pair of imaginary roots whose modulus is equal to the absolute value of three real roots of equal absolute value.
   *c*. Two alternate oscillating columns with exactly the same rates sepa-

rated by an irregular column with an indefinite rate of increase and no possibility of having negative signs.—Two identical pairs of imaginary roots.

*d*. Two oscillating columns with approximately equal rates of oscillation separated by two irregular columns with indefinite rates and no possibility of having negative signs.—Two identical pairs of imaginary roots with modulus equal to the absolute value of a real root.

4. Three oscillating columns.
   *a*. Three adjacent oscillating columns with the outer two having approximately the same rate of oscillation and the intervening column having a very slow rate.—Two pairs of imaginary roots with equal moduli but unequal amplitudes.
   *b*. Two alternate columns with negative coefficients for the first transformed equation and positive coefficients thereafter and with no definite rate of increase, and an intervening oscillating column.—A pair of pure imaginary roots and a pair of imaginary roots with equal moduli. (This is a special case of 4*a*.)
5. Four oscillating columns.
   *a*. Four adjacent oscillating columns with the first and fourth, and second and third having approximately equal rates respectively.—Two pairs of imaginary roots with equal moduli and a real root whose absolute value is equal to the modulus of the imaginary roots.

Pure imaginary roots, as special cases of imaginary roots, constitute exceptions to the above rules. One case, rule 4*b*, has already been given in the above list. In general, the columns which are oscillating for ordinary imaginary roots become, in this case, columns which show a minus sign in the first transformed equation with positive signs for the rest of the transformed equations, and which, in most cases, eventually show the characteristics usually attributed to multiple roots of equal absolute value.

**5. Comparison with Hutchinson's rules.** The rules presented in the above Summary cover all the cases considered by Hutchinson in his Summary and also several other cases not mentioned by him. Some of the rules in my Summary are almost identical with those stated by Hutchinson, except for the terminology used. These are rules 1*a*, 2*a*, 3*a*, and 2*b*, which correspond to Hutchinson's rules 1, 2, 3, and 11, respectively.

In my list of rules I have taken care of all the cases of multiple roots of equal absolute value by a single rule, 1*b*. This rule includes the cases given by Hutchinson in his rules 4, 5, and 7. Furthermore, his rule 6 may be treated as a repeated application of my rule 1*b*. Since the two irregular columns mentioned in his rule 6 are non-adjacent, it being understood that they are separated by at least one regular column, such a case actually involves two groups of columns, each group behaving according to rule 1*b* above. Similarly, the case covered by Hutchinson's rule 8 may be decomposed into two simpler cases, covered by rules 1*b* and 2*a* of my Summary. It involves a pair of real roots of equal absolute

value and a pair of imaginary roots whose modulus differs from the absolute value of the real roots. The peculiarities due to these pairs of roots are exhibited in separate groups of columns since the real and imaginary roots are unrelated as far as their moduli are concerned. Hence each pair of roots can be identified by the simple rule covering such cases.

In his rule 10, Hutchinson has treated the case of two pairs of imaginary roots with equal moduli. Actually this rule covers only the case in which the moduli of the imaginary roots are equal but the amplitudes differ. This case is taken care of in the above Summary by rule *4a*. If the amplitudes as well as the moduli are equal, that is, if the two pairs of imaginary roots are identical, a new rule must be applied, namely, *3c*.

Among the cases not considered by Hutchinson, in addition to that involving two identical pairs of imaginary roots, are those treated in rules *3b*, *3d*, *4b*, and *5a*. These rules deal with various combinations of real and imaginary roots of equal moduli, and a special case involving pure imaginary roots.

The list of rules presented above has been compiled so as to take care of the identification of every possible combination of roots for equations of degree not greater than the fifth. If the system of identification by peculiarly behaving groups of columns is employed, as advocated by the author, the rules may be used for equations of much higher degree except for certain combinations of roots such as those involving three or more pairs of imaginary roots with equal moduli.

**6. An application of the rules.** To illustrate the application of the above set of rules, I shall complete the solution of the equation previously presented. The classification of the columns of coefficients appearing in the table has already been given (see pp. 186 and 187). To determine the nature of the roots, let us examine the groups of peculiarly behaving columns. The first group, made up of two adjacent oscillating columns with equal rates of oscillation, indicates by rule *3a* the presence of a pair of imaginary roots whose modulus is equal to the absolute value of a real root. The next group, made up of a single irregular column with half the regular rate, indicates by rule *1b* a pair of real roots of equal absolute value. In addition, the position of this group (that is, following the other group) shows that the absolute value of each of these real roots is less than the modulus of the imaginary roots.

Let us suppose that the absolute values of the real roots of the given equations are $r$, $a$, and $a$ and of the imaginary roots, $r$. Let the amplitude of the imaginary roots be $\theta$. Then the Encke roots of any transformed equation above will be $r^m$, $r^m e^{im\theta}$, $r^m e^{-im\theta}$, $a^m$, and $a^m$. The equation having such Encke roots may be written

$$(x + r^m)(x + r^m e^{im\theta})(x + r^m e^{-im\theta})(x + a^m)(x + a^m) = 0.$$

or, multiplying out and retaining only the dominant terms (which is sufficiently accurate for $m$ large), we have

$$x^5 + r^m(1 + 2\cos m\theta)x^4 + r^{2m}(1 + 2\cos m\theta)x^3 + r^{3m}x^2 + 2r^{3m}a^m x + r^{3m}a^{2m} = 0.$$

Comparing the coefficients of this equation with the coefficients of the transformed equation for $m = 32$, and using only the coefficients of regular columns for calculations, we have

$$r^{96} = 1.26 \times 10^{67} \quad \text{and} \quad r^{96}a^{64} = 2.33 \times 10^{86}.$$

By the use of logarithms we find $|r| = 5$ and $|a| = 2$. Substituting in the given equations, the real roots are found to be $-5$, $-2$, and $+2$. The absolute value of the imaginary roots is 5. If the real and imaginary components of the imaginary roots are $u$ and $v$ respectively, we can determine $u$ by using one of the relations between roots and coefficients of equations, as follows:

$$-5 - 2 + 2 + 2u = 1, \quad \text{or} \quad u = 3.$$

Then $v = \sqrt{5^2 - 3^2} = 4$. Hence the roots of the given equation are $-5$, $3 \pm 4i$, $-2$, and $+2$.

It is, of course, not advocated that the Graeffe method be used for the solution of equations which, like the one just discussed, can be solved by the use of the factor theorem. The problem was used, however, merely for the purpose of helping to clarify the material presented. The method is to be recommended for the solution of equations containing irrational roots and imaginary roots which are desired only approximately.

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# ON A THREE-DIMENSIONAL PRESENTATION OF FUNCTIONS OF A COMPLEX VARIABLE*

LUISE LANGE, Woodrow Wilson Junior College, Chicago

**1. Introduction.** In the Cartesian scheme of geometrically representing functions of one variable $y = f(x)$ by interpreting corresponding pairs of values $x$, $y$ as coördinates of points in the plane, only real values of $x$ and $y$ are represented. Pairs of complex numbers interpreted as coördinates of a point would, indeed, require a four-dimensional space.

The classical representation of functions of a complex variable, as developed by Gauss and Riemann, uses an altogether different idea, the functional equation $w = f(z)$† being interpreted as a transformation of the points of one two-dimensional continuum onto another. The Cartesian scheme, on the other hand, has also been adapted by plotting in rectangular space coördinates separately the two surfaces $u(x, y)$ and $v(x, y)$, or the surface of the modulus $R(x, y)$.‡

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* Read before the Illinois Section of the Mathematical Association of America on May 14, 1937.

† $z = x + iy = r(\cos\phi + i\sin\phi)$, $w = u + iv = R(\cos\psi + i\sin\psi)$.

‡ This latter scheme has been used and highly recommended in Jahnke und Emde, Funktionentafeln, 2.ed. 1933. For their discussion see Preface, p. II.

In the following a somewhat different method is set forth to adapt the Cartesian scheme to the representation of functions of a complex variable. It consists in presenting on one coördinate axis linear fields of the complex independent variable, and on the other two axes the real and imaginary parts of the dependent variable. The function $w=f(z)$ thereby appears in the form of one-parameter families of space curves. These curves, which may be regarded as three-dimensional sections through the non-presentable four-dimensional loci,* are the complex generalizations of the familiar plane real curves.

Various families of curves (or different sets of sections) for a given function are obtained by different choices of the parameter of the linear complex $z$-field. In the following have been treated some presentations using:

(a) $\phi$ as parameter with $r$ as independent variable ("radial sections"),

(b) $x$ as parameter with $y$ as independent variable ("sections parallel to the imaginary axis"), and

(c) $y$ as parameter with $x$ as independent variable ("sections parallel to the real axis").†

**2. Radial sections, $\phi=0$.** First we consider the special case of radial sections obtained by taking $\phi=0$.

For this case $z$ takes on only real values. This diagram is therefore a mere extension of the usual Cartesian one into the third dimension so as to allow the representation of complex values of $w$ which correspond to real values of $z$.‡

In the following a few simple consequences of the addition of this imaginary dimension are indicated. We shall refer to the $(z, u)$ and $(z, v)$ planes as the "real" and "imaginary" planes respectively.

a. *Conic sections* in the real plane are associated with other conic sections in the imaginary plane whenever the real locus exists for a restricted interval of $z$ only. Thus the ellipse§

$$\frac{z^2}{a^2}+\frac{w^2}{b^2}=1$$

is associated with an hyperbola (axes $a$ and $b$) in the imaginary plane (Fig. 1), and the hyperbola $z^2/a^2-w^2/b^2=1$ with an ellipse in the imaginary plane. The locus of $w^2=kz$ consists of two congruent parabolas in the real and imaginary planes respectively; the "imaginary circle" $z^2+w^2=-a^2$, non-existent in the real plane, is an equilateral hyperbola in the imaginary plane and so on.

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* Or as one-dimensional sections of two dimensional surfaces in four-space. E.J.M.

† The radial sections are identical with lines $\psi=$const. traced on the $R$-surface of the inverse function. Similarly the sections parallel to the imaginary axis are the lines $u=$const. traced on the $v$-surface of the inverse function, and those parallel to the real axis are the lines $v=$const. traced on the $u$-surface of the inverse function. Here, however, these curves are obtained directly.

‡ This scheme of presentation has been used, for instance, by W. O. Pennell, Fourier Series in Three Dimensions, this MONTHLY, vol. 39, 1932, p. 261 ff.

§ The coefficients are throughout regarded as real.

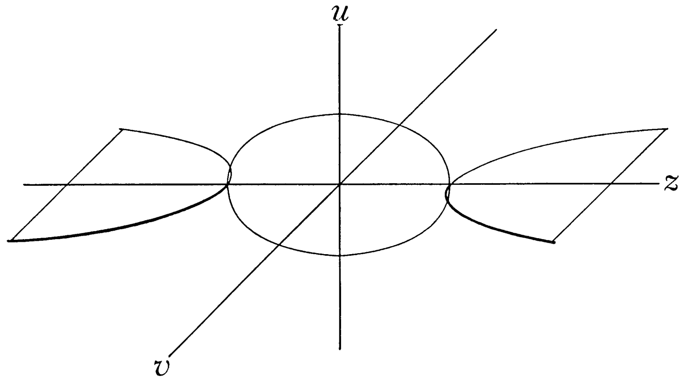


FIG. 1. $\frac{z^2}{a^2}+\frac{w^2}{b^2} = 1, \phi=0.$

b. *Imaginary points of intersection.* When two simultaneous equations in $z$ and $w$ have a solution for some real $z$ this appears as an intersection of the corresponding loci even though the $w$ of the solution may be imaginary. For example, for $z^2+w^2=a^2$ and $(z-b)^2+w^2=a^2$, the two circles intersect in the real plane for $b\leqq 2a$, while for $b\geqq 2a$ the resulting imaginary solutions are illustrated by the intersection of the two hyperbolas (Fig. 2).

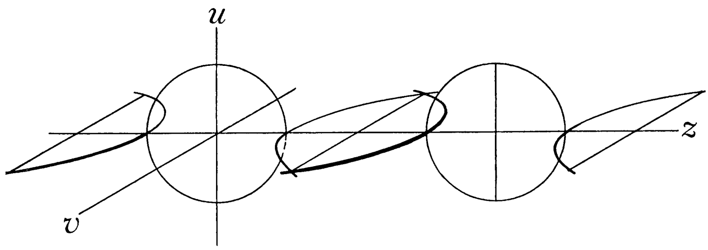


FIG. 2. Imaginary Points of Intersection.

c. *Multi-valued functions* of $z$ appear in this representation in all their multi-valuedness. For example $w=\sqrt[n]{z}$, which in the two-dimensional "real" diagram is one-valued for all $z$, or two-valued for $z>0$, consists, for all $z$, of $n$ congruent branches forming angles of $2\pi/n$ with one another (Fig. 3 shows $w=\sqrt[3]{z}$).

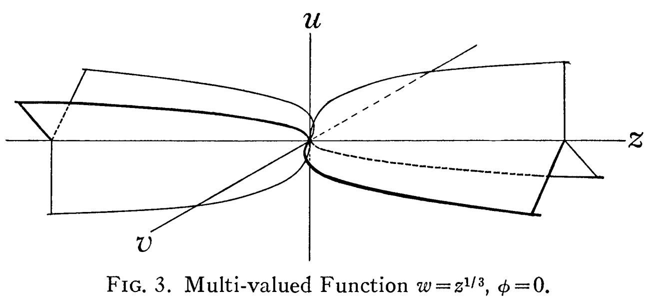


FIG. 3. Multi-valued Function $w=z^{1/3}$, $\phi=0$.

d. *Isolated points* of real loci are seen to belong to imaginary branches of the corresponding three-dimensional loci. For instance the real locus of $z^2+w^2=z^2w^2$ has an isolated point at the origin. In the imaginary plane, however, lie two branches between $|z|<1$, touching the real plane at $z=0$ and thus containing the "isolated" point (Fig. 4). As an island viewed as a two-dimensional surface is an isolated area of land while regarded three-dimensionally it is the peak of a sub-oceanic hill or mountain, so here, what is two-dimensionally an isolated point appears three-dimensionally as the peak of an imaginary formation just reaching into the real plane.

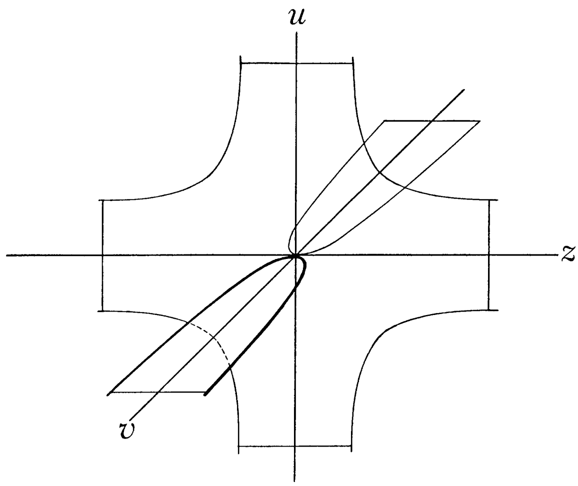


FIG. 4. Isolated Points.

e. *Elementary transcendental functions*. The trigonometric, exponential, and hyperbolic functions are real for all real values of $z$ and hence have no branches in the imaginary plane. The logarithmic function, however, whose real locus exists only for $z>0$, has for $z<0$ a branch symmetrical to the real locus in the plane $v=\pi$.

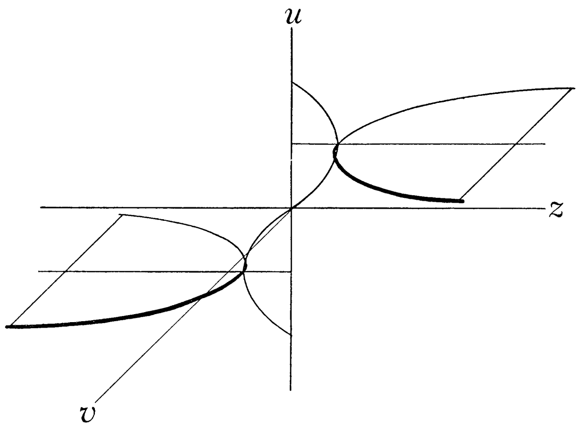


FIG. 5. $w=\text{arc sin } z$, $\phi=0$.

Likewise the arcsine function, whose real locus is restricted to $|z| \leqq 1$ has complex branches for $|z| \geqq 1$. Thus if $w = \arcsin z$, then $z = \sin w$, and $x + iy = \sin u \cosh v + i \cos u \sinh v$; hence

$$x = \sin u \cosh v, \qquad y = \cos u \sinh v.$$

Since $y = 0$ (because $\phi = 0$) it follows that either $v = 0$, or $u = (2n-1)\pi/2$. The former leads to the real curve $x = \sin u$, while the latter implies $x = \pm \cosh v$. Hence in the planes $u = (2n-1)\cdot\pi/2$, which are parallel to the real plane, "imaginary catenaries" are attached to the crests and troughs of the sine wave (Fig. 5).

3. **Radial sections, $\phi$ variable.** The parameter $\phi$ is next supposed to vary. The problem is to find the transformations which the radial sections undergo for given variations of $\phi$ (or for given "rotations" of the $z$-axis into the complex).

a. Consider first the power function $w = az^n$.* Writing $w = Re^{i\psi}$, $z = re^{i\phi}$ we have

$$R = ar^n, \qquad \psi = n\phi.$$

This means that as $\phi$ varies by $\delta\phi$ the locus of the power function rotates through complex space around the $z$-axis by $n\delta\phi$. (For example, as $\phi$ varies from 0 to $2\pi$ each of the three branches of $w = \sqrt[3]{z}$ (Fig. 3) rotates through $2\pi/3$. Hence the configuration as a whole is the same as for $\phi = 0$.) These curves thus allow a simple simultaneous geometrical representation of all four variables entering into the functional relation $u + iv = a\,(r(\cos\phi + i \sin\phi))^n$: while $r$, $u$ and $v$ are the rectangular coordinates of any of its points, $\phi$ is one $n$th of the angle formed between $R$ and the real plane.

b. Next consider the exponential function $w = e^z$. We have $w = R \cdot e^{i\psi} = e^{r(\cos\phi + i \sin\phi)}$, and hence

$$R = e^{r\cos\phi}, \qquad \psi = r \sin \phi.$$

Geometrically this means that for any given $\phi \neq n\pi$ the locus of the exponential function is a helix of pitch $2\pi \operatorname{cosec} \phi$ lying on a horn-shaped exponential surface of revolution which is generated by revolving the curve $u = e^{r\cos\phi}$ around the $z$-axis (Fig. 6).

As $\phi \to \pi/2$ the generating exponential curve approaches the straight line $u = 1$. The horn thus degenerates into a cylinder of radius 1 while the pitch approaches $2\pi$. (The case $\phi = \pi/2$, a helix of pitch $2\pi$ on a cylinder of radius 1, illustrates Euler's formula $e^{ir} = \cos r + i \sin r$.) As $\phi$ becomes greater than $\pi/2$ the horn widens in direction of negative $r$ while the pitch increases indefinitely. For $\phi = \pi$ the helix has transformed itself into the plane exponential curve $u = e^{-r}$.

* It should be noticed that in all these diagrams the modulus $R$ of the function is the distance of the points of the locus from the $z$-axis, while the amplitude $\psi$ is the angle between $R$ and the $(z, u)$ plane.

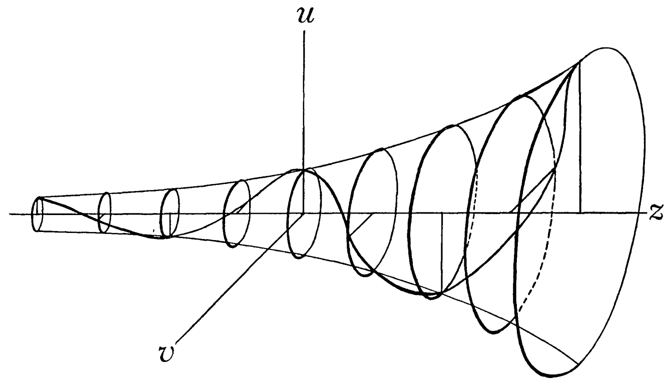


FIG. 6. $w=e^z$, $\phi=80°$.

c. As another example consider the tangent function $w=\tan z$.* From the relation $u+iv=\tan(x+iy)$ one readily obtains

$$u=\frac{1}{2}\,\frac{\sin 2x(1-\tanh^2 y)}{\cos^2 x+\sin^2 x\tanh^2 y},\qquad v=\frac{\tanh y}{\cos^2 x+\sin^2 x\tanh^2 y},$$

where $x=r\cos\phi$, and $y=r\sin\phi$. For $\phi\neq(2n-1)\cdot\pi/2$ the curve $u(r)$ is a modified sine wave with zeros at $r=\frac{1}{2}n\pi\sec\phi$. As $r$ increases the amplitude of the wave approaches zero because of the factor $1-\tanh^2(r\sin\phi)$.† We see that $v(r)$ is positive for $r\sin\phi>0$, and negative for $r\sin\phi<0$, while $v(0)=0$. For a given $\phi$ the function $v(r)$ oscillates between maximum and minimum values $\coth(r\sin\phi)$ and $\tanh(r\sin\phi)$ which occur simultaneously with the zeros of $u(r)$. With increasing $r$ these maxima and minima approach the common limit 1. (Fig. 7 shows $u(r)$ and $v(r)$ dotted.)

Geometrically this means that as soon as the $z$-axis rotates into the complex, adjacent branches of the tangent curve join each other, the "loose ends" meeting in the imaginary plane at $r=\frac{1}{2}(2n-1)\pi\sec\phi, v=\coth(r\sin\phi)$. The locus thus becomes a generalized helix of pitch $\pi\sec\phi$, penetrating the imaginary plane in the maximum and minimum values of $v(r)$ and thus winding in ever tightening coils around the line $v=1$ in the imaginary plane (Fig. 7).‡

As $\phi\to\pi/2$ the pitch of the helix increases indefinitely. At the same time $u\to 0$ (because of the factor $\sin 2(r\cos\phi)$), and $v\to\tanh r$. Thus the helix stretches out more and more along the hyberbolic tangent curve in the imaginary plane which it actually reaches for $\phi=\pi/2$.

* The tangent function has been singled out for presentation in this radial diagram because of the peculiar way in which the discontinuous real locus transforms itself under the variation of $\phi$.

† The factor $\cos^2 x+\sin^2 x\tanh^2 y$ is readily seen to have the effect of rendering the descending branches of the sine wave steeper than the ascending ones, thus shifting the maxima and minima closer to the zeros on the descending branches. Since $\tanh y$ approaches 1 with increasing $r$, the whole factor approaches 1.

‡ This curve has a peculiar resemblance to the one given in Jahnke and Emde, op. cit., p. 111, for a three-dimensional presentation of Fresnel's integrals.

Returning to $\phi=0$ as a limiting case of $\phi\neq 0$, we see that the discontinuous branches of the real tangent curve appear as the broken fragments of the complex tangent spiral, broken under the strain of $u(r)$ and $v(r)$ both becoming infinite at $r=(2n-1)\pi/2$.

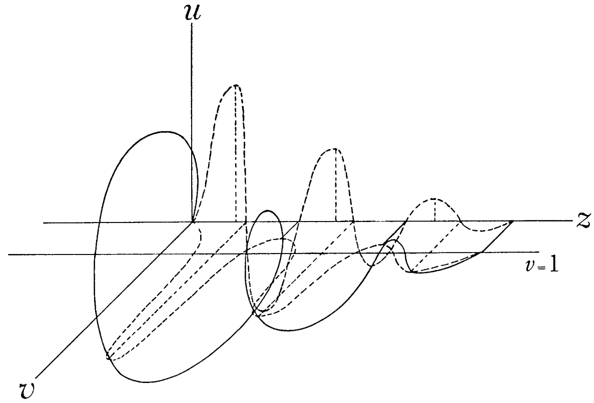


FIG. 7. $w=\tan z$, $\phi=k$, $0<k<\pi/2$.

The other trigonometric, hyperbolic, and anti-trigonometric functions can be treated similarly. The sine wave is then seen gradually to transform itself into the hyperbolic sine curve in the imaginary plane, and vice versa; the arcsine curve with its attached imaginary catenaries changes into a hyberbolic sine curve lying in planes parallel to the imaginary plane; and so on.*

**4. Sections parallel to the real and imaginary axes, $y=$const. and $x=$const.** Some functions have simpler loci in sections parallel to the $x$- or the $y$-axis than in radial sections, notably the exponential and trigonometric functions.†

a. Consider first the exponential function $w=e^z$. We have

$$R = e^x, \qquad \psi = y.$$

Hence for $y=$const. the locus is the original exponential curve rotated by $\psi=y$ around the $z$-axis. (A variation of $y$ by $2\pi$ thus rotates any such curve by a full revolution.) Any point of such a curve allows an immediate simultaneous representation of all four variables entering into the functional relation $u+iv=e^{x+iy}$, $x$, $u$ and $v$ being its rectangular coordinates, while $y$ is the angle formed by $R$ with the $(z, u)$ plane.

* Animated cartoons would be a desirable means to demonstrate this metamorphosis of loci. In a corner of the screen the $z$-axis could be shown rotating through the complex $z$-plane like a clock-hand, illustrating the corresponding variations of $\phi$.

† The loci of polynomial functions with real coefficients are specially simple in sections $x=$const. because these are symmetrical to the origin. This presentation affords a geometrical illustration of the pairs of conjugate complex zeros of these functions. Treating the second degree polynomials in this manner one obtains a simple generalization of the usual graphical presentation of the real roots of a quadratic equation.

According to the above equations, for $x=\text{const.}$ the locus is a helix of pitch $2\pi$ lying on a cylinder of radius $e^x$. (The radial section $\phi=\pi/2$ discussed above is identical with the parallel section $x=0$.)

b. Consider next the sine function $w=\sin z$. We now have

$$u = \cosh y \sin x, \qquad v = \sinh y \cos x.$$

The sections $y=\text{const.}$ are helices of pitch $2\pi$ lying on elliptical cylinders with semi-axes $\cosh y$ and $\sinh y$ in the $u$ and $v$ directions respectively; the foci lie on the lines $u=\pm 1$ in the $(x, u)$ plane. As $y$ approaches zero the cylinders flatten in the $v$-direction, becoming more and more like a strip between $\pm 1$ in the $(x, u)$ plane, the complex sine helix thereby approaching the plane sine wave.

The sections $x=\text{const.}$ are space curves whose projections on the $(y, u)$ plane and $(y, v)$ plane are respectively hyperbolic cosine and hyperbolic sine curves. These curves lie on the surface $u^2/\cosh^2 y+v^2/\sinh^2 y=1$, an elliptical trumpet of width zero at $y=0$ (Fig. 8).* This surface allows a simple simultaneous representation of all four variables entering into the functional relation

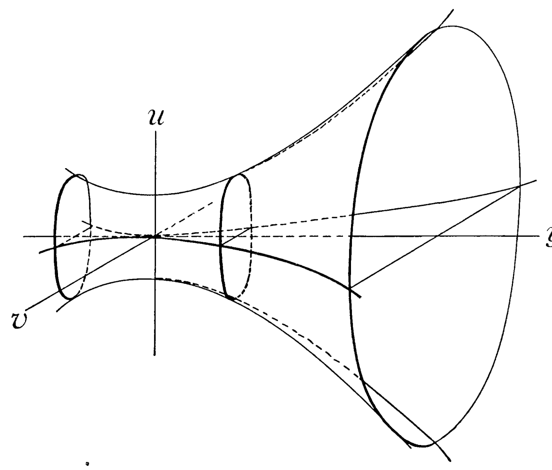


FIG. 8. $\dfrac{u^2}{\cosh^2 y}+\dfrac{v^2}{\sinh^2 y}=1.$

$u+iv=\sin(x+iy)$: while $y$, $u$ *and* $v$ are the rectangular coördinates of any of its points $P$, $x$ is the complement to the eccentric angle of $P$ in the elliptical cross-section $y=\text{const.}$ through $P$ (Fig. 9).

The implied geometric construction of the sine of a complex argument may be regarded as the generalization of the construction of the sine of a real angle $x$. For $y=0$ the ellipse becomes the line segment on the $u$-axis between $\pm 1$. Hence for a given eccentric angle $(90°-x)$ the point $P$ falls on the $u$-axis. If one be-

* This surface is the $v$-surface of the function $w=\arcsin z$.

gins, conversely, with the real angle by projecting, in the usual manner, point $A$ onto the $u$-axis, the transition to the complex argument is made by merely inflating the straight line segment on the $u$-axis into an ellipse of axes cosh $y$ and sinh $y$.

The same surface (Fig. 8) can be used similarly for the representation of all four variables in the function $w = \cos z$ (in which case $x$ is the eccentric angle itself), and $w = \sinh z$ and $w = \cosh z$ (for which $x$ is to be interpreted as the rectangular coördinate and $y$ as the eccentric angle or its complement).

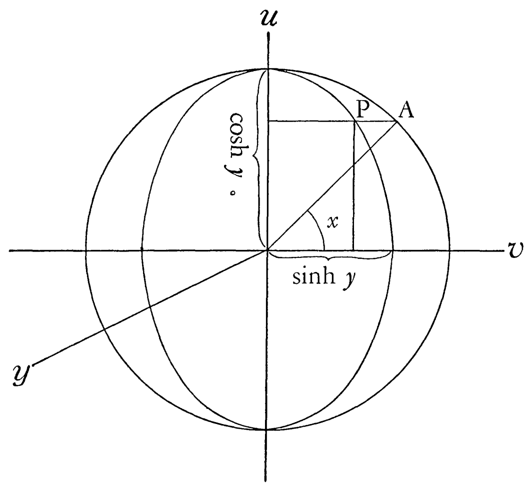


Fig. 9

The repeatedly noticed simultaneous geometric representation of all four variables entering into the functional relation between two complex variables appears in connection with those "sections" which have as a parameter a variable in which the given function is periodic: $y$ in the case of the exponential and hyperbolic functions, $x$ in the case of the circular functions. This periodic parameter appears represented as an angle.

# A TETRAHEDRAL RIEMANN SURFACE MODEL OF A CLOSED FINITE LOCALLY-EUCLIDEAN TWO-SPACE

MICHAEL SADOWSKY, Armour Institute of Technology

The existence of a locally-Euclidean two-space whose fundamental region is a parallelogram, with boundary points identified as indicated in Figure 1, has been shown by Felix Klein.* The aim of the present paper is to show that this fundamental parallelogram may be folded in a Euclidean three-space, remaining metrically invariant, to form a closed finite two-sheeted Riemann surface shaped as a tetrahedron and having no metrically singular points whatsoever as a two-space.

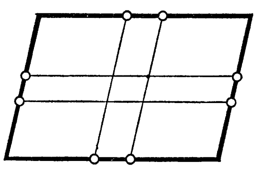

FIG. 1

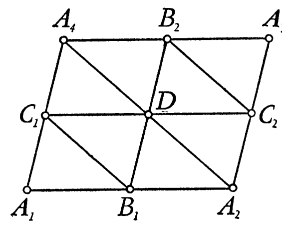


FIG. 2

With $A_1A_2A_3A_4$ as the fundamental parallelogram, let us draw the diagonal $A_2A_4$ joining the vertices $A_2$ and $A_4$ of both obtuse angles, and let us further decompose the parallelogram into eight congruent triangles as shown in Figure 2. Consider the case in which all angles of these triangles are acute. If we make a straight cut from $C_2$ to $D$ and then fold the parallelogram along the seven lines $B_2C_2$, $A_4D$, $DA_2$, $C_1B_1$, $C_1D$, $B_2D$ and $B_1D$ we may form a two-sheeted tetrahedron $ABCD$ such that all 4 points $A_1$, $A_2$, $A_3$ and $A_4$ shall meet at the vertex $A$ of the tetrahedron, while $B_1$ and $B_2$ shall meet at the vertex $B$ of it, $C_1$ and $C_2$ at the vertex $C$, and the center $D$ of the parallelogram shall constitute the vertex $D$ of the tetrahedron (Fig. 3). Both edges of the cut $DC_2$ will meet along the edge $DC$ of the tetrahedron and may be joined again, thus annihilating the cut and closing the surface along $DC$. To join, a self-intersection along $DC$ of the two sheets of the tetrahedral surface becomes necessary, the vertices $C$ and $D$, consequently, becoming branch points of the two-sheeted tetrahedral surface.

After folding, the boundary of the fundamental parallelogram will lie fourfold along the two edges $AB$ and $BC$, and corresponding points of the boundary, as identified by Figure 1, will be coincident. This will permit us to close the surface completely by joining all correspondent parts of the boundary. To perform this closure another self-intersection of the surface along the edge $AB$ shall be

* Felix Klein, Zur nichteuklidischen Geometrie, Mathematische Annalen, vol. 37, 1890; also Collected Papers, vol. 1, p. 371.

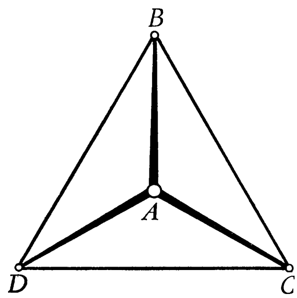


FIG. 3

necessary, and the vertices $A$ and $B$ will become branch points of the two-sheeted surface. This completes the construction.

None of the vertices $A, B, C, D$, considered as points of the closed two-space, is a singular point. The model has no singular points at all (no "stigmata"), being homogeneous. Indeed, each of the four vertices is a branch point of the two-sheeted surface. Consequently, a closed circuit around any vertex in this two-sheeted surface is a rotation through $2\pi$. This proves that the points $A, B, C, D$ are regular points of the locally-Euclidean two-space.

To construct the tetrahedron the triangle $A_1A_2A_4$ (Fig. 2) must have all angles acute, which has been stated previously as an assumption. In case this triangle should have an obtuse angle, folding along $C_1D$, $C_1B_1$, $B_1D$ is useless: the points $A_1$, $A_2$, $A_4$ shall never meet in the three-space, and no tetrahedron may be formed this way. The verification of this is matter of elementary geometry. To find the appropriate method of folding in the obtuse case, let us interpret $\overrightarrow{A_1A_2}=\omega$ and $\overrightarrow{A_1A_4}=\omega'$ as numbers in the complex $\omega$-plane, and let us construct the modulus $\Omega=m\omega+n\omega'$ as shown by Figure 4. The elementary parallelogram $A_1A_2A_3A_4$ may be replaced by any other equivalent elementary parallelogram

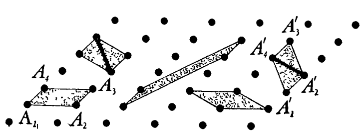


FIG. 4

$A_1'A_2'A_3'A_4'$ of this modulus, which leads us to the question: is it possible to find among all the elementary parallelograms of the modulus such ones as may be bisected by a diagonal into two acute triangles? To analyze, let us use notations

$$A_1'A_2' = \omega_1, \qquad A_1'A_4' = \omega_1',$$

$$\tau = \frac{\omega'}{\omega}, \qquad \tau_1 = \frac{\omega_1'}{\omega_1}\cdot$$

Then

$$\tau_1 = \frac{a\tau + b}{c\tau + d}$$

with real integers $a$, $b$, $c$, $d$, of the elliptic modular group $(ad - bc = 1)$, and

$$J(\tau_1) = J(\tau) = \text{some given number}, \tag{1}$$

since $\tau$ is given.

To realize what the condition of the three acute angles means geometrically, let us turn to Figure 5 which shows the complex $\tau_1$-plane. The notations $A_1'$, $A_2'$, and $A_4'$ correspond in Figures 4 and 5. The points $A_1'$ and $A_2'$ are fixed at 0 and 1 of the $\tau_1$-plane; the possible positions of $A_4'$ are to be discussed now. The condition that $\sphericalangle A_4'A_1'A_2'$ be acute requires that $A_4'$ lie to the right of $A_1'E$; the condition that $\sphericalangle A_4'A_2'A_1'$ be acute requires that $A_4'$ lie to the left of $A_2'F$; the condition that $\sphericalangle A_1'A_4'A_2'$ be acute requires that $A_4'$ lie without the circle $A_1'GA_2'$ whose diameter is $A_1'A_2'$. These three requirements leave for a possible location of $A_4'$ the (open) region shaded in Figure 5. We confine ourselves to the

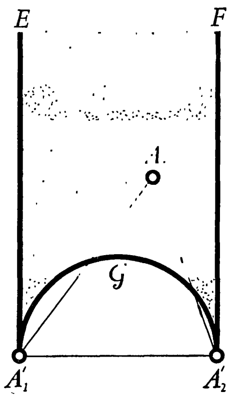


FIG. 5

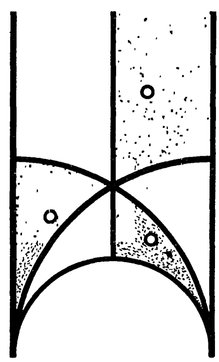

FIG. 6

upper half of the complex $\tau_1$-plane, which is usual in case of elliptic modular functions. The question concerning acute triangles can be reshaped now as follows: does (1) possess any solution in the shaded region of Figure 5?

Figure 6 answers this question as follows: there are three solutions if the modulus of Figure 4 is oblique, and no solution, if it is rectangular. Indeed, any of the three shaded regions is mapped on the upper half-plane by $J(\tau_1)$, and this is a one-to-one mapping for each region. Hence, generally, the number of solutions of (1) is equal to the number of shaded regions which is three. Exceptionally, the solutions might lie on the boundary of the shaded regions and cannot be used then, which happens if the modulus of Figure 4 is rectangular.

The three solutions for $\tau_1$ of the general case do not determine three distinct tetrahedra, but just one single tetrahedron, as explained by Figure 7, because there are three distinct ways to choose $\omega$ and $\omega'$ (with the imaginary part of $\tau$ being positive) with respect to some given face of the tetrahedron. In case of a rectangular modulus there is no proper (three-dimensional) tetrahedron; it becomes flat, a plane rectangle whose four sides and two diagonals form the six edges of the tetrahedron.

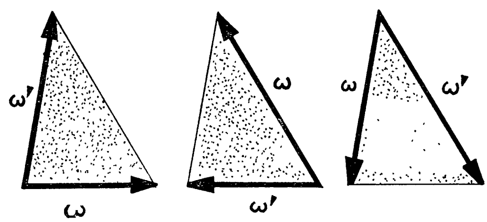


FIG. 7

In case we admit negative values for the imaginary part of $\tau$, we simply interchange $\omega$ and $\omega'$. This produces a tetrahedron which is a symmetric image of the original one (reflection in a plane). This is trivial and shall not be counted a second solution for the tetrahedron. The proof of existence and uniqueness of the tetrahedral Riemann surface has thus been completed. To conclude, Figure 8 shows as an example how to fold (thick lines) an obtuse triangle to obtain a tetrahedron.

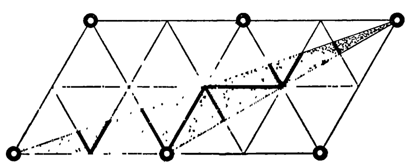

FIG. 8

# A CERTAIN QUADRATIC DIOPHANTINE INVARIANT

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**1. Introduction.** The results of this paper are analogous to those of a joint paper written by the present author with O. E. Brown.* They include as a special case the results of a paper entitled "Arccotangent triads"† written by the present author, but which in view of the results contained in this paper will never be submitted for publication.

An important step in the generalization from "Arccotangent triads" to the present results was taken by my student, Bradford Arnold.

We find it convenient to use the notation of symmetric functions:

$$E_2(x) = \sum_{i,j=1}^{m} x_i x_j,\ i < j;$$

$$E_1(x^2) = \sum_{i=1}^{m} x_i^2.$$

As a result of the theory here developed, it is possible to find an infinite number of integral solutions of such diophantine equations as $E_2(x) = 1$ for any value of $m$. We show that the method gives all the solutions in the case $m = 3$.

**2. The invariant.** We have a set of integers, $x_1, x_2, \cdots, x_m$, called $(x)$ for short, and several linear transformations $\mathfrak{T}(x) = (X)$, and we seek to find quadratic expressions of the form

$$\sum a_{ij} x_i x_j \tag{1}$$

which are invariant under all the transformations $\mathfrak{T}$.

Motivated by the more special and elementary study of arccotangent triads (in a manner which will become obvious later), we limit ourselves to linear transformations $\mathfrak{T}$ of the following forms:

First we have $\mathfrak{T}_-$, which merely changes the signs of all the integers:

$$\mathfrak{T}_-(x) = -x.$$

Any quadratic form (1) is invariant under $\mathfrak{T}_-$, which is introduced merely for the sake of completeness.

Then we have the other $m-1$ trivial, but independent transformations, $\mathfrak{T}_{1,i}$, where $i$ runs from 2 to $m$. $\mathfrak{T}_{1,i}$ merely interchanges the $i$th integer with the first one. Thus:

$$\mathfrak{T}_{1,3}(x_1, x_2, x_3, \cdots, x_m) = (x_3, x_2, x_1, \cdots, x_m).$$

Finally, we have the one non-trivial transformation $\mathfrak{T}_{m,k,n}$ which changes

* Pythagorean sets of numbers. This MONTHLY, May, 1938, pp. 298–301.

† Presented to the American Mathematical Society, June 1936.

the signs of the first $m-k-1$ integers, and adds $n$ times the sum of these $m-k-1$ integers to each of the remaining $k+1$ integers. Thus:

$$\mathfrak{T}_{m,k,n}(x_1, \cdots, x_m) = (X_1, \cdots, X_m)$$

where

$$X_i = -x_i, \text{ for } 0 < i < m-k, \tag{2}$$

and

$$X_i = x_i + n\sum_{j=1}^{m-k-1} x_j, \text{ for } m-k-1 < i < m+1. \tag{3}$$

The non-trivial transformation $\mathfrak{T}_{m,k,n}$ can be distinguished from the trivial transformations $\mathfrak{T}_{1,i}$ since its notation has three subscripts.

Under the transformation $\mathfrak{T}_{1,i}$, expression (1) to be invariant must have the property that $a_{ij}=a_{1j}$, and hence, using all values of $i$ from 2 to $m$, expression (1) assumes the form

$$AE_1(x^2) + BE_2(x). \tag{4}$$

It remains merely to determine $A$ and $B$, or rather their ratio, so that (4) is invariant under the non-trivial transformation $\mathfrak{T}_{m,k,n}$. This $\mathfrak{T}_{m,k,n}$ is not a fixed transformation, because we still have choice of the numbers $m$, $k$, and $n$.

By equating

$$AE_1(x^2) + BE_2(x) = AE_1(X^2) + BE_2(X)$$

and replacing $X_i$ by their values in terms of $x_i$, we are lead to the condition

$$2nA + (nk-2)B = 0, \tag{5}$$

as both necessary and sufficient for (4) to be invariant under $\mathfrak{T}_{m,k,n}$.

The computation by which (5) is derived is interesting, but straightforward, and hence omitted here.

Therefore our invariant may be written in the form

$$I(x_1x_2 \cdots x_n) = (nk-2)E_1(x^2) - 2nE_2(x). \tag{6}$$

**3. Special cases where $A$ vanishes.** For certain choices of $n$ and $k$, we find that $A=0$, and (6) reduces merely to

$$I = E_2(x) \tag{7}$$

after removing the factor $-2n$. For this to happen,

$$nk = 2, \tag{8}$$

whose only integral solutions are $n=1$, $k=2$, and $n=2$, $k=1$. (We note that $k$ cannot be negative.)

Certain choices of $m$, $k$, and $n$ make the transformation $\mathfrak{T}_{m,k,n}$ trivial, and, of course, are to be avoided. Recalling the definition of $\mathfrak{T}_{m,k,n}$, we observe that

if $m-k-1=0$, then $\mathfrak{T}_{m,k,n}$ is merely the identity, and hence trivial. Also, if $n=0$, $\mathfrak{T}_{m,k,n}$ is trivial. While $k$ may be zero, it can not be negative. Were $k=-1$, $\mathfrak{T}_{m,k,n}$ would reduce to $\mathfrak{T}_-$.

*Case* $n=1$, $k=2$, $m=4$. Since $n=1$ or 2, we consider first the case $n=1$. Then $k=2$, and $m>3$, say $m=4$. Then

$$\mathfrak{T}_{4,2,1}(x_1x_2x_3x_4) = (X_1, X_2, X_3, X_4)$$

where

$$X_1 = -x_1, \quad X_2 = x_2 + x_1, \quad X_3 = x_3 + x_1, \quad X_4 = x_4 + x_1,$$

and the invariant may be written

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4. \tag{9}$$

For the particular choice $x_1, x_2, x_3, x_4 = 1, 1, 0, 0$, invariant (9) has the value 1. Let us find other sets of four integers for which invariant (9) has the value 1. In other words, to solve the diophantine equation

$$xy + xz + xw + yz + yw + zw = 1, \tag{10}$$

use the transformations

$$\begin{aligned}
\mathfrak{T}_{4,2,1}(1, 1, 0, 0) &= (-1, 2, 1, 1),\\
\mathfrak{T}_{1,3}(-1, 2, 1, 1) &= (1, 2, -1, 1)\\
\mathfrak{T}_{4,2,1}(1, 2, -1, 1) &= (-1, 3, 0, 2),\\
\mathfrak{T}_{1,2}(-1, 3, 0, 2) &= (3, -1, 0, 2),\\
\mathfrak{T}_{4,2,1}(3, -1, 0, 2) &= (-3, 2, 3, 5),
\end{aligned}$$

and so forth. Thus, we can obtain an unlimited number of sets of solutions of (10) by successive applications of the transformations. The trivial transformations merely rearrange the elements, but it is necessary to use them in combination with the non-trivial transformations in order to generate new solutions.

We could also take $n=2$ and $k=1$. Thus

$$\mathfrak{T}_{4,1,2}(1, 2, -1, 1) = (-1, -2, 5, 7),$$

and so forth. However, $(-1, -2, 5, 7)$ could also have been reached without $\mathfrak{T}_{4,1,2}$ by using merely $\mathfrak{T}_{4,2,1}$ and the trivial transformations.

In a similar manner, an unlimited number of solutions of the equation

$$E_2(x) = 1, \tag{11}$$

for any value of $m$, are readily obtained by starting with the trivial solution $(1, 1, 0, 0, 0, \cdots)$ and transforming by the trivial and the non-trivial transformations subject to condition (8).

4. **Arccotangent triads.** The case $m=3$, $k=1$, and $n=2$ has for its invariant

$$xy + xz + yz. \tag{12}$$

Under $\mathfrak{T}_{3,1,2}$, the sign of the first integer is changed, and twice that integer is added to each of the other two. Thus,

$$\mathfrak{T}_{3,1,2}(1, 1, 0) = (-1, 3, 2),$$
$$\mathfrak{T}_{1,3}(-1, 3, 2) = (2, 3, -1),$$
$$\mathfrak{T}_{3,1,2}(2, 3, -1) = (-2, 7, 3),$$

and so forth. These are all solutions of the diophantine equation,

$$xy + xz + yz - 1 = 0, \tag{13}$$

which may be obtained by taking the tangents of both members of the congruence

$$\operatorname{arccot} x + \operatorname{arccot} y + \operatorname{arccot} z \equiv 0 \pmod{\pi}. \tag{14}$$

In fact (13) and (14) are *equivalent*, provided the arccotangents signify the general values, rather than the principal values. For this reason, solutions of equation (13) are called *arccotangent triads*.

For example, the arccotangent triads (1, 1, 0), $(-1, 3, 2)$, and $(-2, 3, 7)$ lead to the trigonometric congruences

$$2 \operatorname{arccot} 1 + \operatorname{arccot} 0 \equiv 0 \pmod{\pi},$$
$$\operatorname{arccot}(-1) + \operatorname{arccot} 3 + \operatorname{arccot} 2 \equiv 0 \pmod{\pi},$$
$$\operatorname{arccot}(-2) + \operatorname{arccot} 3 + \operatorname{arccot} 7 \equiv 0 \pmod{\pi},$$

the last two of which may be written in the more familiar form:

$$\arctan 1 \equiv \arctan \tfrac{1}{3} + \arctan \tfrac{1}{2} \pmod{\pi},$$

and

$$\arctan \tfrac{1}{2} \equiv \arctan \tfrac{1}{3} + \arctan \tfrac{1}{7} \pmod{\pi},$$

which are useful in finding formulas for computing $\pi$.

**5. Special cases in which neither $A$ nor $B$ vanish.** In studying invariant (6), we have considered special cases in which $A$ vanishes. There are no interesting cases in which $B$ vanishes; for if $B=0$, then $n=0$ by equation (5), and all the transformations are trivial.

If $m$ is small, invariant (6) has only a few terms even if neither $A$ nor $B$ vanish. The simplest case of this is $m=2$, $k=0$, and $n$ arbitrary. (The case $k=1$ is trivial, for $T_{2,1,n}$ is the identity.) Invariant (6) now becomes

$$x^2 + y^2 + nxy. \tag{15}$$

Suppose it is desired to plot the curve

$$x^2 + y^2 + nxy = 1. \tag{16}$$

Immediately, we have the solution (1, 0). Then we have

$$\mathfrak{T}_{2,0,n}(1,\ 0) = (-\ 1,\ n),$$
$$\mathfrak{T}_{1,2}(-\ 1,\ n) = (n,\ -\ 1),$$
$$\mathfrak{T}_{2,0,n}(n,\ -\ 1) = (-\ n,\ -\ 1 + n^2),$$

and so forth. Each solution gives a point on the curve.

**6. Completeness.** Given the diophantine equation (13) and the particular solution (1, 1, 0), we have seen how the transformations $\mathfrak{T}_-$, $\mathfrak{T}_{1,2}$, $\mathfrak{T}_{3,1}$, and $\mathfrak{T}_{3,1,2}$ generate many other solutions. We now show that they generate all integral solutions. Unfortunately we are at present unable to prove the corresponding result for the equation

$$E_2(x) = M \tag{17}$$

except when $m=3$ and $M=1$ or 2. We shall show that in the case $m=3$, $M=3$ all the integral solutions are not generated from one.

DEFINITION. If $(x, y, z)$ is any triad of integers, the value of $xy+xz+yz$ is called the *measure* of $(x, y, z)$, written $M(x, y, z)$.

DEFINITION. A *positive triad* is one having no negative elements.

THEOREM 1. *If $(x, y, z)$ is any triad of positive measure, then by a suitably chosen sequence of the transformations $\mathfrak{T}_-$, $\mathfrak{T}_{1,2}$, $\mathfrak{T}_{1,3}$, and $\mathfrak{T}_{3,1,2}$ allowing repetitions, $(x, y, z)$ can be transformed into a positive triad.*

*Case* 1. If one of the elements $x$, $y$, $z$ vanishes since $xy+xz+yz>0$, the other two elements must have the same sign. If they are both negative, then $\mathfrak{T}_-(x, y, z)$ is the desired positive triad.

*Case* 2. If $x$, $y$, and $z$ all have the same sign, then $(x, y, z)$, or $\mathfrak{T}_-(x, y, z)$ will be a positive triad.

*Case* 3. If two of the elements are negative and one positive, $\mathfrak{T}_-(x, y, z)$ falls under case 4.

*Case* 4. If two of the elements are positive and one negative, then by use of $\mathfrak{T}_{1,2}$, and $\mathfrak{T}_{1,3}$ the elements may be so permuted that $x$ is negative and the other two elements are positive. Then

$$yz = M + (-x)y + (-x)z,$$

and since $(-x)$, $y$, and $z$ are all positive, and $M \geqq 0$, we see that $(-x)$ must be smaller than either $y$ or $z$. Therefore $|2x+y| < y$, and $|2x+z| < z$. But $\mathfrak{T}_{3,1,2}(x, y, z) = (-x,\ 2x+y,\ 2x+z)$, and the $y$- and $z$-elements of the new triad are smaller in absolute value than for $(x, y, z)$ and not both equal to zero, while their $x$-elements are the same.

The transformation $\mathfrak{T}_{3,1,2}(x, y, z)$ will then fall under one of the four cases. If it falls under case 1 or 2, the proof is complete. If it falls under 3 or 4, further steps are required. At each step, two of the elements become smaller in absolute value without becoming zero simultaneously, and since they are integers the

process must terminate. Therefore, after a finite number of steps one arrives at a triad under case 1 or 2, and hence a positive triad.

THEOREM 2. *Each of the transformations* $\mathfrak{T}_-$, $\mathfrak{T}_{1,2}$, $\mathfrak{T}_{1,3}$, $\mathfrak{T}_{3,1,2}$ *is its own inverse.*

THEOREM 3. *If* $(x, y, z)$ *is any integral triad of measure unity, then by a suitably chosen sequence of transformations* $\mathfrak{T}_-$, $\mathfrak{T}_{1,2}$, $\mathfrak{T}_{1,3}$, *and* $\mathfrak{T}_{3,1,2}$, *allowing repetitions*, $(x, y, z)$ *can be transformed into* $(1, 1, 0)$.

*Proof.* By Theorem 1, $(x, y, z)$ can be transformed into $(1, 1, 0)$, $(1, 0, 1)$ or $(0, 1, 1)$ since they are the only positive triads of measure unity.

THEOREM 4. *The triad* $(1, 1, 0)$ *can be transformed into any integral triad of measure* $M=1$ *by a suitable sequence of the transformations of Theorem* 1, *and similarly* $(2, 1, 0)$ *can be transformed into any integral triad of measure* $M=2$.

In case of triads of measure 3, there are $(1, 1, 1)$ and $(3, 1, 0)$ which are both positive, and which can not be transformed into each other by any possible sequence of our transformations.

In fact, every transform of $(1, 1, 1)$ will have three odd integers, and every transform of $(3, 1, 0)$ will have two odd and one even integer. Therefore, when $m=3$ and $M=3$, all the solutions can not be generated from a single solution, but require the two initial solutions, $(1, 1, 1)$ and $(3, 1, 0)$. It is readily seen by trial that all positive integral triads of measure 3 are permutations of one of those two.

The number of initial solutions required before our transformations can give all the integral solutions varies with the value of $M$. For a particular value of $M$, it is merely necessary to find all the positive integral triads with that measure. Since their number is finite, they can all be found by trial. For example, if $M=7$, there would be $(7, 1, 0)$, none starting with a 6, a 5, or a 4, $(3, 1, 1)$, no others with a 3, and none with largest element 2. Therefore, the only positive integral triads of measure 7 would be permutations of $(7, 1, 0)$ and $(3, 1, 1)$. By Theorem 1, all integral triads of measure 7 may be transformed into a positive integral triad of the same measure, and hence into one of the above two.

**7. Conclusion.** We have found transformations under which the quadratic expression (6) in any number $m$ of variables is invariant. These transformations generate an infinite group. Now suppose a change of the coördinate system under which invariant (6) assumes the form of the sum of the squares of the $m$ variables. The change to this new coördinate system brings in imaginaries, so that the transformations of our infinite group are imaginary. Since the invariant is merely the sum of the squares of the variables, the transformations under which it is invariant must all be rotations. Therefore the transformations of our infinite group are merely rotations. Each rotation is a finite rotation. So we have proved that there exists an infinite group of finite imaginary rotations in $m$ dimensions.

# ON THE LOCATION OF THE ROOTS OF REAL POLYNOMIAL EQUATIONS WHEN TWO ROOTS ARE EQUAL*

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**1. Introduction.** If in the real polynomial equation

$$f(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0 \tag{1}$$

we set $z = x + iy$ and expand by Taylor's Formula about the point $x$, we obtain

$$\begin{aligned} f(z) &= f(x) + iyf'(x) - y^2 \frac{f''(x)}{2!} - \cdots + (iy)^n \frac{f^{(n)}(x)}{n!} \\ &= u(x, y) + iy\phi(x, y), \end{aligned}$$

where the polynomials $u$ and $\phi$ are real. If, moreover, $x+iy$ is a root of equation (1), then the point $(x, y)$ lies on the curve $u(x, y) = 0$ and also on either the axis, $y = 0$, or the curve†

$$\phi(x, y) \equiv f'(x) - y^2 \frac{f'''(x)}{3!} + \cdots + (iy)^{\lambda-1} \frac{f^{(\lambda)}(x)}{\lambda!} = 0, \tag{2}$$

where $\lambda$ is the largest odd integer $\leqq n$.

The polynomial $\phi(x, y)$ is independent of the constant term $a_0$ of the equation (1). Hence if the curve (2) is plotted in the $x,y$-plane and then superimposed on the complex $z$-plane (the $x,y$-axes coinciding with the $x,iy$-axes), it gives, together with the real axis, the locus in the complex plane of the roots of equation (1) with $a_0$ considered as a parameter. Moreover, since $a_0$ is the $y$-intercept of the curve $y = f(x)$, increasing or decreasing $a_0$ does not change the shape of this curve but has the effect merely of lowering or raising the $x$-axis with respect to the curve. Because of this relationship between these two curves, $\phi(x, y) = 0$ and $y = f(x)$, with $a_0$ as parameter, we make the following definition:

DEFINITION. *The curve $\phi(x, y) = 0$ is said to be the zero-curve corresponding to the polynomial equation $f(x) = 0$.*

If (1) has a real double root, it occurs in the $z$-plane at an intersection of (2) with the real axis. For a cubic equation the zero-curve is a hyperbola, and the double root occurs at a vertex of this hyperbola. For the cubic the author noticed that when there was a double root at the vertex of one branch of the hyperbola, then the third root was at the *focus* of the other branch. It is the object of this paper to prove that a similar situation holds for all real polynomial equations. In other words, we shall prove the following:

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* Presented to the American Mathematical Society, September 6, 1938.

† J. A. Ward has discussed these curves for the cases $n = 3, 4$ in connection with obtaining a geometrical construction for the complex roots of the cubic and quartic equations. See J. A. Ward, *Graphical Representation of Complex Roots*, National Mathematics Magazine, vol. 11 (1937), pp. 297–303.

THEOREM. *If a polynomial equation of degree n with real coefficients has two of its n roots equal, and if $\alpha=\beta+i\gamma$ is any one of the remaining $n-2$ roots, then the point $(\beta, \gamma)$ is a real focus of the corresponding zero-curve.*

When equation (1) has a double root, let us denote the roots by

$$(3) \qquad \alpha_k = \beta_k + i\gamma_k, \qquad (k = 1, 2, 3, \cdots, n), \text{ where } \alpha_1 = \alpha_2 = \alpha.$$

Then the theorem states that the points $(\beta_k, \gamma_k)$, $(k=3, \cdots, n)$, are real foci of the zero-curve (2). Here $\alpha_k$ may be equal to $\alpha$ for one or more of the values $k=3, \cdots, n$.

**2. Foci of the zero-curve.** The foci of an algebraic plane curve are defined* to be the points of intersection of the isotropic tangents, i.e., the tangents to the curve from the circular points at infinity, $I$ and $J$. If $x+iy=a+ib$ is a tangent from $I:(1, i, 0)$, then $x-iy=a-ib$ is a tangent from $J:(1, -i, 0)$, and these two tangents intersect in the real focus $(a, b)$.

Hence one method of obtaining the real foci is as follows. The pencil of lines through $I$ is

$$(4) \qquad x + iy = c, \quad \text{where} \quad c = a + ib.$$

We eliminate $y$ between (2) and (4), obtaining

$$(5) \qquad \bar{\phi}(x, c) \equiv f'(x) + (c - x)^2 \frac{f'''(x)}{3!} + \cdots + (c - x)^{\lambda-1} \frac{f^{(\lambda)}(x)}{\lambda!} = 0.$$

This is an equation of degree $n-1$ in $x$, whose coefficients contain both the $a_i$. $(i=1, \cdots, n)$ of the original equation and also the parameter $c$. If $c$ is such that (5) has a double root, then we have a line of the pencil (4) which is tangent to the zero-curve, and the point $(a, b)$ is a real focus of that curve. We have to prove then that

$$(6) \qquad \bar{\phi}(x, c) \equiv \phi[x, - i(c - x)] = 0$$

will have a double root whenever $c=\alpha_k$ is a root of $f(z)=0$ and either $\alpha_k$ is not equal to the double root $\alpha$ or $\alpha_k=\alpha$ and $\alpha$ is at least a triple root.

**3. Methods of proof.** The author originally proved this theorem using methods almost entirely algebraic. This proof proceeds as follows: Equation (5) will have a double root if its discriminant is zero. This discriminant, except for a constant factor, can be written down as the resultant of $\bar{\phi}$ and $d\bar{\phi}/dx$. Hence the condition for a double root can be written as a determinant of order $2n-3$, representing a real polynomial equation in $c$. If $c=a+ib$ is a root of this determinant equation, then the point $(a, b)$ is a real focus of the zero-curve. The theorem will therefore be proved if we can show that the $\alpha_k$, $(k=3, \cdots, n)$ are roots of this equation. In examining this determinant with $c=\alpha_k$ the author discovered that if the last column is divided by $(\alpha_k+\alpha)/2$ and added to the next to the

* See, for instance, Hilton, *Plane Algebraic Curves*, p. 69.

last column, this new column divided by $(\alpha_k+\alpha)/2$ and added to the preceding column, and so on, then eventually a determinant is obtained whose left-hand column is composed entirely of zeros. It turns out that to prove this fact, it is sufficient to prove that

$$\bar{\phi}\left(\frac{\alpha_k+\alpha}{2},\ \alpha_k\right)=0 \quad \text{and} \quad \frac{d}{dx}\bar{\phi}\left(\frac{\alpha_k+\alpha}{2},\ \alpha_k\right)=0. \tag{7}$$

This can be done algebraically by setting up the equation $\theta(x)=0$, of degree $n-1$, with roots $\alpha+\alpha$, $\alpha+\alpha_3, \cdots, \alpha+\alpha_n$ and the equation $\psi(x)=0$, of degree $n-2$, with roots $\alpha+\alpha_3, \cdots, \alpha+\alpha_n$. Then it is shown that $\theta(\alpha+\alpha_k)$ and $\psi(\alpha+\alpha_k)$ are identical with the left-hand sides of equations (7), which are therefore numerical identities. The proof of this is carried through by inductive methods and involves the proof and use of numerous identities among the binomial coefficients.

However, equations (7) show that $(\alpha_k+\alpha)/2$ is a double root of (6) when $c=\alpha_k$, $(k=3, \cdots, n)$. Hence to prove the theorem it is sufficient merely to show this: *i.e.*, that with $c=\alpha_k$, $(\alpha_k+\alpha)/2$ *is a double root of* (6). In refereeing this paper, Dr. J. W. Givens pointed out that this could be done directly by analytic methods, quite different from those originally employed by the author. His proof is materially shorter than the author's, and hence has been incorporated here.

**4. Proof of the Theorem.** Since

$$f(x+iy)=u(x,y)+iy\phi(x,y) \tag{8}$$

is an identity in $x$ and $y$, it holds even when $x$ and $y$ are complex. Let us substitute in this equation $x=(\alpha_k+\alpha)/2$ and $y=-i(\alpha_k-\alpha)/2$. This gives

$$0=f(\alpha_k)=u\left(\frac{\alpha_k+\alpha}{2},\ -i\frac{\alpha_k-\alpha}{2}\right)+\frac{\alpha_k-\alpha}{2}\phi\left(\frac{\alpha_k+\alpha}{2},\ -i\frac{\alpha_k-\alpha}{2}\right). \tag{9}$$

Similarly, substituting $x=(\alpha_k+\alpha)/2$ and $y=+i(\alpha_k-\alpha)/2$ gives

$$0=f(\alpha)=u\left(\frac{\alpha_k+\alpha}{2},\ +i\frac{\alpha_k-\alpha}{2}\right)-\frac{\alpha_k-\alpha}{2}\phi\left(\frac{\alpha_k+\alpha}{2},\ +i\frac{\alpha_k-\alpha}{2}\right). \tag{10}$$

Since $u(x,y)$ and $\phi(x,y)$ are *even* functions of $y$, the values of $u$ and of $\phi$ occurring in (9) and (10) are the same. Hence, subtracting, we have that

$$\frac{\alpha_k-\alpha}{2}\phi\left[\frac{\alpha_k+\alpha}{2},\ -i\left(\alpha_k-\frac{\alpha_k+\alpha}{2}\right)\right]=0.$$

If $\alpha_k\neq\alpha$, this proves that $x=(\alpha_k+\alpha)/2$ is a root of (6). If $\alpha_k=\alpha=c$, we observe that for $x=\alpha$, $c-x=0$ and $\phi[x,\ -i(c-x)]$ reduces to $f'(\alpha)$, which is zero. Hence in any case (6) has the root $(\alpha_k+\alpha)/2$.

To prove that $(\alpha_k+\alpha)/2$ is a double root, we show that it is a root of the equation

$$\frac{d}{dx}\phi[x, -i(c-x)] = 0. \tag{11}$$

This derivative can be expressed as $[(\partial\phi/\partial x)+i(\partial\phi/\partial y)]$ *evaluated for* $y=-i(\alpha_k-x)$. One of the Cauchy-Riemann equations is, from (8),

$$\frac{\partial u}{\partial x} = \phi + y\frac{\partial \phi}{\partial y}\cdot$$

Hence (11) can be written in the form

$$\left[\frac{1}{-iy}\left\{-iy\frac{\partial\phi}{\partial x}+\frac{\partial u}{\partial x}-\phi\right\}\right]_{y=-i(\alpha_k-x)} = 0. \tag{12}$$

But

$$-iy\frac{\partial\phi}{\partial x} = (-iy)f''(x) + (-iy)^3\frac{f^{(\mathrm{IV})}(x)}{3!} + \cdots + (-iy)^\lambda\frac{f^{(\lambda+1)}(x)}{\lambda!},$$

and

$$\frac{\partial u}{\partial x} = f'(x) + (-iy)^2\frac{f'''(x)}{2!} + \cdots + (-iy)^{2\mu}\frac{f^{(2\mu+1)}(x)}{(2\mu)!},$$

where $\lambda$ is odd and $\leqq n$ and $2\mu\leqq n$. Hence $-iy(\partial\phi/\partial x)+(\partial u/\partial x)=f'(x-iy)$. Equation (12) can now be written in the form

$$\frac{1}{\alpha_k - x}\left\{f'(2x-\alpha_k) - \phi[x, -i(\alpha_k - x)]\right\} = 0. \tag{13}$$

If $\alpha_k\neq\alpha$, we may substitute $x=(\alpha_k+\alpha)/2$ and find that the equation (13) is satisfied because $2x-\alpha_k=\alpha$ and $f'(\alpha)=0$. The only remaining step is to prove that if $\alpha_k=\alpha$ is at least a triple root, then $x=(\alpha_k+\alpha)/2=\alpha$ is a root of (11). Referring to (5) we see that every term of the derivative of $\phi(x, c)$ will contain a factor $c-x$ (which is zero when $x=\alpha=\alpha_k=c$) with the single exception of the term $f''(x)$, which is zero for $x=\alpha$.

# MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This Department affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

## THE COLLEGE TEACHER OF MATHEMATICS LOOKS AT TEACHER TRAINING*

A. A. BENNETT, Brown University

### PART I

Probably most of us who have been facing classes for over a quarter of a century were from childhood indoctrinated with the view that practically all academic and religious problems admit categorical solutions. The teachers and the preachers knew all the answers. The preacher and all other high minded people (like ourselves) distinguished right from wrong without need of fresh recourse each time to Holy Writ. The teachers also, although in their case by the occasional discreet aid of a surreptitious "answer book," could afford to be no less dogmatic. But such views if they were ever justifiable are hardly acceptable today. I am here to raise questions, not to settle them. Perhaps you know the answers—I do not.

The topic assigned to me is one of methodology. More than one responsible authority has expressed the view that methodological subjects are hardly appropriate for a serious address by a college professor, that such matters are rather to be discussed, if at all, informally after dinner. Certainly the average professor of mathematics is no specialist in any phase of teacher training. My own excuse for speaking is that I am drafted. The explanation is doubtless that my home institution, Brown University, differs from many in that several projects aimed toward teacher training have been in operation there for many years. However, I would offer no apologies for the choice of the topic. As those in this audience well know, there is a perennial and increasing interest in the problems of mathematical education even at the collegiate level as evidenced by numerous articles on this subject in *The Mathematics Teacher*, *The Journal of Engineering Education*, THE AMERICAN MATHEMATICAL MONTHLY, *The Bulletin of the American Association of University Professors*, *The Association of American Colleges Bulletin*, *Proceedings of the Association of American Universities*, *Harvard Educational Review* (formerly *Harvard Teachers Record*); *Science Education*, *Yearbooks* of the National Council of Teachers of Mathematics, Monographs in the series published by Teachers College, Columbia University, entitled *Contributions to Education*, and other journals—not to mention complete books on

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* An address delivered at a joint meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics, Williamsburg, Virginia, December 30, 1938.

the one hand and unpublished addresses on the other. For the periodical literature one may consult the *Education Index*, under the heading "mathematics," and the department conducted by N. Lazar in the current numbers of the *Mathematics Teacher* and that by C. A. Hutchinson in the AMERICAN MATHEMATICAL MONTHLY.

On the relatively rare occasions when the professor of mathematics in a liberal arts or engineering college looks at teacher training he may be thinking of any of, say, five distinct problems, all of which are worth mentioning.

(1) *The state of efficiency of the present secondary school teacher of mathematics* in view of the performance during the first year in college of selected high school graduates. This is a subject which the professor would ordinarily pass over in silence as a remote topic. One may abstractly view with alarm and in egregious cases express exasperation. Rarely does the college instructor pause to commend or investigate, and seldom indeed does the college professor of mathematics offer a constructive suggestion, save for the notable exceptions where extension courses for credit or summer school programs bring the high school teacher and the college professor into direct contact with each other. In such contacts both often show up at their worst. To put it bluntly, the secondary school teacher may haggle over marks and grades, seem unreceptive to new ideas, dogmatic in his errors, and blind to the beauties of the subject; while the college professor is likely to be careless of the rudiments of good teaching, smugly ignorant of the problems of the secondary school, coldly superior, and may seem to place a grossly exaggerated emphasis upon the minutiae of logical rigor. If the high school teacher stacks up poorly against the college undergraduate in mental elasticity, the college professor often shows himself as several stages below a ninth grade pupil in arithmetical speed and accuracy and entirely unsuited to a teaching career with respect to ideals of educational methodology. With these ungracious words depicting an all too common situation, let us turn to the second of the five problems which the college professor may have in mind when he speaks of teacher training.

(2) *The curriculum of colleges of education.* Most professors of mathematics in a liberal arts college do not question for a moment the importance of systematic training in educational technique for prospective secondary school teachers. Beyond this they usually show little interest and no knowledge of the details of the problem. The professor of mathematics is likely to view with amazement and some disgust the rapid rise of institutions and institutional regulations which seem to devote so much repetitive drill to the theory of method and yet require so little well-grounded knowledge of subject matter and of its background. Common decency suggests that the college professor, such as myself, either make a careful study of the problem of teacher preparation as conducted in the colleges of education or refrain from passing judgment. An appeal to the example of foreign countries is, of course, decidedly inappropriate during the current trend in international developments. The need for cooperation is obvious. One might consult, for example, E. R. Hedrick. Desirable co-

operation between educationists and mathematicians, *School and Society*, vol. 36, 1932, pp. 769–777, and R. J. Havighurst, Can mathematicians and educationists cooperate?: *Mathematics Teacher*, vol. 30, 1937, pp. 211–213.

But the professor of mathematics may come nearer home and consider the offerings of his own department, and so we have

(3) *The offerings of the college which contribute toward preparation of secondary school teachers.* Relatively few liberal arts curricula make specific provision for the prospective teacher of high school mathematics, although in many a college a large proportion of the students concentrating in the mathematical field are planning to teach in secondary schools. The usual attitude is easy to explain. Specific training in routine details of methodology seems incompatible with the ideal of a liberal course in the humanities. Free the student's mind of narrow preconceptions by such liberalizing courses as projective geometry or modern algebra or number theory, give him a taste of the power and beauty of functional analysis. Let him take a course in logic with the philosophers, and courses in history. Let him learn at first hand from physics of the need of mathematics in modern science—but a course in how to teach factoring or Euclid pruned down to a shorter catechism—never! The contrast is exhibited by such books as J. W. Young,* *Fundamental concepts of algebra and geometry*, and the Carus monographs on the one hand, and J. W. A. Young,† *The Teaching of Mathematics* or Joseph Seidlin,‡ *A critical study of the teaching of elementary college mathematics*, on the other. This line of thought sounds plausible, but one can at least argue other views. Is a mathematics course—planned at every stage to push students along the thorny road to a doctorate—really liberal? May it not be deep but narrow and professional, sacrificing the normal intellectual interests of the majority for the professional needs of a select minority? Is there a place in the modern college, whose graduates will spread into fields of business and the arts, for the student who would like to understand what he can of the meaning and value of mathematics, but will not spend seven to ten year-courses in this pursuit? Is his culture best furthered by a course in analytic geometry and the calculus? Is the ideal freshman text to be found among the carefully written (but to many critics unmotivated) drill-books with which we have all been so long conversant, such as say Granville's *Calculus*, or is even a freshman entitled to more inspiration and a broader view, such as being promised at the University of Chicago, or is to be expected from the text,§ *Introduction to mathematics* by Cooley, Gans, Kline, and Wahlert. Should the department of mathematics give courses in statistics or the theory of investment, for students of limited mathematical aptitude? In particular, what of the future high school teacher who finds little significance in differential equations and Fourier series as these are taught, and could never even pass a respectable course in real variable

* Macmillan, 1935.

† Longmans, 1906, 1924.

‡ Contributions to Education, No. 482. Teachers College, Columbia University, 1931.

§ Houghton Mifflin, 1937.

theory. Is he a better teacher of high school mathematics for being rejected by the mathematics department and turned over to others to train? May it not be better tactics either to see that a student who avoids mathematics in college shall be prevented from teaching the subject in high school, or failing this power of prevention, see that he who plans to teach in high school shall be encouraged to secure a broad mathematical background in college? Most college undergraduates either show definite suggestion of ability to start specialized graduate study or else never learn much of the history of mathematics, of modern views of geometry, of even rudimentary number theory, or of the concept of an algebraic field. What ought the college to do?—Perhaps you know the answer.

The college professor may come still nearer home where the directness of the problem becomes obvious and his own responsibility becomes clear. The professor of mathematics may think of the problem of teacher-training as involving

(4) *The prerequisites as to teaching performance for appointment to a college faculty.* Most college authorities at present certainly view with suspicion and would combat with vigor any attempt to impose the requirements favored by many educationists, as prerequisite for appointment to a college faculty. At the collegiate level, surprisingly many problems of school management practically dissappear. The students are selected, more mature, and all terminal examinations are imposed from within the staff. The college usually feels little more than a disciplinary responsibility toward those in the student body who are disinclined to profit by the richness of their academic opportunities. The college staff itself is a group of individualists who may indeed be the ornament of the community, but who intend to offer little cooperation along lines of elaborate administrative routine.

In many colleges good teaching as such or even ready mastery of elementary topics is not the prime ideal. While seeming success with students is acknowledged as valuable, it has been neither a necessary nor a sufficient condition for tenure or promotion. Not a few colleges can afford to take on only experienced teachers—persons whose previous teaching record as gathered from personal impressions of their superiors is at least passable. The college may do nothing officially about improving the quality of teaching. Ordinary good manners and a fair level of teaching performance are taken for granted. Serious deviations from accepted standards may be the basis for private interview or in extreme cases for official reprimand or even dismissal. Of course, some professors have atrocious table manners, or are extremely slovenly as to dress, or succeed very poorly in transmitting and evoking ideas in the class room. But such disagreeable details are often condoned in the presence of more valued attributes. An old graduate whose young hopeful, after a highly gratifying previous year as senior in high school, comes home from college at the close of his freshman year with a depressing list of D's and E's, is inclined to say that the first duty of the college is effective instruction. "Let the professor engage in research to his heart's content, and may the world be richer therefor; but for

Heaven's sake give the starry-eyed freshman a square deal in return for his idealism, if not for his high tuition!"

Finally the professor of mathematics may think of teacher training as concerned with

(5) *The supervision to be exercised over the inexperienced young graduate assistant in his first year or two of instructional practice.* In many cases the young instructor is left to his fate, to blunder through somehow with scarce a word of advice. He may be too awe-struck, too inexperienced, and too sensitive to ask direction in matters very easily adjusted but often momentous in their immediate consequences to his own advancement and the success of his charges. If the young instructor is of the sort to develop into a good teacher, he will welcome encouraging supervision and profit greatly by it. He can be warned against typical mistakes and learn many simple and effective methods which might not suggest themselves to him otherwise. I shall not seek to deliver a schedule of *do*'s and *don't*s. It will suffice to mention merely the following recent articles, an excellent two-page summary of "Advice to the graduate assistant" by A. D. Campbell of Syracuse University in the AMERICAN MATHEMATICAL MONTHLY for last January, vol. 45, 1938, pages 32–33; a survey, "Preparation of College Teachers" by Homer L. Dodge in the *Journal of Engineering Education* for last September, vol. 29 (new series), 1938, pp. 62–72; and a mimeographed manual by Professor Ralph Beatley of Harvard University.

## PART II

Thus far I have viewed the topic assigned me in its generality, and have touched upon five distinct interpretations of the proposed subject. Each of these is far too extensive in its implications and relations to make it possible to do justice to even one of them. Instead, with your indulgence I shall talk upon only two special problems. (1) the selection of prospective teachers, (2) the object of examinations.

Among the many thousands of topical headings listed in the *Education Index*, I find none that seems devoted to this matter of the selection of prospective teachers. Of course many young people prepare for a teaching career and fail to find a vacancy in their own small community, which for their own personal reasons and also because of the local pride of other communities is the only region to be considered. Are there tendencies at work in the educational process which weed out the unfit and encourage only those best adapted to a teaching career? Some forces are surely obvious—but whether they are what the community needs may not be so clear.

The stupid do not survive a severe course of advanced mathematical concentration in college. They fall by the wayside in the educational struggle—but often teach mathematics just the same, because any mathematics is widely accepted as an easy subject to assign to any teacher. The erratic but brilliant student may have too much imagination and too little routine accuracy to stand well the steady grind of daily drill. He may become restive under a pro-

gram intended to drag the mediocre along and decide that teaching holds no attractions for him. The vivacious student full of life, fond of good times and pleasant company is soon displaced by the grind who burns his mid-night oil and never cuts a class. Among the college seniors there are many young persons with high ambitions, many with culture and charm, with vigor and originality. How many of these plan to go into teaching?—some, but pathetically few. The college too often encourages into a teaching career, primarily the social misfit, the conscientious student with narrow ambitions, and limited social poise, the young man who studies in the absence of stronger temptations, and proposes to teach in the absence of any other prospects. At the collegiate level, what is the picture that the student shares with the community as the portrait of the typical professor? At the risk of giving only ridiculous caricatures let me suggest a few typical concepts.

1. *The professor as autocrat.* This picture, apparently quite frequently realized some decades ago, presents the professor as a man of cold temper and incisive tongue, a stern moralist, a practical exponent of the disciplinary value of scholarship—a local Jove, well versed in all traditional lines of study, but unknown beyond his own campus and too absorbed in academic duties to condescend to authorship. He lectures, perhaps, but hardly explains. Like his textbook and his lectures, he changeth not. Openly feared, he is also furtively liked. His mannerisms are a by-word, but he is reputed to see through human frailties of students with vision trained by long discipline, and beneath a frosty exterior to harbor human sentiments. He believes in the value of character, sets high standards, but does not expect to see them fulfilled. He asks for obedience, and faith, not for originality, skeptical criticism, or exercise of personal judgment. For professors of this type the college of tomorrow may have no place, and yet much of our process of academic selection tends to produce just such characters.

2. *The absent-minded professor.* This age-old fiction is dear to the hearts of our citizens. The professor is pictured as the specialist of minutiae; one who can read a dozen languages, but fails to understand plain English; one who in daily affairs is more helpless than a child; one who is unconscious of persons and of social amenities; who can cover a blackboard while numbling to himself, but can neither ask nor answer an intelligent question in an intelligible way. Many a small school has pedagogues suggestive of this type, and a university can well have many; but such a professor, if one exists, has no place trying to teach freshmen. But what does the college do to train mathematics students to act like human beings, to speak simply and clearly, to respond to simple-minded questions with kindly and appropriate answers, to mingle with strangers readily on an easy conversational basis? In most cases it does nothing. Rather it holds up for scholastic praise and recognition many whose social maturity is about on a 14-year level. What is more, these often become professors, and if not too absent-minded, deans.

3. *The tutorial drudge.* The previous two types are at least respected by the student. But one often meets teachers in college whose patience is proverbial,

who endure with the sluggish student to the last bitter trial and endure from him the disrespect of broken appointments, palpably false excuses, and the contempt one is likely to extend toward a hired hand. Most of us have known many such hopeless and helpless individuals, perhaps somewhat stupid, but certainly over-worked; knowing the text-book from cover to cover, but showing never a thrill at anything mathematical. Why any one would wish to look forward to such a life may be hard to explain, and yet are we not definitely contributing toward continuing this pattern? The senior with his daily chore of set problems to work whether or not he understands the subject; the graduate assistant eking a half-livelihood by grading innumerable daily exercises—is this the picture of the American scholar imbued with the thirst for scientific discovery and the thrill of imparting the glad tidings to eager followers?

Such at their worst, perhaps, are three extreme types. Doubtless all of you recognize in others, if not in yourselves, something of resemblance to one or more of these—although let me hasten to assure you in the self-protective language of the not so silent screen, that any resemblance to historical personages living or dead is purely coincidental.

All of us have heard the remark, "Those who can, do; those who cannot, teach." The humor, if any, of such a quip has long since grown stale for us, leaving only a lingering hint of intended malice. Out of our own need for self-respect and the sincere appreciative understanding of the work of our fellow teachers, we are inclined to dismiss such reflections as based in ignorance or jealousy. And yet may there not be some excuse for the phrase, which we would do well to admit if only in private. Of course, even to raise the question places this particular address in an awkward light—for if the teachers be those who could not survive the public competitive struggle to carry on the practical work of the world, what of him who waxes didactic on the very subject of teaching? Why bother, indeed, about infinitesimals of the second order? But let us be concrete.

In collegiate circles, what of the numerous deans, vice-presidents, and other administrative officers? Judging from salaries, and prestige, might not one be excused for saying, "Those who can administer, do, those who cannot, teach." In our engineering schools, the outsider is very likely to remark: "Those who can build bridges do, those who cannot, teach." With regard to schools of business management, medical schools, or dramatic schools, similar views are widely held. But what of the specifically mathematical field? The dean of the faculty and head of the mathematics department in one of our most distinguished universities remarked not very long ago in my presence: "Oh, you can find good teachers everywhere. What we want is productive research." In other words those who can do research, do; those who cannot, teach. Of course, countless thousands of young women who have taught school from one to three years look back from amid the realities of marriage and motherhood to their unattached sisters still disciplining the children of strangers, and find justification for their own lot in the smug remark, "Those who can marry well, do; those

who cannot, teach." We who teach are more inclined to formulate the situation in a manner more flattering to ourselves. We think perhaps that most young people are intellectually slow, unwilling to study books, incapable of mastery of abstract topics, while we ourselves are among the cream of the academic crop. From among the select group of high-standing students, most have been eager for immediate financial returns, quick to enter the sordid competition for wealth whether in business, or medicine, or law. A few, very few, inspired by the noblest altruistic aims have devoted their lives to the enlightenment of their fellows, content with the solace of scholarship and the benedictions of the rising generations. These select heroes of culture and intellectual freedom—they are, of course, ourselves, as we would like to see us.

Let me not pass judgment; let me only pause to ask some questions, perhaps irrelevant. Do not scholars perhaps tend overmuch to encourage in a new generation, those like unto themselves? Are not such socially essential attributes of personality as say, dissatisfaction with economic want, love of home life, contagious merriment, preoccupation with the tangible and palpable, artistic imagination, and hosts of other often desirable traits, systematically suppressed by us as teachers, who perhaps are ourselves conspicuously deficient in these very essentials?* When the young man who is "dumb" in mathematics succeeds brilliantly in his chosen field, is it the student who is unintelligent, or ourselves and our system? The college must of course encourage scholarship, and lay a sound framework for later professional training for those who are to enter the learned professions. But the modern college seems in the main to have a wider and very different mission, that of enriching the minds and lives of citizens who are to occupy widely varied stations in the civilization of a tomorrow already dawning. Is the college teacher, however expert he may be in juggling with the notions of his own subject, justified in demanding from the student memorization of the views of teacher or textbook? Is routine drill in technical mathematical manipulations of lasting value to the average citizen? Does cramming for examinations make him better equipped to settle the questions of his own business or the affairs of the community? Perhaps there is no simple general answer.

That the young graduate student fresh from his doctoral examination and innocent of any exposure to courses called education, has much to learn during his first year or two of college teaching is obvious. But a sensitive sympathy, an honest desire to do a good job of teaching, and the patience to devote many hours to preparation for classes will undoubtedly pull him through after a fashion. He will learn also from experience—although here as elsewhere "experience is a dear teacher." What of the fledgling versus the experienced teacher? Of

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* See, for example, Sir John Adams, The teacher as humorist, Harv. Teach. Rec. 2, 1932, pp. 101–7. H. W. Holmes, Indoctrination, the teacher in politics and the social aim of education, Harv. Teach. Rec. 3, 1933, pp. 157–160. G. B. Price, A program for the association, this MONTHLY, vol. 45, 1938, pp. 531–536.

course, experience if reasonably successful is a form of insurance against costly blunders. The experienced teacher takes his job seriously, and he cannot be easily trapped into foolish answers. The argument that the young graduate has been so recently a student that he will understand the undergraduate point of view has probably little weight. The most supercilious, impatient, sarcastic, unprepared, and easily routed teacher is likely to be the young Ph.D. in his first teaching assignment. But the whole theory of teacher training, whether in subject matter, studied in professional courses in education, or in practice teaching, seems to assume that good teaching can be learned and that the experienced teacher is the most effective. From the administrative point of view the value of a tried and true member of the staff is not open to question. But may it not be the case that the qualities of personality likely to be most effective in dealing with elementary classes, have little to do with the form of training? What of the dishonesty which seeks futilely to disguise ignorance, or which evades admitting the real basis of grading? What of the ignorance that fails to see interconnections, the stolidity that never vibrates with enthusiasm, the lack of sympathetic intuition that fails to appreciate a student's real difficulties, the idiosyncracies that disgust the well-bred? Do these fatal defects wear off with age? Is it not rather a question of careful selection with many criteria including but not consisting exclusively of scholarly achievement? I raise the question of course in connection with teaching freshmen, and not with regard to purely graduate instruction.

Let us turn next to the subject of examinations about which so very much has been written. Surely the prospective teacher at college level should have some philosophy of testing. Examinations loom large in the student's eyes, and take much of the instructor's time. But ordinarily in college circles they remain for the professor as routine matters, since it is he who sets and grades examinations for his own students. In some lecture subjects, the widely-spaced examinations give the only clue to the student's active participation in the course; but not so in mathematics. Examinations here are hardly needed for the purpose of giving an acceptable grading of the class. They can, however, serve a specific educational end. Usually they involve a longer span of subject matter than is practicable in short tests. They usually provide records of the speed, accuracy and retentiveness of the student along assigned mathematical lines. They do not offer opportunity for much original thought, for ability in self-expression, or for framing a problem. They give no clue as to the student's powers in using sources of information, his appreciation of mathematical methods, his ability to transfer formal training to novel stituations, his self-confidence amid distractions, his understanding of basic interrelationships among varied fields of thought. Does the examination test what the course is intended to develop? Or have the objectives of the course never perhaps been formulated by teacher or student? For the sake of discussion let me formulate a few conditions that some modern (perhaps ultra-modern) teachers might hope to find fulfilled. Since these views are so rarely proposed and practically never adopted, their utility here may lie in merely directing attention in non-tradi-

tional lines. Perhaps they show no vision, being merely visionary.

1. Every test should be so phrased that the expert in the field could complete the test perfectly without knowing either the teacher or the textbook.

2. Any test designed for college freshmen should be so fundamental in its nature and so little dependent upon the trivial and easily forgotten details that any well educated person admitting ordinary conversance with the field should be able at least to pass the test. In particular any engineer should be able to pass easily all freshman tests in mathematics.

3. The testing time should be ample to allow the student to check and recheck his work, and to rephrase his results—originality, analytic power, and ability in expression, rather than speed or memory being demanded.

4. The series of tests as a whole, if not each test separately, should provide an objective evaluation of achievement toward each of the aims laid down for the course and previously discussed with the class. These objectives should be far-reaching and basic, and acceptable to the class. Records should be kept of these results, so that remedial work may be planned in the specific areas requiring further attention. Among the objectives in mathematical instruction might well be included some overlapping other fields.

5. The tests in a course should be prepared and graded where possible by persons other than the teacher, and without being shown to him prior to the examination. The teacher as friend and guide should be spared the role of executioner.

6. The relative proportion of A's, B's, C's, D's, and failures, should not be regarded as arranged solely by the instructor or determined wholly impersonally by recorded averages, but should respect departmental or general collegiate considerations. In particular, the instructor's entire duty with respect to final grades might be covered by his ranking his students in order of decreasing attainment, suitably bracketing those of essentially the same standing, while indicating also a suggested level for passing the course.

7. The course should change in its details from year to year, by varying the textbook or the portions covered—not for the sake of the students, but to keep the teacher out of a rut.

Are these suggestions sound, or are they mere sound? Perhaps none of you see any merit in any of them. Do you feel prepared to defend your present practice? You perhaps know; I do not.

In closing, let me thank you for your generous attention. If I have touched sore spots,* it is in the sincere hope for a rapid cure. If I have remained remote from our common problems, it is due to my inability to make of this opportunity an example of good teaching. Surely you will agree, at least, that the subject merits discussion at the hands of those of you who are experts therein.

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* The teacher of college freshman mathematics will recognize many of his own failings exposed quite pitilessly in Joseph Seidlin: A critical study of elementary college mathematics. Teachers College Columbia University Contributions to Education, Number 482, 1931.

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# QUESTIONS, DISCUSSIONS, AND NOTES

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*The Department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

## POSITIVE DETERMINANTS

D. G. BOURGIN, University of Illinois

The following result seems worthy of explicit mention:

If (a) $a_{ij} \geqq 0$, $i, j = 1, 2, \cdots, n$; (b) $1 > \rho > \sum_{j=1}^{n} a_{ij}$; (c) $c_m > 0$, $c_i \geqq 0$, $i \neq m$; then $D^{(m)} > 0$, where $D^{(m)}$ is the determinant obtained from

$$\Delta_n = \begin{vmatrix} 1 - a_{11} & -a_{12} \cdots & -a_{1n} \\ -a_{21} & 1 - a_{22} \cdots & -a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ -a_{n1} & -a_{n2} \cdots & 1 - a_{nn} \end{vmatrix}$$

by replacing the elements of the $m$th column by $c_1, c_2, \cdots, c_n$.

H. T. Davis (*Theory of Linear Operators*, p. 124), gives a theorem, which states that when (a) and (b) hold then

$$\Delta_n \neq 0. \tag{1}$$

Actually, it is known that

$$\Delta_n > 0. \tag{2}$$

Since the proof of this inequality is a simple by-product of the considerations of this note, it has been incorporated in the argument.

Consider the equation system

$$x_i - \sum_{j=1}^{n} a_{ij} x_j = c_i. \tag{3}$$

The formal Neumann series is

$$x_i = c_i + \sum_{j}^{n} a_{ij} c_j + \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} a_{jk} c_k + \cdots. \tag{4}$$

The right side of (4) is dominated by the convergent geometric series

$$c(1 + \rho + \rho^2 + \cdots), \qquad c \geqq \max (c_k). \tag{5}$$

The series of (4) therefore converges and defines a solution, as is evident on substitution in (3). Since the terms are non-negative and $c_m > 0$, we have

$$x_m > 0. \tag{6}$$

In view of (1) the solution of (4) is unique. With this in mind, we have from Cramer's rule

$$x_m \Delta_n = D^{(m)}. \tag{7}$$

The special case $c_i = \delta_{i1}$ ($\delta_{ik} = 0$, 1 accordingly as $i \neq k$, $i = k$) implies

$$x_1 = \Delta_{n-1}/\Delta_n > 0, \tag{7.1}$$

where $\Delta_{n-1}$ is an $n-1$ rowed determinant formed from $\Delta_n$ by eliding the first row and first column. To prove $\Delta_n > 0$, we remark that, by (7.1), $\Delta_n$ is of the same sign as $\Delta_{n-1}$. Consideration of the equation system with $n-1$ unknowns $x_2', \cdots, x_n'$ and $c_i = \delta_{i2}$, leads to

$$0 < x_2' = \Delta_{n-2}/\Delta_{n-1}.$$

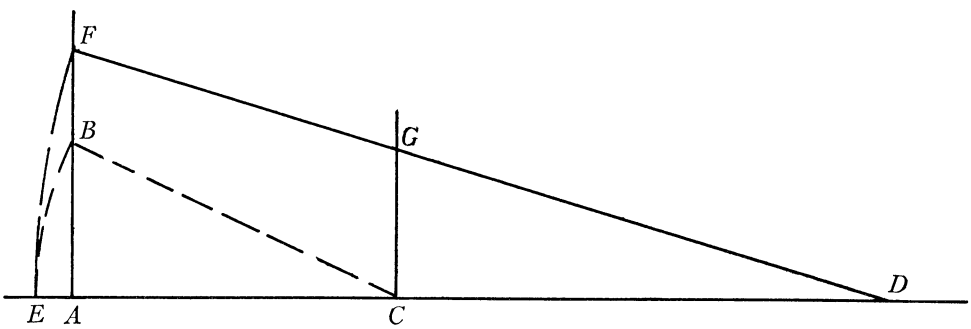
Hence $\Delta_n$, $\Delta_{n-1} \cdots \Delta_1$ are of the same sign. Since $1 - a_{nn} > 0$ it follows that $\Delta_n > 0$. This requires, according to (7), that $D^{(m)} > 0$, which is the assertion of our theorem.

## AN APPROXIMATE RECTIFICATION OF THE CIRCLE

L. S. JOHNSTON, University of Detroit

The construction below gives the circumference of a circle of given radius more conveniently and far more accurately than does any other construction with which the writer is familiar.

Let the radius $r$ of the given circle be the segment $AB$ of the figure below. The figure is set up by the steps shown in the order named:




Let $AC = 2r$, $CD = 3r$, $CE = CB$, $DF = DE$, with perpendiculars as appearing in the figure. Then $DF = (3 + \sqrt{5})r$, and $DG = 0.6(3 + \sqrt{5})r = 0.6(5.236068)r = 3.1416408r$. The semicircumference of the circle of radius $r$ is, to seven decimals, $3.1415927r$, whence $DG$ exceeds the semicircumference by $0.0000481r$, or approximately $r/20{,}000$, a difference beyond detection by any drafting room methods and beyond any ordinary method of detection except the most accurate precision gauges.

Conversely, we may find with the same degree of approximation the radius of a circle if the circumference be given. If one-sixth of the given circumference be laid off on $DG$ and projected on $DA$, this projection is the radius of the corresponding circle.

## REDUNDANCY OF THE COMMUTATIVE-ADDITION POSTULATE

C. J. EVERETT, JR., University of Wisconsin

E. H. Moore has noted* that the postulate $a+b=b+a$ is redundant in a quasi-field. This is true under much weaker conditions. Let $S$ $(0, a, b, \cdots)$ be an additive group, with a well-defined multiplication satisfying a two-sided distributive law relative to addition. For such a system it is easily shown that $0 \cdot a=0=a \cdot 0$ and thence that $x(^-y)=^-(xy)=(^-x)y$, $^-x$ signifying the additive inverse of $x$. Define $N$ as the set of elements $z$ of $S$ such that $zS=0$. Then if $N=(0)$, $a+b=b+a$ ($a$, $b$ in $S$). For let $n=(a+b)+^-(b+a)$. For every $s$ of $S$,

$$\begin{aligned} ns &= (a+b)s + [^-(b+a)]s = (a+b)s + (b+a)(^-s) \\ &= as + bs + ^-(bs) + ^-(as) = 0. \end{aligned}$$

Hence $n=0$ and $a+b=b+a$. It is interesting to note that associativity of multiplication is not assumed, and also that $N=(0)$ is not necessary for commutative addition, as is shown by the example: $S$ any commutative group, $a \cdot b=0$ (all $a$, $b$ in $S$).

## NOTE ON FRACTIONS

J. P. BALLANTINE, University of Washington

The prime purpose of this note is to call attention to the excellent paper by L. R. Ford, in the November 1938 issue of this MONTHLY. He constructed circles above the $x$-axis and tangent to it at all its rational points. It is interesting that the circles which he used were just large enough to be mutually tangent without intersecting. Another way to carry out this construction is given here.

Construct the circles $C_i$, with diameter unity, above the $x$-axis and tangent to it at the points $(i, 0)$, where $i$ assumes all integral values. Obviously the circles $C_i$ are tangent for each consecutive pair of values of $i$.

Next construct circles $I_i$, with radius unity, and centers at the points $(i, 0)$. Now invert all the circles $C_i$ in all the circles $I_i$. Each resulting circle is tangent to the $x$-axis at a point $(p/q, 0)$, where $p$ and $q$ are integers. Call these circles $C_{p,q}$. All rational values $p/q$ will not be obtained at the first step, but if the circles $C_{p,q}$ obtained are again inverted in all the circles $I_i$, and so forth, there will result a set of circles $C_{p,q}$ containing one for each rational number. The radius of $C_{p,q}$ is readily found to be $1/(2q^2)$, and this value does not have to be known before the construction. Since the circles $C_i$ do not intersect or include each other, the same will be true of the circles $C_{p,q}$.

The circles $C_{p,q}$ and $C_{r,s}$ coincide provided that the two fractions $p/q$ and $r/s$ are equal, even though one of them is not in its lowest terms. Otherwise one circle would include the other. The formula given above for the radius requires $p/q$ to be in lowest terms.

The circle $C_i$ inverted in $I_i$ becomes the "circle" $y=1$, mentioned by Ford as corresponding to the fraction $1/0$, and hence in our notation, $C_{1,0}$.

* E. H. Moore, Memoirs of American Philosophical Society, General Analysis, Part I, p. 45.

The set of circles $C_{p,q}$ inverts into itself in any of the circles $I_i$, the inversion merely rearranging the circles of the set. The circles $C_{i+1}$ and $C_{i-1}$ are each unchanged, and $C_i$ and $C_{1,0}$ are interchanged.

There are other circles besides $I_i$ under which the set $C_{p,q}$ inverts into itself. Define $I_{r,s}$ where $r/s$ is in its lowest terms, as a circle of radius $1/s$ and center $(r/s, 0)$. If $s$ is limited to 2, the set $C_{p,q}$ inverts into itself. I have not checked it for other values of $s$, but I know it is not true for some. The set $C_{p,q}$ also inverts into itself in the "circles" $x=i$ and $2x=2i+1$.

It did not serve Ford's purposes, but I was interested in filling up all the area between $y=0$ and $y=1$ with circles. This is done by moving the circles $I_i$ one unit in the $y$-direction, thus forming the circles $J_i$. Repeated inversion of the circles $C_i$ in both $I_i$ and $J_i$ gives the desired set of circles completely filling that area.

*Note by the Editor.* It is readily found by direct calculation that if $C_{p,q}$ is inverted in $I_{r,s}$ the resulting circle is tangent to the $x$-axis at $[pr-q(r^2-1)/s]/(ps-qr)$ and has radius $1/2(ps-qr)^2$. Hence for the set $C_{p,q}$ to invert into itself in $I_{r,s}$ it is necessary and sufficient that for every relatively prime pair $p$, $q$, we must have $pr-q(r^2-1)/s$ an integer prime to $ps-qr$. A necessary condition is therefore that $s$ divide $r^2-1$, which is equivalent to the condition that $r$ be of the form $\alpha s\pm 1$. To show that this condition is also sufficient, let $r=\alpha s+\epsilon$, $\epsilon=\pm 1$. If we put

$$P = pr - q(r^2 - 1)/s = p(\alpha s + \epsilon) - q(\alpha^2 s + 2\alpha\epsilon),$$
$$Q = ps - qr = ps - q(\alpha s + \epsilon),$$

then from the identities

$$p = (\alpha s + \epsilon)P - (\alpha^2 s + 2\alpha\epsilon)Q, \qquad q = sP - (\alpha s + \epsilon)Q,$$

we see that $P$ and $Q$ can have no common factor, since $p$ and $q$ have none.

Since the line $y=0$ is uniquely specified by the fact that all $C_{p,q}$ are tangent to it, it must be invariant under any inversion which preserves the set $C_{p,q}$. Hence the circles $I_{\alpha s\pm 1,s}$ are the only ones under which the set $C_{p,q}$ inverts into itself.—R.J.W.

## ON THE EQUATION $x^2+y^2=z^2$

MANNIS CHAROSH, Lafayette High School, Brooklyn, N. Y.

The following solution of the above quadratic Diophantine equation seems to be new. It has the advantage of being related to the simple theory of the quadratic equation, and is therefore suitable for presentation to a bright high school class in intermediate algebra. The author has successfully presented it to just such a class—with, of course, great care spent on the details.

Let $d$ be the H.C.F. of $x$, $y$, $z$. If $x=da$, $y=db$, $z=dc$, then $a^2+b^2=c^2$, and $a$, $b$, and $c$ are prime each to each.

As in the usual solution, we may assume $a$ to be odd and $b$ even. Let $b=2n$; then $c^2-4n^2=a^2$. Therefore the roots of $u^2-cu+n^2=0$ are integral. If the roots

of this quadratic equation be $r_1$ and $r_2$, then $c=r_1+r_2$; and $n^2=r_1r_2$. Since $c$ and $n$ are prime to each other, $r_1$ and $r_2$ must also be prime to each other. Therefore we may let $r_1=p^2$ and $r_2=q^2$. Therefore $c=p^2+q^2$; $b=2pq$; $a=p^2-q^2$. And finally $x=d(p^2-q^2)$; $y=d(2pq)$; $z=d(p^2+q^2)$.

## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 513 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Research and Statistical Methodology Books and Reviews of* 1933–1938. By O. K. Buros. New Brunswick, N. J., Rutgers University Press, 1938. 6+100 pages. $1.25.

*An Introduction to the Theory of Numbers.* By G. H. Hardy and E. M. Wright. Oxford, Clarendon Press, 1938. 16+403 pages. $8.00.

*An Introduction to Modern Statistical Methods.* By P. R. Rider. New York, John Wiley and Sons; London, Chapman and Hall, 1939. 9+220 pages.

*Science in a Tavern.* Essays and Diversions on Science in the Making. By C. S. Slichter. Madison, University of Wisconsin Press, 1938. 9+186 pages. $3.00.

*Der Deutsche Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts.* E.V. 1891–1938. By W. Lorey. Frankfurt am Main, Otto Salle, 1938. 165 pages. RM. 3.00.

*Seis Conferencias.* (Junta Para Ampliacion de Estudios e Investigaciones Cientificas). By C. Barinaga. Madrid, Nuevas Graficas, S.A., 1938. 79 pages.

*The Engineer's Manual.* By R. G. Hudson. Second Edition. New York, John Wiley and Sons; London, Chapman and Hall, 1939. 6+340 pages.

*Geometrisieren im Bereiche wichtiger Kurvenformen.* By E. L. Locher. Zürich and Leipzig, Orell Füssli, 1938. 64 pages. fr.4.80.

*Coordinate Solid Geometry.* By R. J. T. Bell. London, The Macmillan Company, 1938. 13+175+43 pages. $2.25.

*Your Chance to Win: The Laws of Chance and Probability.* By H. C. Levinson. New York and Toronto, Farrar and Rinehart, 1939. 343 pages. $2.50.

*Lezioni di Analisi Matematica.* Parte Prima, Quarta Edizione. By F. Tricomi. Padova, Cedam, 1939. 8+328 pages.

*Trigonometry.* With Tables. By H. K. Hughes and G. T. Miller. New York, John Wiley and Sons; London, Chapman and Hall, 1938. 8+189+79 pages. $2.00.

*Formelsammlung zur praktischen Mathematik.* (Sammlung Göschen, 1110.) Berlin and Leipzig, de Gruyter, 1937. 147 pages. RM 1.62.

*College Algebra.* By L. J. Rouse. Second Edition. New York, John Wiley and Sons; London, Chapman and Hall, 1939. 13+462 pages. $2.25.

## REVIEWS

*Über die einfachen Konfigurationen der euclidischen und der projectiven Ebene.* By R. Klee. Verlag Akad. Buchhandlung Focken u. Dresden, Oltmanns, 1938. 67 pages.

A *simple* figure in a plane consist of $n$ non-parallel lines such that through each point there are two lines and on each line there are $(n/2)$ points. The $n$ lines divide the plane into regions bounded (or unbounded) by various numbers of lines. By means of a Bow notation such a figure can be represented by a regular polygon covered by rhombi of different angles. The booklet is devoted to the calculation of the numbers of different types of figures for the cases of $n$ up to and including 8, and in the reviewer's opinion is not worth even a brief perusal.

M. S. KNEBELMAN

*Zur Begründung der Geometrie im begrenzten Ebenstück.* By Emanuel Sperner. Schriften der Königsberger Gelehrten Gesellschaft. Heft 6, 1938. Pp. 121–143.

As the title of this article indicates, it deals with the postulates of a geometry of a bounded portion of the plane. The postulates are those of Hilbert for projective geometry, slightly modified. They insure convexity of the region, but two lines need not have a common point, and Desargue's theorem is postulated to hold if corresponding sides of two perspective triangles intersect. The main problem is to arithmetize such a geometry, and the last half of the article is devoted to embedding this arithmetic in a non-commutative field.

One can hardly recommend this article to a beginner, and for one acquainted with (by now) classical projective geometry, there is very little of novelty or interest.

M. S. KNEBELMAN

*Introductory Mathematical Analysis.* By Joel S. Georges and Jacob M. Kinney. New York, The Macmillan Company, 1938. 15+605 pages. $3.00.

This text "is the result of experimentation to determine the aims of mathematical instruction for college students, the selection and organization of instructional materials for the attainment of those aims, and the methods and modes of instruction and of evaluation of instruction."

Unfortunately, the authors do not tell us in the preface what the aims of mathematical instruction are. They have included in the six hundred pages, however, an abundance of material which they estimate as sufficient for a five-hour course for two semesters.

Roughly, one-fourth of the text is algebra, one-third is analytic geometry (both plane and solid), one-sixth is trigonometry, and one-eighth is calculus. These are joined together by the function concept into a unified whole which ought to make a rather thorough foundation in mathematics for those who intend to major in science. The book is not a "survey" course for the ordinary

student, but each of the subjects mentioned above is pretty thoroughly covered. We find all the topics of the ordinary college algebra, such as complex numbers, theory of equations, determinants, logarithms, and even interest and annuities. In analytic geometry, there are full treatments of the straight line, the circle, the conics, transformation of coördinates, parametric equations, and solid geometry.

Though the treatment of plane trigonometry is brief, it is adequate for general uses in science. There are abundant exercises in trigonometric equations and formulas. The treatment of the calculus includes both differential and integral, and these are applied in algebra and analytics to the advantage of all.

There are tables of squares and square roots, of natural trigonometric functions, of logarithms of natural numbers, of logarithms of trigonometric functions, and of natural logarithms. Abundant exercises are provided with each topic, and these furnish not only adequate drill but typical applications to practical problems in physics, chemistry, and business.

The authors suggest that the first six chapters, 257 pages, may be used as a five-hour semester course. Such a course would embrace a good elementary course in analytic geometry and college algebra, together with a short introduction to the calculus.

Those who like unified courses will find this an excellent text, in which the unifying principle of function has been used in such a skilful way as to hide the old lines of separation. Those who have not tried the unified course will find this a good text on which to begin.

The publishers have done their work well and have produced an excellent volume. There are few errors, and the teacher should find it most satisfactory.

R. P. STEPHENS

*Euclide. L'Optique et la Catoptrique.* By Paul Ver Eecke. (Oeuvres traduites pour la première fois du grec en français, avec une introduction et des notes. Ouvrage publié sous les auspices de la Fondation Universitaire de Belgique.) Paris et Bruges, Desclée de Brouwer et Cie, 1938. 47+126 pages. 75 fr.

Mr. Ver Eecke has obliged students of Greek mathematics by a series of competent translations into French. First appeared a complete translation of Archimedes, then came the *Conic Sections* of Apollonius, Diophantos, the *Spherics* of Theodosius of Tripolis, the books on the *Section of a Cylinder* and the *Section of a Cone* by Serenus of Antinoeia, and the *Collection* of Pappus. Now he has added to this remarkable set a translation of Euclid's *Optics*, Theon's recension of this book, and the *Catoptrics* which is in the Euclidean tradition and is included in Heiberg's standard edition of Euclid's works (vol. VII).

The two versions of the *Optics*, one Euclid's own text, the other a modification by Theon of Alexandria (4th century A.D.), contain a kind of elementary perspective. They deal with the way simple geometrical figures look when seen from one point or from two points. As an example, we can take Proposition

XXVIII of Euclid's *Optics*: "If a cylinder is seen by a single eye at an arbitrary point, less than half the cylinder can be seen."

The *Recension* seems to be Theon's modification of Euclid's *Optics*, perhaps for the purpose of instruction. It begins with a "Prologue," which has the appearance of an inaugural lesson, written down by a listener.

The *Catoptrics* is a theory of reflection in a flat, a concave, or a convex circular mirror.

The *Optics* and *Catoptrics* exist in several old Latin translations, the first of which was published in 1505. It already contained Theon's *Recension*. The translator was B. Zamberti. The first translation in a modern language was E. Danti's Italian version of 1573. There exist old versions in Spanish and French. In 1918 G. Ovio published, in Milan, an Italian translation of the *Optics*, taking as his text, however, an old Latin translation by Pena (1557). Ver Eecke's work is the only direct translation into a modern language from the original Greek text as published by Heiberg in 1895.

The value of Mr. Ver Eecke's achievement could be improved if he would consent to an elaborate index of subject matter and authors' names. We must add that the publishers have again done an excellent technical job, so that it is not only an intellectual, but also an aesthetic pleasure to read in this venerable document of ancient learning.

D. J. Struik

*One Hundred Problems in Consumer Credit.* By Charles H. Mergendahl and Le Baron R. Foster. (Pollak Pamphlet, no. 35.) Newton, Massachusetts, Pollak Foundation for Economic Research, 1938. 55 pages. $.10.

This book contains 100 problems in consumer credit for high school and for college courses in the mathematics of finance. It first gives an adequate introduction to the meaning of consumer's credit and an idea about the principal concepts and calculations involved. The problems are very well thought out, interesting and stimulating. The book can be recommended for supplementing textbooks on the mathematics of finance in a course on this subject.

Gerhard Tintner

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.*

*Correction by the Editor.* In the December 1938 number of the MONTHLY, p. 689, there occurred, unfortunately, the following error: in referring to W. W. R. Ball's *Mathematical Recreations and Essays*, the title was erroneously given as *Mathematical Excursions and Essays.*—E.H.C.H.

## CLUB TOPICS

The following titles with accompanying bibliography were contributed by C. B. Read, University of Wichita. The references as a whole have been taken from the AMERICAN MATHEMATICAL MONTHLY, *Mathematics Teacher*, and *School Science and Mathematics.* For brevity, only volume number and pages have been given, with the obvious abbreviations AMM, MT, and SSM.

24. *Calculating Machines and Devices*

Cheng, D. C. The use of computing rods in China. AMM 32: 492–499.
Datta, B. The science of calculation by the board. AMM 35: 520–529.
Gandz, S. Did the Arabs know the abacus? AMM 34: 308–317.
Lehmer, D. H. A photo-electric number sieve. AMM 40: 401–406.
Locke, L. L. The history of modern calculating machines, an American contribution. AMM 31: 422–429.
Richardson, L. J. Digital reckoning among the ancients. AMM 23: 7–13.

35. *The History of American Mathematics*

Cajori, F. American contributions to mathematical symbolism. AMM 32: 414–416.
Coolidge, J. L. Robert Adrian, and the beginnings of American Mathematics. AMM 33: 61–76.
Finkel, B. F. The human aspect in the early history of the American Mathematical Monthly. AMM 38: 305–320.
Silverman, L. L. John Wesley Young, his life and scientific activities. AMM 39: 311–314.
Simons, L. G. Algebra at Harvard College in 1730. AMM 32: 63–70.
Yates, R. C. Sylvester at the University of Virginia. AMM 44: 194–201.

36. *The Origin of Various Mathematical Terms and Symbols*

Cajori, F. American contributions to mathematical symbolism. AMM 32: 414–416.
Cajori, F. Origin of the name "Mathematical Induction." AMM 25: 197–201.
Cajori, F. Rahn's algebraic symbols. AMM 31: 65–71.
Cajori, F. The origin of the symbols for "degrees, minutes, and seconds." AMM 30: 65–66.
Committee on Symbols. American standard mathematical symbols. AMM 35: 300–304.
Datta, B. On Mûla, the Hindu term for root. AMM 34: 420–423.
Datta, B. On the origin of the Hindu terms for "root." AMM 38: 371–376.
Gandz, S. On the origin of the term "root." AMM 33: 261–265; 35: 67–75.
Gandz, S. The origin of the term "algebra." AMM 33: 437–440.
Gandz, S. On three interesting terms relating to area. AMM 34: 80–86.

37. *Historical Items Pertaining to the Calculus*

Bryan, N. R. The first attempt at a table of integrals. AMM 29: 392–394.
Cajori, F. Grafting of the theory of limits on the calculus of Leibniz. AMM 30: 223–234.
Cajori, F. The early history of partial differential equations and of partial differentiation and integration. AMM 35: 459–467.
Cajori, F. The history of Zeno's arguments on motion. AMM 22: 1–6; 39–47; 77–82; 109–115; 143–149; 179–186; 215–220; 253–258; 292–297.

Wren, F. L. and Garrett, J. A. The development of the fundamental concepts of infinitesimal analysis. AMM 40: 269–281.

39. *Numerals and Number Systems*

Cajori, F. Fanciful hypotheses on the origin of the numeral forms. MT 18: 129–133.
Cajori, F. Sexagesimal fractions among the Babylonians. AMM 29: 8–10.
Cajori, F. Spanish and Portugese symbols for "thousands." AMM 29: 201–202.
Datta, B. Early literary evidence of the use of zero in India. AMM 33: 449–455; 38: 566–572.
Ganguli, S. The elder Aryabhata and the modern arithmetical notation. AMM 34: 409–415.
Ganguli, S. The Indian origin of the modern place value arithmetical notation. AMM 39: 251–256; 389–393; 40: 25–31; 154–157.
Ginsburg, J. On the early history of the decimal point. AMM 35: 347–349.
Miller, G. A. On the history of common fractions. SSM 31: 138–145.
Richeson, A. W. The number system of the Mayas. AMM 40: 542–546.
Sanford, Vera. Roman numerals. MT 24: 22–27.
Sherman, C. P. Origin of our numerals. MT 16: 398–401.
Smith, D. E. and Mourad, S. The dust numerals among the ancient Arabs. AMM 34: 258–260.
Smith, D. E. and Ginsburg, J. Rabbi ben Ezra and the Hindu-Arabic problem. AMM 25: 99–108.

40. *Mathematics in the Ancient World*

Archibald, R. C. Babylonian mathematics with special reference to recent discoveries. MT 29: 209–219.
Das, S. R. Coordinates used in Hindu astronomy. AMM 35: 535–540.
Das, S. R. The equation of time in Hindu astronomy. AMM 35: 540–543.
Evans, G. W. The Greek idea of proportion. AMM 34: 354–357.
Fields, Margaret. Practical mathematics of Roman times. MT 26: 77–84.
Karpinski, L. C. Algebraical developments among the Egyptians and Babylonians. AMM 24: 257–265.
Miller, G. A. A few theorems relating to the Rhind Mathematical Papyrus. AMM 38: 194–197.
Miller, G. A. Theorems relating to the pre-Grecian mathematics. AMM 38: 496–500.

41. *History of Algebra*

Bussey, W. H. The origin of mathematical induction. AMM 24: 199–207.
Cairns, W. D. Napier's logarithms as he developed them. AMM 35: 64–67.
Cajori, F. Absurdities due to division by zero. MT 22: 366–368.
Funkhouser, H. G. A short account of the history of symmetric functions of roots of equations. AMM 37: 357–365.
Ginsburg, J. Rabbi Ben Ezra on permutations and combinations. MT 15: 347–356.
Miller, G. A. On several points in the history of algebra. SSM 29: 404–410.
Miller, G. A. On the history of determinants. AMM 37: 216–219.
Thompson, A. J. Henry Briggs and his work on logarithms. AMM 32: 129–131.
Turetsky, M. Permutations in the sixteenth century Cabala. MT 16: 29–34.

## THE MATHEMATICAL SAGA OF LINNIE R. E. QUASHUN*

Once upon a time, there lived in the far (1) ______ of our country which (2) ______ call our western (3) ______, a lovely damosel known as Linnie R. E. Quashun. No beauty (4) ______ nor (5) ______ (minus the first *s*) needed she, for her lips were red as (6) ______, her (7) ______ orbs were bright as any (8) ______ intelligence (9) ______, her permanent wave was

* The numbered blank spaces are to be filled in with words occurring in mathematics, such as "series," "rational," "power," "altitude," *etc.* Answers will be published in the next number of the MONTHLY.

as straight as any (10) , and her nose of good (11) Her (12) also was the envy of those who knew her, the (13) of her shoulders was as bewitching as the (14) of her waist was (15) , and her (16) so small that the size of her shoe was number (17) Indeed, all knew her as a perfect (18) -(19) .

But not all of her charm was in her (20) . She was sent to a modern school where whe learned many things of no (21) value. Her teachers were not only able to correlate, but also to (22) . As a result she could discern the (23) between a (24) and a tree top, a (25) and a chair. There was little that was (26) to her. She also knew the nature of (27) , so that she often went off on a (28) , but on the whole, her efforts were more or (29) in the same (30) as those of her fellow students. This made her reasoning a bit (31) ; in fact, at times it reached the (32) of the ridiculous. But her teachers were (33) , and her principal was heard to declare that any (34) , however small, if (35) to her knowledge, would cause her to (36) above (37) and well beyond the (38) . There were times when she felt that she was going about in (39) , that her capacity was definitely approaching a (40) . There were even times when she was (41) to wish for a better (42) , but on the (43) , considering the (44) under which she was working, she appeared quite (45) . Really, this is only an (46) of her charm. She was an (47) of all the freak cults of the day.

Dear reader, you must not for one moment be (48) that this school did not develop and (49) her physical abilities. In the gymnasium, she performed on the (50) bars, and in the open (51) she was known to prefer ice. "(52) , (52) ," she would cry whenever it was cold enough. At such (53) her (54) would cover a considerable (55) of time. Her movements were found to (56) perfectly. Socially her charms were even (57) , for each afternoon the dean served (58) and tea from a (59) on a small (60) . At such times she loved to dance, and as she seized her friend Ho (61) De Rhomboid, a (62) if there ever was (63) , for the swing, she would whistle "(64) ."

But the (65) of all evil was the high (66) of her car. No one was able to (67) its speed, least of all the (68) who changed the (69) for her. This frequently (70) her to tears; but, nothing daunted, she filled her (71) (minus the rhom) with her friends, and sought a (72) of new adventures. Even with their combined (73) knowledge, many considered this (74) off their (75) . This, no doubt, was due to the (76) , a (77) for which there is no (78) . They must have been (79) ; you can see for yourself that there was little (80) to their procedure.

One day her father, who paid all the bills, came to investigate her education and training. "(81) spend my money like this," he lisped. "I must (82) her stay in school. She must be (83) with me at any (84) ." He packed her (85) and (86) home, so that others might say (87) things about her. For even when his regard for her was at a (88) , his hopes (89) about her. Her opinion (90) with his, so with a (91) she spent the (92) of her days resting, sometimes (93) , but usually (94) . And herewith endeth this (95)

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 375. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A rectangular block or beam has for its three dimensions different odd prime numbers of inches. The numbers expressing its volume and total surface area in cubic and square inches are respectively a three- and a four-place number. Find the dimensions and show that the solution is unique.

E 376. *Proposed by V. W. Graham, Harcourt Street High School, Dublin, Ireland.*

Show that if $1/[(1-x)(1-x^2)(1-x^4)(1-x^8)]$ is expanded in a series of positive powers of $x$, then the coefficients will run as follows:

$$
\begin{array}{llll}
x^{8n} \text{ and } x^{8n+1} & \text{will have} & (n+1)(2n+1)(2n+3)/3, & \\
x^{8n+2} \text{ and } x^{8n+3} & \text{will have} & (n+1)(n+2)(4n+3)/3, & \\
x^{8n+4} \text{ and } x^{8n+5} & \text{will have} & (n+1)(n+2)(4n+6)/3, & \text{and} \\
x^{8n+6} \text{ and } x^{8n+7} & \text{will have} & (n+1)(n+2)(4n+9)/3. &
\end{array}
$$

E 377. *Proposed by V. Thébault, Le Mans, France.*

Find two perfect cubes which, considered jointly, contain the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 once each. Is the solution unique?

### SOLUTIONS

E 336 [1938, 319]. *Proposed by W. B. Campell, Drexel Institute.*

The equation $x^2-bx+c=0$ has two positive integer roots, $p$ and $q$. Let $r$ equal $p$ with its (two) digits interchanged, and $s$ equal $q$ with its (two) digits interchanged. Show that the following four statements are consistent, and lead to a unique solution, and that any three of them lead to the same unique solution. (a) In the equation whose roots are $r$ and $s$, the two digits of $b$ are interchanged. (b) In the revised equation, $b=78$. (c) In the revised equation, $c$ is unchanged. (d) $r$ and $s$ differ by 6.

*Solution by L. S. Johnston, University of Detroit.*

The problem as stated is not provable, for the two equations,

$$x^2-87x+1472=0 \qquad \text{and} \qquad x^2-87x+1512=0$$

satisfy the first three statements, but only the latter equation satisfies the fourth statement.

We can exhibit all the equations for which (a) and (c) are true, and among them will be found the two already shown. Under (a) neither $p+q$ nor $r+s$ can exceed 99, and neither $p$ nor $q$ can consist of identical digits. Let the digits of $p$ be $AB$ and the digits of $q$ be $DE$. Then under (c) we have $AB \cdot DE = BA \cdot ED$, whence $(10A+B)/(10B+A) = (10E+D)/(10D+E)$, from which we easily derive the proportion, $A/B = E/D$. The following sets, and no others (except, of course, that $p$ and $q$ can be interchanged, as can $r$ and $s$) satisfy this proportion, together with the restriction that both $p+q$ and $r+s$ must be less than 100:

| $A$ | $B$ | $E$ | $D$ | $p=AB$ | $q=DE$ | $r=BA$ | $s=ED$ | Equations |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 4 | 12 | 42 | 21 | 24 | $x^2-54x+504=0$, $x^2-45x+504=0$ |
| 1 | 2 | 3 | 6 | 12 | 63 | 21 | 36 | $x^2-75x+756=0$, $x^2-57x+756=0$ |
| 1 | 3 | 2 | 6 | 13 | 62 | 31 | 26 | $x^2-75x+806=0$, $x^2-57x+806=0$ |
| 1 | 2 | 4 | 8 | 12 | 84 | 21 | 48 | $x^2-96x+1008=0$, $x^2-69x+1008=0$ |
| 1 | 4 | 2 | 8 | 14 | 82 | 41 | 28 | $x^2-96x+1148=0$, $x^2-69x+1148=0$ |
| 2 | 3 | 4 | 6 | 23 | 64 | 32 | 46 | $x^2-87x+1472=0$, $x^2-78x+1472=0$ |
| 2 | 4 | 3 | 6 | 24 | 63 | 42 | 36 | $x^2-87x+1512=0$, $x^2-78x+1512=0$ |

For every one of these seven pairs of equations, both (a) and (c) are true; for the last two pairs (a), (b) and (c) are true; for only the last pair are all four statements true.

Also solved by W. E. Buker, E. P. Starke and the proposer.

E 337 [1938, 319]. *Proposed by V. Thébault, Le Mans, France.*

$(S)$ and $(T)$ are two fixed intersecting spheres, and $(X)$ and $(Y)$ are two variable spheres, tangent to each other at $P$, and each tangent to both $(S)$ and $(T)$. Show that the locus of $P$ is the surfaces of two more spheres which cut each other orthogonally. Examine the cases in which $(S)$ and $(T)$ are internally or externally tangent, or have no points in common.

*Solution by the Proposer.*

Let $A$ and $B$ be the ends of a diameter of the circle of intersection of $(S)$ and $(T)$, and let $(A)$ be the sphere centered at $A$, with radius $AB$. Transform the given configuration by inversion with respect to the sphere $(A)$. The spheres $(S)$ and $(T)$ become the planes $M$ and $N$ respectively, each passing through $B$, and respectively perpendicular to the lines $AS$ and $AT$. The spheres $(X)$ and $(Y)$ become two new spheres $(X')$ and $(Y')$, tangent to the planes $M$ and $N$, and tangent to each other at $Q$, the transform of $P$. But $Q$ lies on one of the two perpendicular planes bisecting the dihedral angles formed by planes $M$ and $N$. Consequently $P$, the inverse of $Q$, must lie on one of the two orthogonal spheres obtained by inverting these two perpendicular planes with respect to the sphere $(A)$. These orthogonal spheres pass through $A$ and $B$, but since $AB$ was any diameter of the circle of intersection of $(S)$ and $(T)$, these orthogonal spheres intersect in that same circle.

In case $(S)$ and $(T)$ are tangent, we call their point of tangency, $A$, and choose any convenient radius for sphere $(A)$. Now the planes $M$ and $N$ are parallel, and the real locus of $Q$ is the single plane midway between them. The locus of $P$ is then a single sphere, tangent to $(S)$ and $(T)$ at $A$, and containing the smaller of them. (The second plane on which $Q$ moved in the general case becomes the plane at infinity in this case, and its transform, the second orthogonal sphere, shrinks to the point $A$.)

In case $(S)$ and $(T)$ have no real points in common, the sphere $(A)$ is imaginary, and so are the inversion transforms of all the spheres in our original configuration. But $M$ and $N$ are still planes, though imaginary, and the locus of $Q$ is two perpendicular, imaginary planes, which invert back into two orthogonal spheres, one of which is real and one imaginary. This real sphere is the desired locus of $P$, and contains the smaller of the spheres $(S)$ and $(T)$.

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## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3909. *Proposed by Bêla de Sz. Nagy, Szeged, Hungary.*

Let $P_1, P_2, \cdots, P_{n+2}$ be $n+2$ points in $n$ dimensional space, $n \geqq 2$, no three of the points on the same straight line. Let the symbol $[P_{i_1}P_{i_2} \cdots P_{i_s}]$ denote the least convex polyhedron containing in its interior the points indicated. Show that if $n=2, 3$, then there exist always subscripts $i, k$, such that $[P_iP_k]$ is not an edge of $[P_1P_2 \cdots P_{n+2}]$. Show that if $n>3$ then this statement does not remain true.

3910. *Proposed by E. T. Bell, California Institute of Technology.*

If all the letters $x_1, \cdots, x_m, y_1, \cdots, y_m, u$ denote polynomials, with rational coefficients, in the $r$ independent variables $a_1, \cdots, a_r$, and $n$ is an integer greater than unity, there is a solution of

$$(x_1 + \cdots + x_m)^n - u(y_1 + \cdots + y_m)^n \equiv x_1^n + \cdots + x_m^n - u(y_1^n + \cdots + y_m^n)$$

when $r=n=m=3$, and when $r=4$, $n=2$, $m=4$. Find all solutions.

3911. *Proposed by J. R. Musselman, Western Reserve University.*

If $N$ be the center of the nine-point circle of triangle $ABC$ and $L'$, $M'$, $N'$ be the symmetrics of $A$, $B$, $C$ respectively as to $N$, show that the circles $AM'N'$,

$BN'L'$, and $CL'M'$ meet on the circumcircle of $ABC$ at $\Phi$, the point of Feuerbach for the tangential triangle of $ABC$, that is the triangle formed by the tangents to the circumcircle of $ABC$ at its vertices.

3912. *Proposed by J. R. Musselman, Western Reserve University.*

If $G$ be the centroid of the triangle $ABC$, and $L'$, $M'$, $N'$ be the symmetrics of $A$, $B$, $C$ respectively as to $G$, show that the circles $AM'N'$, $BN'L'$ and $CL'M'$ meet on the circumcircle of $ABC$ at the Steiner point of the triangle $ABC$.

## SOLUTIONS

3819 [1937, 111]. *Proposed by V. Thébault, Le Mans, France.*

If four spheres passing respectively through the vertices of a tetrahedron $ABCD$ intersect in pairs on the edges joining corresponding vertices, they meet in a point $P$. (S. Roberts, *Mathesis*, 1881, p. 95.) The straight lines from an arbitrary point $Q$ to the vertices cut the corresponding spheres again in $E$, $F$, $G$, $H$. Show that the points $P$, $Q$, $E$, $F$, $G$, $H$ lie upon a sphere.

*Solution by the Proposer.*

Let $(\omega_a)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ be the spheres passing through the vertices of the tetrahedron $ABCD$ and intersecting in pairs in the points $a$, $b$, $c$, $a'$, $b'$, $c'$ on the edges $BC$, $CA$, $AB$, $DA$, $DB$, $DC$. The proof that the four spheres meet in a point $P$ is well known; a synthetic proof was given by J. Neuberg, *Mathesis*, 1881, p. 95. Let $\alpha$, $\beta$, $\gamma$, $\delta$ be the points where $QA$, $QB$, $QC$, $QD$ cut the corresponding given spheres. By the theorem of Roberts $(Q\beta\gamma\delta)$, the circumsphere of the tetrahedron $Q\beta\gamma\delta$, and the spheres $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ intersect in $P$. Similarly, the spheres $(Q\gamma\delta\alpha)$, $(Q\delta\alpha\beta)$, $(Q\alpha\beta\gamma)$ pass through $P$. Since the spheres $(Q\alpha\beta\gamma)$ and $(Q\gamma\delta\alpha)$ circumscribe the tetrahedron $Q\alpha\gamma P$, they coincide. The same is true of the circumscribing spheres of $Q\beta\gamma\delta$ and $Q\delta\alpha\beta$, and these coincide with the first two. Thus the points $P$, $Q$, $\alpha$, $\beta$, $\gamma$, $\delta$ lie on the same sphere $(\omega)$.

This theorem generalizes the one by Roberts and extends to the tetrahedron a proposition by Mannheim, *Nouvelles Annales de Mathématique*, 1890, p. 239. We have extended this theorem to a polyhedron of $n$ vertices and considered the case where the point $Q$ is at infinity in any direction. These results will appear in another publication.

*Editorial Note.* The above reasoning with more detail seems to be as follows: Let $Q$ be any point which does not lie in any one of the three planes $ABC$, $DBC$, $PBC$; and consider first the tetrahedron $QBCD$. The sphere $(Q\beta\gamma\delta)$ passes through the vertex $Q$ of tetrahedron $QBCD$ and through the points $\beta$, $\gamma$, $\delta$ on its edges $QB$, $QC$, $QD$. Also $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ pass through $\beta$, $\gamma$, $\delta$, respectively, and intersect in pairs on the other three edges. Hence $(Q\beta\gamma\delta)$, $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ satisfy the conditions for the theorem of Roberts, and they must meet in a point $P'$. But $(\omega_b)$, $(\omega_c)$, $(\omega_d)$ meet in just two points, a point $A'$ in the plane of $BCD$ and the point $P$. Hence $P'\equiv P$, and we may now write the first sphere as $(QP\beta\gamma\delta)$. In a similar manner, considering the tetrahedron $(Q\alpha\beta\gamma)$, we have the sphere

$(QP\alpha\beta\gamma)$. These two spheres must coincide since they circumscribe the actual tetrahedron $QP\beta\gamma$; and hence we have the sphere $(QP\alpha\beta\gamma\delta)$, and the proof is complete. In Court's *Modern Pure Solid Geometry* a proof of the theorem of Roberts is given on page 228; and on page 302 there are references to other proofs, one of which is by M. W. Haskell in this MONTHLY, vol. 10 (1903), p. 30 with an extension to $n$ dimensions. The proof in the text cited above may be extended to $n$ dimensions.

3820 [1937, 179]. *Proposed by Paul Erdös, The University, Manchester, England.*

Let $a_1 < a_2 < \cdots < a_n < 2n$ be positive integers such that no one of them is divisible by any other member of the sequence; then $a_1 \geqq 2^k$, where $k$ is defined by the inequalities $3^k < 2n < 3^{k+1}$. This estimate for $a_1$ is the best possible.

*Solution by Emma Lehmer, Cambridge, England.*

If we write $a_\nu = 2^{b_\nu} c_\nu$, where the $c$'s are odd, then the $c$'s are distinct, for if two $a$'s had the same $c$, one of the $a$'s would divide the other. Therefore the $a$'s can be given by

$$a_\nu = 2^{b_\nu} c_\nu, \quad \text{where} \quad c_\nu = 1, 3, 5, \cdots, 2n-1 \text{ in some order}.$$

Consider first a subsequence of the $a$'s for which $c = 1, 3, 3^2, \cdots, 3^k$. These $a$'s can be written

$$2^{\beta_i} 3^i, \quad \text{where} \quad i = 0, 1, \cdots, k,$$

and where we must have $\beta_i > \beta_{i+1}$ in order to avoid divisibility. It follows therefore that $\beta_i \geqq k - i$ and hence

$$2^{\beta_i} 3^i \geqq 2^{k-i} 3^i \geqq 2^k.$$

If $a_1$ belongs to this subset the theorem of the problem is proved, if it does not then $c_1 \geqq 5$.

Let us now suppose, contrary to the statement of the problem, that

$$a_1 = c_1 2^{b_1} < 2^k, \quad \text{where} \quad c_1 \geqq 5,$$

and therefore that

$$c_1 < 2^{k-b_1}, \quad \text{where} \quad k - b_1 \geqq 3.$$

Since we have shown that the $c$'s are the distinct odd integers less than $2n$, the numbers

$$c_1 3^{\lambda-1}, \qquad \lambda = 1, 2, 3, \cdots, b_1 + 2$$

are $c$'s, since the largest of them is such that

$$3^{b_1+1} c_1 < 3^{b_1+1} 2^{k-b_1} < 3^{b_1+1} 3^2 2^{k-b_1-3} < 3^k < 2n.$$

These $c$'s determine a subset of the $a$'s

$$a_\lambda = c_1 3^{\lambda-1} 2^{b_\lambda}, \qquad \lambda = 1, 2, \cdots, b_1 + 2,$$

which will have at least one of its terms divisible by another unless the $b$'s are distinct and each $b_\lambda < b_1$. But this is impossible since $\lambda$ assumes $b_1+2$ values and there are at most $b_1+1$ distinct non-negative integers not exceeding $b_1$.

To show that this estimate for the least term is the best possible we exhibit a set of $a$'s for every $n$ whose least element is $2^k$. This set can be defined as a two dimensional array as follows:

$$a_{ij} = 2^{k_i - j}3^j\omega_i, \quad \text{where} \quad \begin{cases} \omega_i < 2n \text{ and prime to } 6; \\ 3^{k_i} < \dfrac{2n}{\omega_i} < 3^{k_i+1}; \\ j = 0, 1, \cdots, k_i. \end{cases}$$

It can be easily verified that this set satisfies the conditions of the problem and has $2^k$ for its least element. For example for $n=15$ we have as our set the numbers 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 21, 23, 25, 27, 29.

*Editorial Note.* M. T. Bird proved that the original statement of this problem with the inequality $3^k < 2^n < 3^{k+1}$ is incorrect.

3821 [1937, 179]. *Proposed by V. Thébault, Le Mans, France.*

Given an orthocentric tetrahedron $ABCD$ and two variable points $M$ and $M'$ diametrically opposite on the circumsphere, the parallels to $AM$, $BM$, $CM$, $DM$ and to $AM'$, $BM'$, $CM'$, $DM'$, drawn from the orthocenter $H$ cut, respectively, the corresponding faces in four points on planes $\pi$ and $\pi'$. Show that: (a) The planes $\pi$ and $\pi'$ envelope a quadric surface of revolution $Q$ inscribed in the tetrahedron and concentric with the second sphere of twelve points. (b) The planes $\pi$ and $\pi'$ intersect in a line $\Delta$ in the directrix plane of $Q$ relative to its focus $H$. (c) The chord of contact turns about $H$ remaining perpendicular to the plane $(H, \Delta)$.

*Generalization by R. Goormaghtigh, Bruges, Belgium.*

The proposed question is a special case of the following property, obtained by developing our note in Mathesis, 1926, p. 196 on a generalization of Droz-Farny's theorem:

*Let $ABCD$ and $A_1B_1C_1D_1$ be two tetrahedrons polar reciprocal with respect to a sphere $\Gamma$ with center $N$; if $M$, $M'$ are two points diametrically opposite on the sphere $ABCD$, the parallels drawn from $N$ to $AM$, $BM$, $CM$, $DM$, and $AM'$, $BM'$, $CM'$, $DM'$, cut, respectively, the corresponding faces of $A_1B_1C_1D_1$ in four points on planes $\pi$ and $\pi'$. These planes envelop a quadric surface of revolution $Q$ inscribed in the tetrahedron $A_1B_1C_1D_1$ and having $N$ as focus; $\pi$ and $\pi'$ intersect in a line $\Delta$ in the directrix plane of $Q$ relative to the focus $N$; the chord of contact passes through $N$ and is perpendicular to the plane $(N, \Delta)$.*

The conjugate to $AM'$ with respect to $\Gamma$ is the intersection of the plane $B_1C_1D_1$ with the plane passing through $N$ and perpendicular to $AM'$; but this

last plane contains the parallel to $AM$ drawn from $N$. Hence the polar plane $\pi$ of $M'$ with respect to $\Gamma$ contains the intersections of the faces of the tetrahedron $A_1B_1C_1D_1$ with the parallels drawn from $N$ to $AM$, $BM$, $CM$, $DM$.

When $M$, $M'$ move on the sphere $ABCD$, the plane $\pi$ envelopes the polar reciprocal of the sphere $ABCD$ with respect to $\Gamma$, *i.e.* a quadric surface of revolution $Q$ inscribed in $A_1B_1C_1D_1$, having as focus $N$ and as axis the straight line joining $N$ to the center $O$ of the sphere $ABCD$.

If, similarly, $\pi'$ is the plane corresponding to $M'$, *i.e.* the polar plane of $M$ with respect to $\Gamma$, the intersection $\Delta$ of $\pi$ and $\pi'$ is the conjugate to $MM'$. Therefore $\Delta$ is in the polar plane $p$ of $O$ with respect to $\Gamma$. But, the polar plane of $O$ with respect to the sphere $ABCD$ being the plane $q$ at infinity, the pole of $p$ with respect to $Q$ is the pole of $q$ with respect to $\Gamma$, *i.e.* the point $N$; hence $p$ is the directrix plane of $Q$ relative to the focus $N$.

The straight line $\Delta$ belongs to the plane $\sigma$ passing through $N$ and perpendicular to $MM'$; this plane contains also the line at infinity, intersection of the tangent planes at $M$ and $M'$ to the sphere $ABCD$ and the conjugate to that intersection, i.e., the contact-chord of $\pi$ and $\pi'$ with $Q$, is therefore the perpendicular to $\sigma$ at $N$.

This completes the proof of the generalized theorem.

The proposed question corresponds to the case when $ABCD$ is an orthocentric tetrahedron and when $\Gamma$ is the conjugate sphere; $Q$ is then the inscribed quadric of revolution having as foci the orthocenter and its isogonal point, i.e. the orthocenter of the medial tetrahedron, formed by the centroids of the faces; its center is therefore the mid-point of the segment between these two points; i.e. the center of the second twelve points-sphere.

Solved also by the proposer.

*Editorial Note.* The above generalization and its proof by means of the polar theory apply to a non-degenerate simplex $S$ in $n$ dimensions with $n+1$ vertices $A$, $B$, $C$, $D$, $\cdots$ and its polar reciprocal simplex $S_1$ with the vertices $A_1$, $B_1$, $C_1$, $D_1$, $\cdots$ with respect to the sphere $\Gamma$ with the center $N$. The circumsphere $(O)$ of $S$ with the center $O$ has for its polar reciprocal a quadric surface of revolution $Q$ tangent to the $n+1$ faces of $S_1$, with one focus at $N$ and with its other focus $N'$ and its center on $ON$, which is the axis of revolution. The polar plane $p$ of $O$ with respect to $\Gamma$ is the directrix plane corresponding to the focus $N$. The inverse of $(O)$ with respect to $\Gamma$ is the auxiliary sphere of $Q$ corresponding to the focal diameter, in other words it is tangent to $Q$ at the extremities of its focal diameter. The intersection $\Delta$ of the tangent planes $\pi$ and $\pi'$ is a space of $n-2$ dimensions, and it is a straight line if and only if $n=3$; but the reasoning for the case where $\Delta$ is a straight line is not altered essentially for the general case $n \geqq 3$, and $\Delta$ has the orthogonal properties in the theorem. Also $\Gamma$ may be a sphere with a real center $N$ but with a radius length which is a pure imaginary $ir$, as is the case in the specialization for one of the two types of the orthocentric $S$ where $N$ is taken as the orthocenter $H$. In such cases we may

replace $\Gamma$ by a real sphere $\Gamma'$ with the real radius $r$ and with the same center $N$; the polar plane of a given point with respect to $\Gamma'$ is then reflected in $N$, and the result is the same as the analytical process of finding the polar plane of the given point with respect to the imaginary sphere $\Gamma$. The same kind of remark applies to inversion with respect to $\Gamma$ when it is imaginary.

The equations in vector form are simple and are easily obtained. If $N$ is chosen as origin of vectors in $n$ dimensions, the equation of $\Gamma$ is $\mathbf{x}^2-m=0$, where $m$ is real, positive or negative. If $\mathbf{c}$ denotes the vector of $O$ and if $R$ is the radius of $(O)$, the equation of $Q$ is $R^2\mathbf{x}^2=(m-\mathbf{c}\cdot\mathbf{x})^2$. The inverse of $(O)$ has the equation $(\mathbf{c}^2-R^2)\mathbf{x}^2-2m\mathbf{c}\cdot\mathbf{x}+m^2=0$, and the vectors of the center and the other focus of $Q$ are easily obtained from this result.

For the theorem of the problem in $n$ dimensions we suppose that $S$ is orthocentric with $H$ as the orthocenter. Let the vectors of its vertices be denoted by $\mathbf{a}_i$, where $H$ is now the origin of vectors. Then $\mathbf{a}_i\cdot\mathbf{a}_j$, has the same value, say $m$, for every pair of distinct values of $i, j=1, 2, \cdots, n+1$; and we take for $\Gamma$ the sphere whose equation is $\mathbf{x}^2-m=0$, the conjugate sphere of $S$; $S$ is then self-polar with respect to $\Gamma$. If $\mathbf{g}$ is the vector of the centroid of $S$, then $2\mathbf{c}=(n+1)\mathbf{g}$, and $\mathbf{c}^2-R^2=nm$. The equation of the inverse of $(O)$ may now be written in the simpler form $n\mathbf{x}^2-2\mathbf{c}\cdot\mathbf{x}+m=0$. The center of $Q$ is given by the vector $\mathbf{c}/n$, while the other focus $H'$ is at $2\mathbf{c}/n$; the radius of the inverse sphere is $R/n$ and this is the semi-focal diameter of $Q$. Thus $(O)$ and its inverse have $H$ as center of similitude with the ratio $n:1$. Thus the inverse sphere cuts each of the segments $HA_i$, $A_i$ denoting a vertex of $S$, in points which divide $HA_i$ in the ratio $1:n$; it also passes through the $2(n+1)$ points which are the projections of $H$ and the other focus $H'$ on the faces of $S$. It is thus a $3(n+1)$ point sphere for $S$; for $n=2$ it is the nine-point circle. These facts are easily deduced synthetically.

The last statement of the above solution results from the following theorem: Let $H$ be the Monge point of a simplex $S$ with circumcenter $C$, and let $S_g$ be the simplex whose vertices $G_i$ are the centroids of all the vertices of $S$ except $A_i$. Then the Monge point of $S_g$ is at $H'$ where $HH'=2HC/n$; and, if $S$ is orthocentric, so is $S_g$, and conversely. For, $n\mathbf{g}_i=(n+1)\mathbf{g}-\mathbf{a}_i=2\mathbf{c}-\mathbf{a}_i$, where $\mathbf{a}_i$, $\mathbf{g}_i$, $\mathbf{g}$ are the vectors of $A_i$, $G_i$, $G$, the vertices of $S$, the vertices of $S_g$, and the centroid of $S$ with $H$ as origin. This may be written $(\mathbf{g}_i-\mathbf{c}/n)=-(\mathbf{a}_i/n-\mathbf{c}/n)$: and this says that, if $S_n$ is the simplex similar to $S$ with the center of similitude $H$ and the ratio $1/n:1$, then $S_g$ is symmetric to $S_n$ with respect to the end point of vector $\mathbf{c}/n$. Hence $H'$ is given by $2\mathbf{c}/n$; and obviously, if $S$ is orthocentric, so is $S_g$; and conversely. $S$ and $S_g$ have the same centroid $G$.

The fact that $H$ and $H'$, the foci of $Q$, are isogonal conjugates with respect to $S$ is a special case of the theorem that any point $P$ has an isogonal conjugate $P'$ with respect to a simplex $S$, whether orthocentric or not; and $P$ and $P'$ are the foci of a quadric surface $Q$ tangent to the faces of $S$. The $2(n+1)$ projections of $P$ and $P'$ on the faces of $S$ lie on a sphere which is the auxiliary sphere of $Q$ with respect to its focal diameter. For the definition of isogonal conjugate points

it is convenient to use a system of normal homogeneous coördinates for a given point $x_1, x_2, \cdots, x_{n+1}$ which are proportional to the distances of the point from the faces of $S$. Thus, if the coördinates of $P$ are denoted by $y_i$, the plane through $P$ and all of the vertices of $S$ except $A_1$ and $A_2$ is $x_1/y_1 = x_2/y_2$; the plane which bisects the angle between the faces opposite to $A_1$ and $A_2$ has the equation $x_1 = x_2$; and hence the plane which is symmetric to the first plane with respect to the bisecting plane is given by $x_1/y_2 = x_2/y_1$. It then follows that the $(n+1)n/2$ symmetric planes intersect in the point $P'$, $y_1^{-1}, y_2^{-1}, \cdots, y_{n+1}^{-1}$, the isogonal conjugate of $P$. Consider the sphere $(C)$, or plane in certain special cases, passing through the $n+1$ projections of $P$ on the faces of $S$; let $C$ be its center; and let $P'$ be the symmetric of $P$ with respect to $C$. Let the projections of $P$ and $P'$ on a face, say the face opposite $A_i$, be $M$ and $M'$: then the two-dimensional plane through $M$, $M'$, $P$, $P'$ cuts $(C)$ in a circle in that plane passing through $M$ and $M'$. Let $M'P'$ cut the circle again in $K$, so that $P'K = MP$. Then $MP \cdot M'P' = R^2 - c^2$, where $R$ is the radius of $(C)$ and $PC = c$. If the coördinates of $P$ and $P'$ are denoted $y_i$ and $y_i'$, then this says that we may take $y_i y_i' = R^2 - c^2$. Hence $P'$ is the isogonal conjugate of $P$, and $P$ and $P'$ are the foci of a quadric surface of revolution tangent to the faces of $S$. The types of the quadric surface are given by the sign of $R^2 - c^2$; and the case where $(C)$ is a plane may be regarded as the limit case where the center and a focus are at infinity.

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## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

The following mathematicians were starred for the first time in the sixth edition of *American Men of Science*: L. V. Ahlfors, Garrett Birkhoff, Solomon Bochner, Alonzo Church, Richard Courant, H. T. Davis, Harold Hotelling, B. O. Koopman, E. J. McShane, A. D. Michal, Hans Rademacher, H. P. Robertson, Otto Szász, Gabriel Szegö, J. M. Thomas, W. J. Trjitzinsky, Hermann Weyl, Hassler Whitney, Aurel Wintner, Oscar Zariski.

Dean G. D. Birkhoff of Harvard University has been appointed to a Walker-Ames Professorship of Mathematics at the University of Washington for the Summer Quarter, 1939.

### SUMMER COURSES

The following courses in mathematics are announced for the summer of 1939.

*University of California at Los Angeles.* In addition to the usual courses in trigonometry, algebra, and calculus, the following advanced courses will be offered: By Dr. A. E. Taylor: Harmonic functions; By Professor W. L. Ayres: Seminar in topology; By the staff: Special problems in mathematics.

*Catholic University of America.* In addition to the usual elementary courses

the following advanced courses will be offered: By J. Daly: Fundamental concepts of mathematics; By Dr. E. J. Finan: Theory of equations, Theory of numbers; By Dr. J. N. Rice: Advanced calculus; By Dr. O. J. Ramler: Differential equations, Analytical projective geometry.

*University of Chicago.* During the Summer Quarter of 1939 the department of mathematics at the University of Chicago will offer a well-balanced program, but will place especial emphasis on the field of analysis. A conference on the calculus of variations is to be held June 27 to June 30 at which a number of eminent mathematicians will present papers. In addition to courses in the calculus, theory of equations and differential equations the following courses will be offered: By Professor G. A. Bliss: A seminar on the calculus of variations; By Professor E. J. McShane: Modern theories of integration; By Professor L. E. Dickson: Analytic theory of numbers; By Professor A. C. Lunn: Partial differential equations of mathematical physics, Relativity; By Professor L. M. Graves: Functions of lines, Fundamental concepts of mathematics; By Professor Mayme I. Logsdon: Higher plane curves, Analytic projective geometry; By Professor W. T. Reid: Functions of a complex variable; By Professor M. R. Hestenes: Topology.

*University of Colorado.* In addition to the usual elementary courses the following advanced courses will be offered: By Professor Hutchinson: Real variables; By Professor Kempner: Non-Euclidean geometry, Differential equations, Squaring the circle.

*Duke University. First term, June 12 to July 22.* By Professor Brinkmann: Modern developments in mathematics, Theory of equations; By Professor Carlitz: Projective geometry, Thesis seminar; By Dr. Dressel: Advanced calculus, Integral equations; By Professor Gergen: Real variables; By Professor Rankin: Teaching of mathematics. *Second term, July 25 to September 2.* By Professor Gergen: Thesis seminar; By Professor Miles: Advanced calculus, Probability; By Professor Roberts: Projective geometry, Real variables.

*University of Illinois.* In addition to the usual elementary courses, the following advanced courses will be offered: By Professor P. W. Ketchum: Graphical and numerical methods, Topology; By Dr. E. L. Welker: Elementary statistics; By Professor Arnold Emch: Advanced aspects of Euclidean geometry, Geometric transformations in real and complex spaces; By Dr. Josephine Chanler: Introduction to higher algebra; By Professor W. J. Trjitzinsky: Advanced calculus, Analysis; By Dr. G. E. Moore: Introduction to higher geometry; By Professor H. R. Brahana: Theory of groups, Higher algebra; By members of the department: Thesis course.

*University of Michigan.* June 26 to August 18. In addition to elementary courses and the standard courses in differential equations, theory of equations, advanced solid analytic geometry, and advanced calculus, the following courses will be offered: By Professor Bradshaw: Descriptive geometry; By Professor Churchill: Harmonic analysis; By Professor Copeland: Theory of probability, Introduction to the foundations of mathematics; By Professor Craig: Theory of

statistics, I; Advanced theory of statistics, II; By Professor Dwyer: Social statistics, Theory of statistics, II; By Professor Field: Analytic projective geometry; By Dr. Greville: Finite differences; By Professor Hildebrandt: Theory of functions of a real variable, Partial differential equations; By Professor Karpinski: Teachers' seminar in geometry, History of arithmetic and algebra; By Professor Miller: Point-set topology; By Professor Poor: Vector analysis; By Professor Rainich: Continued fractions, Mathematics of relativity; By Professor Rouse: Graphical methods; By Dr. Thrall: Finite groups. In addition there will be an orientation seminar, a seminar in pure mathematics conducted by Professors Hildebrandt and Rainich, and one in statistics by Professor Craig.

*University of Minnesota.* In addition to the usual undergraduate courses, the following graduate courses will be offered: *First term*: By Professor Gladys Gibbens: Advanced theory of equations; By Professor Dunham Jackson: Vector analysis, Introduction to the theory of probability; By Professor A. L. Underhill: Intermediate calculus, Differential equations; By Professors Jackson and Underhill: Reading in advanced mathematics. *Second term*: By Professor Elizabeth Carlson: Reading in advanced mathematics.

*University of North Carolina.* In addition to the regular freshman and sophomore courses in analytic geometry and the calculus, the following courses will be offered: *First term*: By Professor Winsor: History of mathematics; By Professor Linker: Differential equations; By Professor Lasley: Analytic projective geometry; By Professor Henderson: Analytic geometry of space; Foundations of geometry. *Second term*: By Professor Garner: Differential equations (continued); By Professor Mackie: Advanced calculus, Theory of functions of a complex variable; By Professor Browne: Introduction to higher algebra.

*Northwestern University.* In addition to courses in plane analytical geometry and the calculus, the following courses will be offered: By Dr. Walter Leighton: Probability; By Professor Wall: Functions of several variables, Number systems; By Professor Garabedian: Solid analytic geometry; By Professor Moulton: Ordinary differential equations; By Professor Dines: Topics in algebra and geometry.

*Ohio State University.* June 19 to September 1. In addition to the usual elementary courses the following courses will be offered: By Professor Blumberg: Introduction to the theory of relativity, Point sets and real functions; By Professor Kuhn: Differential equations, Fundamental ideas in algebra and geometry; By Professor Weaver: Vector analysis, Differential geometry.

*University of Pennsylvania.* In addition to the regular undergraduate courses the following advanced courses will be given: By Professor Hallett: Advanced calculus, Theory of groups of finite order; By Professor Babb: Modern analytic geometry; By Professor Shohat: Theory of functions of a complex variable; By Professor Rademacher: Elliptic functions.

*Stanford University.* By Professor Artin: Topology, Modern algebra; By Professor Dresden: Fundamental survey of mathematics.

*Teachers College, Columbia University.* By Professor W. D. Reeve: Teaching algebra in secondary schools, The reorganization of secondary school mathematics; By Professor J. R. Clark: Teaching intuitive geometry in junior high schools, Teaching geometry in secondary schools; By Professor Max Black (of the University of London): Teaching mathematics in the secondary schools of England, An introduction to mathematical methods; By Professor J. A. Lauwerys (of the University of London): The correlation of mathematics and science in secondary schools; By Professor C. N. Shuster: Modern business arithmetic, Field work in mathematics; By Dr. J. A. Swenson: Professionalized subject matter in senior high school mathematics; By Miss E. Sutherland: Professionalized subject matter in junior high school mathematics, Teaching arithmetic in primary grades (first three weeks), Teaching arithmetic in intermediate grades (second three weeks); By Dr. Nathan Lazar: Teaching algebra in junior high schools, Experimental demonstration class in demonstrative geometry.

*Texas Technological College.* In addition to the usual elementary courses the following advanced courses will be offered. *First term, June 5 to July 14.* By Professor E. L. Thompson: Modern higher algebra, Mathematics of finance; By Professor J. N. Michie: Advanced algebra; By Professor F. W. Sparks: Applied calculus, Reading and research course. *Second term, July 17 to August 24.* By Professor R. S. Underwood: Elementary theory of numbers; By Professor J. N. Michie: Reading and research course.

*University of Texas.* In addition to the regular elementary courses the following advanced courses will be offered: *First term, June 6 to July 17.* By Professor P. M. Batchelder: Advanced calculus; By Professor R. L. Moore: Introduction to the foundations of geometry, Theory of sets; By Professor H. S. Vandiver: Elementary number theory, Finite groups; By Professor E. L. Dodd: Mathematical statistics, Theory of functions of real variables; By Professor H. V. Craig: Vector and tensor analysis, Applications of tensor analysis; By Professor E. G. Keller: Advanced applied mathematics, Non-linearity in engineering and physics. *Second term, July 17 to August 28.* By Dr. F. B. Jones: Advanced calculus; By Professor R. G. Lubben: Non-Euclidean geometry; By Professor H. J. Ettlinger: Descriptive geometry, Calculus of variations; By Professor A. E. Cooper: Group theory of differential equations; By Professor R. N. Haskell: Dynamics, Potential theory.

*University of Virginia. First term, June 19 to July 29.* By Professor Deane Montgomery: Elementary analysis, Functions of a complex variable. *Second term, July 31 to September 2.* By Professor G. T. Whyburn: Functions of a complex variable, Foundations of geometry.

*University of Washington.* In addition to elementary courses the following courses will be offered: By Dean G. D. Birkhoff: Differential equations of dynamics, a series of evening lectures on "Mathematical and philosophical thought"; By Dr. Mary E. Haller: Mathematics in modern education; By Professor R. M. Winger: Projective geometry, Topics in modern mathematics:

By Professor J. P. Ballantine (first term) and Dr. A. H. Taub (second term); Advanced analysis, Modern algebra.

*University of Wisconsin.* In addition to the usual elementary courses the following courses will be offered: By Professor Evans: Advanced calculus, Differential equations, Topics in the history of mathematics; By Professor Ingraham: Theory of equations, Mathematics of educational statistics, Basic number systems of mathematics; By Dr. Jackson: Elliptic integrals, Differential geometry; By Professor Trump: College geometry; By Professor MacDuffee: Finite groups (six or nine weeks), Theory of numbers (six or nine weeks).

## EXAMINATION QUESTIONS FOR THE SECOND WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, MARCH 4, 1939

MORNING SESSION: 9:00 A.M. to 12:00 NOON. (*Answer the questions in any order and by any method. Show all your work, and indicate your answers clearly. No tables or other books permitted.*)

1. Find the length of the curve $y^2=x^3$ from the origin to the point where the tangent makes an angle of 45° with the $x$-axis.

2. A point $P$ is taken on the curve $y=x^3$. The tangent at $P$ meets the curve again at $Q$. Prove that the slope of the curve at $Q$ is *four* times the slope at $P$.

3. Find the cubic equation whose roots are the cubes of the roots of

$$x^3+ax^2+bx+c=0.$$

4. Find the equations of the *two* straight lines each of which cuts all the *four* straight lines

$$x=1,\ y=0;\ y=1,\ z=0;\ z=1,\ x=0;\ x=y=-6z.$$

5. TAKE EITHER ($a$) OR ($b$).

($a$) Solve the system of differential equations

$$\frac{dx}{dt}=x+y-3, \qquad \frac{dy}{dt}=-2x+3y+1,$$

subject to the conditions $x=y=0$ for $t=0$.

($b$) A heavy particle is attached to the end $A$ of a light rod $AB$ of length $a$. The rod is hinged at $B$ so that it can turn freely in a vertical plane. The rod is balanced in the vertical position above the hinge and then slightly disturbed. Prove that the time taken to pass from the horizontal position to the lowest position is

$$\sqrt{\frac{a}{g}}\log_e(1+\sqrt{2}).$$

6. TAKE EITHER ($a$) OR ($b$).

($a$) A circle of radius $a$ rolls on the inner side of the circumference of a circle of radius $3a$. Find the area contained within the closed curve generated by a point on the circumference of the rolling circle.

($b$) A shell strikes an airplane flying at a height $h$ above the ground. It is known that the shell was fired from a gun on the ground with a muzzle-velocity of magnitude $V$, but the position of the gun and its angle of elevation are both unknown. Deduce that the gun is situated within a circle whose center lies directly below the airplane and whose radius is

$$\frac{V}{g}\sqrt{V^2-2gh}.$$

(Neglect the resistance of the atmosphere.)

7. TAKE EITHER (*a*) OR (*b*).

(*a*) Find the curve touched by all the curves of the family

$$(y - k^2)^2 = x^2(k^2 - x^2).$$

Make a rough sketch showing this curve and two curves of the family.

(*b*) If the expansion in powers of $x$ of the function

$$\frac{1}{(1 - ax)(1 - bx)}$$

is given by

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

prove that the expansion in powers of $x$ of the function

$$\frac{1 + abx}{(1 - abx)(1 - a^2x)(1 - b^2x)}$$

is given by

$$c_0^2 + c_1^2x + c_2^2x^2 + c_3^2x^3 + \cdots.$$

AFTERNOON SESSION: 2:00 P.M. to 5:00 P.M. (*Answer the questions in any order and by any method. Show all your work, and indicate your answers clearly. No tables or other books permitted.*)

8. From the vertex $(0, c)$ of the catenary

$$y = c \cosh \frac{x}{c}$$

a line $L$ is drawn perpendicular to the tangent to the catenary at a point $P$. Prove that the length of $L$ intercepted by the axes is equal to the ordinate $y$ of the point $P$.

9. Evaluate the definite integrals

(i) $$\int_1^3 \frac{dx}{\sqrt{(x - 1)(3 - x)}},$$ (ii) $$\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}}.$$

10. Give the power-series

$$a_0 + a_1x + a_2x^2 + \cdots$$

in which

$$a_n = (n^2 + 1)3^n,$$

show that there is a relation of the form

$$a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0,$$

in which $p$, $q$, $r$ are constants independent of $n$. Find these constants and the sum of the power-series.

11. Find the equation of the parabola which touches the $x$-axis at the point $(1, 0)$ and the $y$-axis at the point $(0, 2)$. Find the equation of the axis of the parabola and the coordinates of its vertex.

12. TAKE EITHER (*a*) OR (*b*).

(*a*) Prove that

$$\int_1^a [x]f'(x)dx = [a]f(a) - \{f(1) + \cdots + f([a])\},$$

where $a$ is greater than 1 and where $[x]$ denotes the greatest of the integers not exceeding $x$.

Obtain a corresponding expression for

$$\int_1^a [x^2]f'(x)dx.$$

(*b*) A particle moves on a straight line, the only force acting on it being a resistance proportional to the velocity. If it started with a velocity of 1,000 ft. per sec. and had a velocity of 900 ft. per sec. when it had travelled 1,200 ft., calculate to the nearest hundredth of a second the time it took to travel this distance.

13. TAKE EITHER (*a*) OR (*b*).

(*a*) Let $f(x)$ be defined for $a \gtreqless x \gtreqless b$. Assuming appropriate properties of continuity and derivability, prove for $a<x<b$ that

$$\frac{\dfrac{f(x)-f(a)}{x-a}-\dfrac{f(b)-f(a)}{b-a}}{x-b}=\tfrac{1}{2}f''(\xi),$$

where $\xi$ is some number between $a$ and $b$.

(*b*) Calculate the mutual gravitational attraction of two uniform rods, each of mass $m$ and length $2a$, placed parallel to one another and perpendicular to the line joining their centers at a distance $b$ apart.

In your answer let $a$ approach zero, and comment on the form of the result.

14. TAKE EITHER (*a*) OR (*b*).

(*a*) If

$$u=1+\frac{x^3}{3!}+\frac{x^6}{6!}+\cdots,$$

$$v=\frac{x}{1!}+\frac{x^4}{4!}+\frac{x^7}{7!}+\cdots,$$

$$w=\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots,$$

prove that

$$u^3+v^3+w^3-3uvw=1.$$

(*b*) Consider the central conics

$$(ax^2+by^2)+2(px+qy)+c=0,$$
$$(ax^2+by^2)+2\lambda(px+qy)+\lambda^2c=0,$$

where $\lambda$ is given positive constant.

Show that if all radii from the origin to the first conic are changed in the ratio $\lambda$ to 1 the tips of these new radii generate the second conic.

Let $P$ be the point with coordinates

$$x=-\frac{p}{a}\,\frac{2\lambda}{1+\lambda},\qquad y=-\frac{q}{b}\,\frac{2\lambda}{1+\lambda}.$$

Show that if all radii from $P$ to the first conic are changed in the ratio $\lambda$ to 1 and then reversed about $P$ the tips of these new radii generate the second conic.

Comment on these results in case $\lambda=1$.



NOTE. Chairmen of the mathematics departments may obtain copies of the examination questions for the Putnam Competition for 1938 and for 1939 by writing for them to Professor W. D. Cairns, Oberlin, Ohio.

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NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the Secretary-Treasurer, W. D. Cairns, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.

## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

Allegheny Mountain, Greenville, Pa., May 13.
Illinois, Galesburg, May 12–13.
Indiana, Muncie, April 28–29.
Iowa, Ames, April 21–22.
Kansas, Topeka, April 1.
Kentucky, Murray, April 28–29.
Louisiana-Mississippi, Baton Rouge, La., March 3–4.
Maryland-District of Columbia-Virginia, Aberdeen Proving Ground, Md., May 13.
Michigan, Ann Arbor, March 18.
Minnesota, Northfield, May 13.
Missouri, Springfield, April 28.
Nebraska, Lincoln, May 5.
Ohio, Columbus, April 8
Oklahoma, Tulsa, February 10.
Philadelphia, Bethlehem, Pa., December 2.
Rocky Mountain, Laramie, Wyo., April 28–29.
Southeastern, Charleston, S.C., March 24–25.
Southern California, Whittier, March 4.
Southwestern, Alpine, Texas, May 2–3.
Texas, Abilene, March 31–April 1.
Wisconsin, Milwaukee, May 6.

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Following a meeting held at San Francisco January 28, 1939, for the purpose of organizing a Northern California Section of the Association, By-Laws were submitted to the officers of the Association. These By-Laws have been approved by action of the Trustees, and the Northern California Section is thereby established.

W. D. CAIRNS, *Secretary-Treasurer*

# LIMITS TO THE CHARACTERISTIC ROOTS OF A MATRIX*

E. T. BROWNE, University of North Carolina

**1. Introduction.** Let $A=(a_{rs})$ be a square matrix of order $n$ whose elements are real or complex numbers. If $I$ is the unit matrix and $\lambda$ is a scalar indeterminate, the equation obtained by equating to zero the determinant $|A-\lambda I|$ is called the *characteristic equation* of $A$. The roots of this equation are called the *characteristic roots* (*latent roots*, or *characteristic values*) of $A$. The characteristic equation is one of the most important equations in modern algebraic analysis and has received a great deal of attention within the past seventy-five or hundred years.

If $A$ is a matrix of some particular type, certain definite statements can be made as to the nature of its characteristic roots. For example, if $A$ is *real symmetric*, or *Hermitian*, its characteristic roots are all *real*; if $A$ is *real skew-symmetric*, its characteristic roots are all *pure imaginary* or zero; if $A$ is *real orthogonal*, or *unitary*, its characteristic roots have *modulus unity*. Although it is not possible to make any definite statement as to the nature of the characteristic roots of the general matrix $A$, several authors have given limits to the roots. It is the purpose of this paper to list and compare the various limits that have been given and to sketch proofs of the more interesting and important of them.

**2. Notations and definitions.** By $A'$ we denote the *transpose* of $A$; *i.e.*, the matrix obtained from $A$ by interchanging its rows and columns. By $\bar{A}$ we denote the *conjugate* of $A$; *i.e.*, the matrix obtained from $A$ by replacing each element $a_{rs}$ by its conjugate imaginary $\bar{a}_{rs}$. Hence, $\bar{A}'$ is the *conjugate* of the *transpose* of $A$. By $A^{-1}$ we denote the *inverse* of $A$, which exists only if $A$ is non-singular. With these notations we say that $A$ is *real* if $A=\bar{A}$; $A$ is *symmetric* if $A=A'$; $A$ is *skew-symmetric* if $A=-A'$; $A$ is *Hermitian* if $A=\bar{A}'$; $A$ is *orthogonal* if $A^{-1}=A'$; $A$ is *unitary* if $A^{-1}=\bar{A}'$.

**3. Bendixson's theorem.**† In 1900 Bendixson proved the following theorem:

THEOREM I. *If $\alpha+i\beta$ is a characteristic root of a real n-square matrix $A$, then*

$$\text{(a)}\qquad |\beta| \leqq g''\left\{\frac{n(n-1)}{2}\right\}^{1/2};$$

$$\text{(b)}\qquad \rho_1 \geqq \alpha \geqq \rho_n;$$

*where $g''$ is the largest of the quantities $|a_{rs}-a_{sr}|/2$, and $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all real) of the real symmetric matrix $\frac{1}{2}(A+A')$.*

It will be noted that the limit furnished by the first part of this theorem is

* An address delivered by invitation before the Mathematical Association of America at Williamsburg, Virginia, December 31, 1938.

† Bendixson, Sur les racines d'une équation fondamentale, Acta Mathematica, vol. 25, 1902, pp. 359–365.

in terms of the elements of the matrix itself, while those furnished by the second part are in terms of the characteristic roots of an auxiliary matrix. Moreover, the second part is a special case of a theorem given later by Hirsch and Bromwich, and follows as an immediate consequence of a general lemma which we shall prove. We shall, therefore, confine our attention here to a proof of the first part only.

Since $\lambda=\alpha+i\beta$ is a characteristic root of $A$, there exists a set $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$, such that

$$\sum a_{rs}x_s = \lambda x_r, \qquad (r = 1, \cdots, n).$$

If we put $\lambda=\alpha+i\beta$, $x_s=\xi_s+i\eta_s$ multiply out and equate real parts and imaginary parts, we obtain

$$\sum_s a_{rs}\xi_s = \alpha\xi_r - \beta\eta_r, \qquad (r = 1, \cdots, n), \tag{1}$$

$$\sum_r a_{sr}\eta_r = \alpha\eta_s + \beta\xi_s, \qquad (s = 1, \cdots, n), \tag{2}$$

where in the last equation we have interchanged the subscripts $r$ and $s$.

Multiply (1) through by $\eta_r$ and sum as to $r$; similarly, multiply (2) through by $\xi_s$ and sum as to $s$. On subtracting the resulting equations member for member, we obtain:

$$\beta\sum_{s=1}^{n} (\xi_s^2 + \eta_s^2) = \sum_{r,s}^{1\cdots n} (a_{sr} - a_{rs})\xi_s\eta_r.$$

Since the matrix of the bilinear form on the right is skew-symmetric, it is easy to see that this may be written:

$$\beta\sum (\xi_s^2 + \eta_s^2) = \sum_{r<s} (a_{sr} - a_{rs})(\xi_s\eta_r - \xi_r\eta_s),$$

where now the summation extends over all the $n(n-1)/2$ combinations $r<s$ of the numbers $1, \cdots, n$ taken 2 at a time. If then $g''$ denotes the largest of the $\frac{1}{2}|a_{sr}-a_{rs}|$, we have on substituting and squaring,

$$\beta^2\{\sum (\xi^2 + \eta^2)\}^2 \leqq 4g''^2\{\sum | \xi_s\eta_r - \xi_r\eta_s | \}^2. \tag{3}$$

Now

$$\{\sum (\xi^2 + \eta^2)\}^2 = \{\sum \xi^2 - \sum \eta^2\}^2 + 4\sum \xi^2\sum \eta^2 \geqq 4\sum \xi^2\sum \eta^2.$$

Also by Lagrange's identity,

$$\sum \xi_r^2 \sum \eta_r^2 = (\sum \xi_r\eta_r)^2 + \sum (\xi_s\eta_r - \xi_r\eta_s)^2 \geqq \sum (\xi_s\eta_r - \xi_r\eta_s)^2,$$

where the last summation extends over all the $n(n-1)/2$ combinations $r<s$. Hence

$$\{\sum (\xi^2 + \eta^2)\}^2 \geqq 4\sum_{r<s} (\xi_s\eta_r - \xi_r\eta_s)^2.$$

But by an easily established inequality for $m$ real quantities $k_1, \cdots, k_m$,

$$\left\{\sum_1^m k_i\right\}^2 \leqq m \sum_1^m k_i^2,$$

we have, since there are $n(n-1)/2$ terms in the summation on the right in (3),

$$\beta^2 \sum (\xi_s\eta_r - \xi_r\eta_s)^2 \leqq \frac{n(n-1)}{2} g''^2 \sum (\xi_s\eta_r - \xi_r\eta_s)^2.$$

Hence, if for at least one combination $(r, s)$, $\xi_s\eta_r - \xi_r\eta_s \neq 0$, we have

$$|\beta| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}; \tag{4}$$

while if every $\xi_s\eta_r - \xi_r\eta_s = 0$, then $\beta = 0$ by (3) so that (4) still holds.

The proof of part (a) of the theorem is therefore complete. The proof of part (b) we defer until later.

In particular, if $A$ is real symmetric, then each $a_{sr} - a_{rs} = 0$ so that $g'' = 0$, and therefore $\beta = 0$. We therefore have as a corollary the well-known theorem:

COROLLARY. *The characteristic roots of a real symmetric matrix are all real.*

The limit (4) given by Bendixson is sometimes actually attained if $n = 2$ or $n = 3$, but never (for $g'' \neq 0$) if $n > 3$, as will be seen by reference to Bromwich's and Pick's theorems which we shall state in §§4, 6. For example the limit is attained in the case of the following two matrices:

$$A = \begin{pmatrix} 0 & g'' \\ -g'' & 0 \end{pmatrix}; \qquad A = \begin{pmatrix} 0 & g'' & g'' \\ -g'' & 0 & g'' \\ -g'' & -g'' & 0 \end{pmatrix}.$$

**4. Hirsch's and Bromwich's theorems.** The case in which the elements of $A$ are complex numbers was considered in 1902 by Hirsch,* and again in 1904 by Bromwich.† For convenience in formulating our results, let us write

$$b_{rs} = \frac{1}{2}(a_{rs} + \bar{a}_{sr}); \qquad c_{rs} = \frac{1}{2i}(a_{rs} - \bar{a}_{sr});$$

*i.e.*,

$$B = \frac{1}{2}(A + \bar{A}'); \qquad C = \frac{1}{2i}(A - \bar{A}').$$

* Hirsch, Acta Mathematica, vol. 25, 1902, p. 367.

† Bromwich, On the roots of the characteristic equation of a linear substitution, Acta Mathematica, vol. 30, 1906, pp. 295–304.

The following note is due to Dr. E. Hellinger: O. Toeplitz, Das Algebraische Analogon zu einem Satze von Féjer, *Mathematische Zeitschrift*, vol. 2, 1918, pp. 187–197, gave the following extension of the theorem of Bendixson and Hirsch, which can be applied to forms of infinitely many variables and to integral equations: *The characteristic roots of an n-square matrix A belong to the set of all complex values, which the form* $\sum a_{rs}x_r\bar{x}_s$ *takes, if* $\sum x_r\bar{x}_r = 1$; *the boundary of this set is convex.*—E.J.M.

It is easy to verify that both $B$ and $C$ are Hermitian matrices.

Hirsch's theorem is as follows:

THEOREM II (a). *If $\alpha+i\beta$ is a characteristic root of an n-square matrix $A$, and if we denote by g the greatest of the $|a_{rs}|$, by $g'$ the greatest of the $|b_{rs}|$, and by $g''$ the greatest of the $|c_{rs}|$, then*

$$|\alpha+i\beta| \leqq ng, \qquad |\alpha| \leqq ng', \qquad |\beta| \leqq ng''.$$

(b) *In the case where the $a_{rs}$ are such that the $a_{rs}+a_{sr}$ are all real, this last inequality can be replaced by*

$$|\beta| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}.$$

(c) *If $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic roots of $B=\frac{1}{2}(A+\bar{A}')$, then always*

$$\rho_1 \geqq \alpha \geqq \rho_n.$$

We prove part (a) of the theorem first. Since $\lambda=\alpha+i\beta$ is a characteristic root of $A$, there exists a set $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ which may be normalized so that $\sum x_r\bar{x}_r=1$, and which satisfies the relations

$$\lambda x_r = \sum_{s=1}^{n} a_{rs}x_s, \qquad (r = 1, \cdots, n).$$

Multiplying these equations through by $\bar{x}_r$ and summing as to $r$, we have

$$\lambda = \alpha + i\beta = \sum a_{rs}\bar{x}_r x_s. \tag{5}$$

If in this last equation we take the conjugates of both members and interchange subscripts, we have

$$\bar{\lambda} = \alpha - i\beta = \sum \bar{a}_{sr}\bar{x}_r x_s;$$

whence, on adding and subtracting, we obtain:

$$\alpha = \sum b_{rs}\bar{x}_r x_s, \tag{6}$$

$$\beta = \sum c_{rs}\bar{x}_r x_s. \tag{7}$$

From the equations (5), (6) and (7) we can prove all three parts of Hirsch's Theorem II (a). Since the proofs of the three parts are practically identical, we shall prove the first part only.*

Let $\eta_r=|x_r|$ so that $\sum\eta_r^2=\sum x_r\bar{x}_r=1$. Then since $\eta_r\eta_s \leqq \frac{1}{2}(\eta_r^2+\eta_s^2)$, we have from (5)

$$|\lambda| \leqq \sum |a_{rs}||\bar{x}_r||x_s| \leqq \tfrac{1}{2}g\sum_{rs}(\eta_r^2+\eta_s^2)$$

$$= \tfrac{1}{2}g\sum_{s=1}^{n}\sum_{r=1}^{n}\eta_r^2 + \tfrac{1}{2}g\sum_{r=1}^{n}\sum_{s=1}^{n}\eta_s^2 = \tfrac{1}{2}ng + \tfrac{1}{2}ng = ng.$$

* In fact, the last two parts of Theorem II (a) follow as direct consequences of the first part and of Theorems II (c) and III.

This limit for $|\lambda|$ is actually attained if $A$ is a *real* matrix every one of whose elements is equal to the same number $g$; and indeed this limit is not attained in any other case.

The proof of Hirsch's Theorem II (b) is quite similar to that of Theorem I (a) as given by Bendixson.

Before proceeding to a proof of Theorem II (c), we shall establish a lemma, which will be useful also in proving Bromwich's theorem and Theorem VII.

LEMMA I. *If $H=(h_{rs})$ is an Hermitian matrix with characteristic roots $R_1 \geqq R_2 \geqq \cdots \geqq R_n$, and if the equality*

$$\sum h_{rs}\bar{x}_r x_s = \sigma \sum x_r \bar{x}_r \tag{8}$$

*is satisfied by a set $(x) \neq (0)$, then*

$$R_1 \geqq \sigma \geqq R_n,$$

The proof of the lemma is very simple. For by a conjunctive transformation $x_r = \sum u_{rs} y_s$, where the matrix $U=(u_{rs})$ is unitary, the equality (8) is transformed into*

$$R_1 y_1 \bar{y}_1 + R_2 y_2 \bar{y}_2 + \cdots + R_n y_n \bar{y}_n = \sigma \sum y_r \bar{y}_r.$$

Hence, if $R_1$ is the greatest and $R_n$ the smallest of the $R$'s, we have since each $y_r \bar{y}_r \geqq 0$,

$$R_1 \sum y_r \bar{y}_r \geqq \sum R_r y_r \bar{y}_r = \sigma \sum y_r \bar{y}_r \geqq R_n \sum y_r \bar{y}_r,$$

so that

$$R_1 \geqq \sigma \geqq R_n.$$

Now let us return to equations (6) and (7). Since both the matrices $B$ and $C$ are Hermitian, it follows immediately from the lemma that if $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic roots of $B$ while $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$ are the characteristic roots of $C$, then

$$\rho_1 \geqq \alpha \geqq \rho_n,$$

$$\mu_1 \geqq \beta \geqq \mu_n.$$

We have, therefore, established not only Hirsch's Theorem II (c) but also the following theorem due to Bromwich:

THEOREM III. *If $\alpha + i\beta$ is a characteristic root of $A$ and if $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$ are the characteristic roots (all real) of the Hermitian matrix $C=(A-\bar{A}')/(2i)$, then*

$$\mu_1 \geqq \beta \geqq \mu_n. \tag{9}$$

In particular, if $A$ is a real matrix so that $K = iC = \frac{1}{2}(A - A')$ is real skew-symmetric with non-zero characteristic roots

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* Turnbull and Aitken, Theory of Canonical Matrices, London, 1932, p. 106.

$$\pm i\mu_1, \ \pm i\mu_2, \cdots, \pm i\mu_t \qquad (2t \leqq n), \tag{10}$$

Bromwich shows that the limit $|\beta| \leqq |\mu|$, where $|\mu|$ is the greatest of the $|\mu_r|$, is more restricted than that given by Bendixson. For the sum of the products two at a time of the roots in (10) is easily seen to be

$$\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2.$$

But this is equal to the coefficient of $\lambda^{n-2}$ in the expansion of $|\lambda I - K|$; *i.e.*,

$$\mu_1^2 + \mu_2^2 + \cdots + \mu_t^2 = \sum_{r<s} \begin{vmatrix} 0 & k_{rs} \\ -k_{rs} & 0 \end{vmatrix} = \sum_{r<s} k_{rs}^2,$$

where the summation extends over the $n(n-1)/2$ combinations $r<s$ taken two at a time. If then $g''$ is the largest of the $|k_{rs}|$ and $|\mu|$ is the largest of the $|\mu_r|$, we have

$$|\mu|^2 \leqq \sum |\mu_r|^2 = \sum k_{rs}^2 \leqq \frac{n(n-1)}{2} g''^2,$$

so that

$$|\mu| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}.$$

Thus, the limit on the imaginary part of the characteristic root of a real matrix $A$ as given by Bromwich is (in general) more restricted than that given earlier by Bendixson. However, the limit given by the former is in terms of the characteristic roots of an auxiliary matrix, while the limit given by the latter is in terms of the elements of the matrix itself.

**5. Schur's proofs.** In 1909 I. Schur* gave a demonstration of Hirsch's Theorem II (a) and of Bendixson's Theorem I (a) which seems to be of sufficient interest to be included here. Schur first establishes the following lemma:

LEMMA II. *If $U$ is any unitary matrix and if $\bar{U}'AU = T$, then*

$$\sum_{r,s}^{1\cdots n} |t_{rs}|^2 = \sum_{r,s}^{1\cdots n} |a_{rs}|^2;$$

*that is, if two matrices $A$ and $T$ are unitary equivalent, the sum of the squares of the absolute values of all the elements of $A$ is equal to the corresponding sum for $T$.*

For from

$$\bar{U}'AU = T, \tag{11}$$

we have on taking conjugate transposes,

$$\bar{U}'\bar{A}'U = \bar{T}'; \tag{12}$$

* I. Schur, Über die characteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, Mathematische Annalen, vol. 65, 1909, pp. 488–510.

whence

$$T\bar{T}' = \bar{U}'AU\bar{U}'\bar{A}'U = U^{-1}A\bar{A}'U, \quad \text{since} \quad \bar{U}' = U^{-1}.$$

Hence, the matrices $T\bar{T}'$ and $A\bar{A}'$ are *similar*, and therefore have the same trace.* But the trace of $T\bar{T}'$ is seen to be precisely

$$\sum_{r,s}^{1,\cdots,n} t_{rs}\bar{t}_{rs} = \sum_{r,s}^{1,\cdots,n} |t_{rs}|^2.$$

The lemma is therefore established.

Now let the characteristic roots of $A$ be

$$\lambda_r = \alpha_r + i\beta_r, \qquad (r = 1, \cdots, n).$$

Then by an easily established and well-known theorem,† we know that $A$ is unitary equivalent to a *triangular* matrix of the form

$$T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots 0 & 0 \\ t_{21} & \lambda_2 & 0 & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ t_{n-1\,1} & t_{n-1\,2} & t_{n-1\,3} & \cdots \lambda_{n-1} & 0 \\ t_{n\,1} & t_{n\,2} & t_{n\,3} & \cdots t_{n\,n-1} & \lambda_n \end{pmatrix}.$$

Hence, by the lemma, we have

$$(13) \qquad \sum |\lambda_r|^2 \leqq \sum |\lambda_r|^2 + \sum_{r>s} |t_{rs}|^2 = \sum |a_{rs}|^2.$$

If then $|\lambda|$ is the greatest of the $|\lambda_r|$ and $g$ is the greatest of the $|a_{rs}|$, we have at once

$$|\lambda|^2 \leqq \sum |\lambda_r|^2 \leqq n^2g^2,$$

whence

$$|\lambda| \leqq ng.$$

It is clear that the equality sign holds in (13) if, and only if, $U$ can be so chosen that $t_{rs}=0$, $r\neq s$, that is, $A$ is unitary equivalent to a diagonal matrix or is *normal*.

Moreover, with $T$ in the form above, we have on combining (11) and (12),

$$(14) \qquad \bar{U}'\left(\frac{A \pm \bar{A}'}{2}\right)U = \frac{T \pm \bar{T}'}{2},$$

where the diagonal elements of the matrix on the right are in the respective

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* By the *trace* of a matrix we mean the sum of the elements in the principal diagonal of the matrix. It is invariant under *similarity* transformations. Cf. MacDuffee, The Theory of Matrices Berlin, 1933, p. 19; also p. 69.

† MacDuffee, *loc. cit.*, p. 75.

cases $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $i\beta_1, i\beta_2, \cdots, i\beta_n$, the absolute values of the remaining elements being $|t_{rs}|/2$. Hence by the lemma we have

$$\sum |\alpha_r|^2 \leqq \sum |\alpha_r|^2 + \sum \left|\frac{t_{rs}}{2}\right|^2 = \sum |b_{rs}|^2, \tag{15}$$

$$\sum |\beta_r|^2 \leqq \sum |\beta_r|^2 + \sum \left|\frac{t_{rs}}{2}\right|^2 = \sum |c_{rs}|^2, \tag{16}$$

where $b_{rs}$ and $c_{rs}$ have the meaning given in section **4**. If then we denote by $g'$ and $g''$, respectively, the greatest of the $|b_{rs}|$ and $|c_{rs}|$, we have immediately

$$|\alpha_r| \leqq ng', \qquad |\beta_r| \leqq ng''.$$

But further, if $A$ is *real*, then in (16) $c_{rr}=0$, so that

$$\sum |\beta_r|^2 \leqq n(n-1)g''^2;$$

and since complex roots occur in pairs so that each $\beta \neq 0$ occurs at least twice, we have Bendixson's result

$$|\beta_r| \leqq g''\left(\frac{n(n-1)}{2}\right)^{1/2}.$$

6. **Pick's Theorem.** In 1922 Pick* gave a proof of Hirsch's Theorem II (a) and II (c), and in addition for a *real* matrix $A$ gave a limit for $|\beta|$ which is in general more restricted than that given by Bendixson. Pick's theorem is as follows:

THEOREM V. *If $\alpha+i\beta$ is a characteristic root of a real n-square matrix $A$ and if $g''$ is the largest of the $|k_{rs}| = |a_{rs}-a_{sr}|/2$ then*

$$|\beta| \leqq g'' \cot \frac{\pi}{2n}\cdot$$

To prove Pick's theorem we recall that the characteristic roots of $K=\frac{1}{2}(A-A')$ are all pure imaginary or zero. If $|\mu|$ is the greatest of the absolute values of the roots, then since by Bromwich's Theorem $|\beta| \leqq |\mu|$, we need only show that

$$|\mu| \leqq g'' \cot \frac{\pi}{2n}\cdot$$

Since $i\mu$ is a characteristic root of $K$, there exists a set $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$ such that

$$i\mu x_r = \sum k_{rs}x_s,$$

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* Pick, Georg, Über die Wurzeln der characteristischen Gleichungen von Schwingungsproblemen, Zeitschrift für angewandte Mathematik und Mechanik, vol. 2, 1922, pp. 353–357.

whence

$$i\mu \sum x_r \bar{x}_r = \sum_{r,s}^{1,\cdots,n} k_{rs} \bar{x}_r x_s. \tag{17}$$

If any one of the numbers $x_r$ has an amplitude $\theta_r$ such that $\pi/2 < \theta_r \leqq 3\pi/2$ we can change its sign thus obtaining a set such that each $x_r$ has amplitude $\theta_r$ where $-\pi/2 < \theta_r \leqq \pi/2$. We now apply, if necessary, a permutation to the $x$'s so that the resulting set has the property that for $r < s$ we have $\theta_r \leqq \theta_s$.

These last two changes can be looked upon as real orthogonal transformations on the $x$'s and every such transformation leaves the left member of (17) unaltered, while it transforms the right member into an expression of the same type, where the $k$'s differ at most in sign or order from the original $k$'s. We may then write (17)

$$\mu \sum x_r \bar{x}_r = \sum_{r<s} k_{rs} \left( \frac{\bar{x}_r x_s - \bar{x}_s x_r}{i} \right)$$

where the summation on the right extends to all the $n(n-1)/2$ combinations $(r < s)$ and where $\theta_r \leqq \theta_s$ for $r < s$. It is easy to see that the expression in the parentheses is never negative. Hence, if $g''$ is the greatest of the $|k_{rs}|$, we have

$$|\mu| \sum x_r \bar{x}_r \leqq g'' \sum_{r<s} \left( \frac{\bar{x}_r x_s - \bar{x}_s x_r}{i} \right);$$

*i.e.*,

$$\frac{|\mu|}{g''} \sum x_r \bar{x}_r \leqq \frac{1}{i} \sum_{r,s}^{1,\cdots,n} d_{rs} \bar{x}_r x_s,$$

where $D = (d_{rs})$ is the real skew-symmetric matrix with $d_{rs} = 1$ for $r < s$ and $d_{rs} = -1$ for $r > s$. Since $D/i$ is Hermitian, by Lemma I, $|\mu|/g''$ cannot exceed in absolute the greatest characteristic root of $D$. The characteristic equation of $D$ is

$$\begin{vmatrix} -\lambda, & 1, & 1, \cdots, & 1, & 1 \\ -1, & -\lambda, & 1, \cdots, & 1, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1, & -1, & -1, \cdots, & -\lambda, & 1 \\ -1, & -1, & -1, \cdots, & -1, & -\lambda \end{vmatrix} = 0,$$

which on expansion gives $\sum_{j=0}^{n} \sigma_j (-\lambda)^{n-j} = 0$, where $\sigma_j$ is the sum of all the $j$-rowed principal minor determinants $\Delta_j$ of $D$. Every principal minor determinant $\Delta_{2s+1}$ of odd order is zero, while each $\Delta_{2s}$ of even order has the value $+1$. This last statement is easily established by induction. For it is obviously true if $2s = 2$. Moreover, if in $\Delta_{2s}$ we add the last column to the first and in the resulting determinant add the first row to the last, we have $\Delta_{2s} = \Delta_{2s-2}$, so that $\Delta_{2s} = 1$.

Since there are exactly $\binom{n}{j}$ principal minor determinants $\Delta_j$ of order $j$ in $D$, the characteristic equation of $D$ can therefore be written

$$(-1)^n\left\{\lambda^n + \binom{n}{2}\lambda^{n-2} + \binom{n}{4}\lambda^{n-4} + \cdots\right\} = 0,$$

where the final term on the left is 1 or $n\lambda$ according as $n$ is even or odd. But this is seen to be precisely the same as*

$$\tfrac{1}{2}\{(\lambda + 1)^n + (\lambda - 1)^n\} = 0. \tag{18}$$

On solving (18) we obtain

$$\lambda = \frac{\alpha + 1}{\alpha - 1}, \quad \text{where} \quad \alpha^n = -1; \tag{19}$$

*i.e.*,

$$\alpha = \cos\frac{(2k+1)\pi}{n} + i \sin\frac{(2k+1)\pi}{n}, \qquad (k = 0, 1, \cdots, n-1).$$

From (19) it is easy to show that

$$\lambda = -i \cot\frac{(2k+1)\pi}{2n}, \qquad (k = 0, 1, \cdots, n-1).$$

Now the largest of these roots in absolute value is $\cot \pi/2n$. Hence, we have

$$|\mu| \leqq g'' \cot\frac{\pi}{2n}\cdot$$

This limit is in general more restricted than that given by Bendixson. For the roots of (18) are

$$\lambda_1 = -i\cot\frac{\pi}{2n}, \ \lambda_2 = -i\cot\frac{3\pi}{2n}, \cdots, \lambda_n = -i\cot\frac{(2n-1)\pi}{2n}\cdot$$

Moreover, from (18) we have

$$\sum \lambda_r = 0, \qquad \sum \lambda_r\lambda_s = \frac{n(n-1)}{2},$$

whence

$$\sum \lambda_r^2 = -n(n-1).$$

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* Cf. Hölder, Leipziger Berichte, vol. 65, pp. 110–120; also vol. 66, pp. 98–102. Cf. also Hirsch, Mathematische Annalen, vol. 52, pp. 150 ff.

that is,

$$\cot^2\frac{\pi}{2n}+\cot^2\frac{3\pi}{2n}+\cdots+\cot^2\frac{(2n-1)\pi}{2n}=n(n-1).$$

But

$$\cot^2\frac{\pi}{2n}=\cot^2\frac{(2n-1)\pi}{2n},$$

whence

$$\cot^2\frac{\pi}{2n}\leqq\frac{n(n-1)}{2}\cdot$$

**7. More recent limits to the roots.** In 1930 the author* gave a refinement of Hirsch's Theorem II (a) which was further refined by Parker† in 1937.

Returning to equations (5) let us write as before $|x_r|=\eta_r$, so that $\sum\eta_r^2=1$. We then have

$$\begin{aligned}|\lambda| &\leqq \sum|a_{rs}|\eta_r\eta_s\leqq\tfrac{1}{2}\sum_{r,s}^{1,\cdots,n}|a_{rs}|(\eta_r^2+\eta_s^2)\\ &=\tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}|a_{rs}|\eta_r^2+\tfrac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n}|a_{rs}|\eta_s^2.\end{aligned}$$

Let us now denote by $\sigma_j$ $(\tau_j)$ the sum of the absolute values of the elements in the $j$th row (column) of $A$; *i.e.*, let

$$\sigma_j=\sum_{s=1}^{n}|a_{js}|,\qquad \tau_j=\sum_{r=1}^{n}|a_{rj}|,$$

and let $\sigma$ $(\tau)$ be the greatest of the $\sigma_j$ $(\tau_j)$. We then have

$$\begin{aligned}|\lambda| &\leqq \tfrac{1}{2}\sum_r\sigma_r\eta_r^2+\tfrac{1}{2}\sum_s\tau_s\eta_s^2=\tfrac{1}{2}\sum(\sigma_r+\tau_r)\eta_r^2\\ &\leqq\tfrac{1}{2}(\sigma+\tau)\sum\eta_r^2=\tfrac{1}{2}(\sigma+\tau).\end{aligned}$$

In a similar manner it follows from (6) and (7) that if $\sigma'$ is the greatest of the $\sigma_j'=\sum|b_{jr}|=\sum|b_{rj}|$, and if $\sigma''$ is the greatest of the $\sigma_j''=\sum|c_{jr}|=\sum|c_{rj}|$, then

$$|\alpha|\leqq\sigma',\qquad |\beta|\leqq\sigma''.$$

It is clear that these limits for $|\lambda|$, $|\alpha|$, $|\beta|$ are never greater, and are in general less, than those given by Hirsch.

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* Browne, The characteristic roots of a matrix, Bulletin of the American Mathematical Society, vol. 36, 1930, pp. 705–710.

† Parker, The characteristic roots of a matrix, Duke Mathematical Journal, vol. 3, 1937, pp. 484–487.

THEOREM VI. *If* $\lambda=\alpha+i\beta$ *is a characteristic root of an n-square matrix* $A$, *and if we denote by* $\sigma$, $\tau$, $\sigma'$, $\sigma''$ *respectively, the greatest of the*

$$\sigma_j = \sum_s |a_{js}|, \qquad \tau_j = \sum_r |a_{rj}|,$$

$$\sigma_j' = \sum_r \tfrac{1}{2}|a_{jr}+\bar{a}_{rj}|, \qquad \sigma_j'' = \sum_r \tfrac{1}{2}|a_{rj}-\bar{a}_{jr}|,$$

*then*

$$|\lambda| \leqq \tfrac{1}{2}(\sigma+\tau), \qquad |\alpha| \leqq \sigma', \qquad |\beta| \leqq \sigma''.$$

**8. Limits in terms of the matrix** $A\bar{A}'$. The limits to the characteristic roots of a matrix $A$ which we have given up to this point have been either in terms of the elements of the matrix itself or in terms of the characteristic roots of the auxiliary matrices $B=\frac{1}{2}(A+\bar{A}')$ and $C=(A-\bar{A}')/(2i)$. We shall now give a limit in terms of the characteristic roots of the positive definite matrix $A\bar{A}'$, which in some cases, notably in the case of a *unitary* matrix, or of a *real orthogonal* matrix, gives limits which are more restricted than those given earlier.

The matrix $A\bar{A}'=M$ is clearly Hermitian; i.e., $\bar{M}'=M$. Moreover, its characteristic roots are all $\geqq 0$ and are the same as the characteristic roots of $\bar{A}'A$.* Suppose then that $\lambda$ is a characteristic root of $A$, so that there exists a set $(x)\neq(0)$ such that

$$\sum_t a_{ts}x_t = \lambda x_s, \qquad (s=1,\cdots,n).$$

Taking conjugates of both members, we have

$$\sum_r \bar{a}_{rs}\bar{x}_r = \bar{\lambda}\bar{x}_s.$$

Multiplying these two equations, member for member, and summing as to $s$, we have

$$\sum_{r,t}^{1,\cdots,n}\left(\sum_s a_{ts}\bar{a}_{rs}\right)x_t\bar{x}_r = \lambda\bar{\lambda}\sum x_s\bar{x}_s.$$

Since the matrix of the Hermitian form on the left is $A\bar{A}'$, it follows from the lemma of section 4 that $\lambda\bar{\lambda}$ lies between the greatest and the least of the characteristic roots of $A\bar{A}'$. We have then the following theorem:

THEOREM VII. *If* $\lambda$ *is a characteristic root of a matrix* $A$, *and if* $R_1\geqq R_2\geqq\cdots\geqq R_n$ *are the characteristic roots (all* $\geqq 0$) *of* $A\bar{A}'$, *then*

$$R_1 \geqq \lambda\bar{\lambda} \geqq R_n.$$

By this theorem the characteristic roots of a matrix are restricted to an annular region around the origin in the complex plane, with inner and outer radii $R_n^{1/2}$ and $R_1^{1/2}$, respectively. This region of course enlarges to a circle if $A$ is singular.

* Browne, The characteristic equation of a matrix, Bulletin of the American Mathematical Society, vol. 34, 1928, pp. 363–368.

In the particular case in which $A$ is unitary, or real orthogonal, we have $A\bar{A}'=I$, so that $R_1=R_n=1$. In this case the annular region shrinks to the periphery of a circle of radius unity. As a corollary to Theorem VII we therefore have the well known result:

COROLLARY. *The characteristic roots of a unitary matrix are of modulus unity.*

**9. Alternate proofs of Theorems II(c), III and VII.** The proofs of Theorems II (c), III and VII which we have given were all based on Lemma I concerning an Hermitian matrix. Alternate proofs of these theorems can be given which are so simple as to justify their inclusion here. These proofs are based on the following lemma:*

LEMMA III. *If $R_1 \geqq R_2 \geqq \cdots \geqq R_n$ are the characteristic roots of the Hermitian matrix $H=(h_{rs})$, then*

$$R_1 \geqq h_{rr} \geqq R_n;$$

*that is, the matrix has at least one characteristic root which is not less than the greatest diagonal element, and at least one characteristic root which is not greater than the least diagonal element.*

The proof of this lemma is based on the well known fact that every diagonal element of a positive definite Hermitian matrix is $\geqq 0$.

First, suppose that $h_{kk}>R_1$. If then $\sigma$ is a real number such that $h_{kk}>\sigma>R_1$, the form

$$\sigma \sum \bar{x}_r x_r - \sum h_{rs}\bar{x}_r x_s$$

is positive definite. This follows directly from the argument employed in the proof of Lemma I. But at least one diagonal element $\sigma - h_{kk}$ is negative, which is impossible.

Similarly, if $h_{kk}<R_n$ and if $h_{kk}<\sigma<R_n$, the form

$$\sum h_{rs}\bar{x}_r x_s - \sigma \sum \bar{x}_r, x_r$$

is positive definite, while at least one diagonal element in its matrix is $<0$.

Now, to apply this lemma to the proof of our theorems, we choose the unitary matrix $U$ such that $\bar{U}'AU=T$, where $T$ is triangular as in section 5. We then have as in (14)

$$\bar{U}'\left(\frac{A+\bar{A}'}{2}\right)U = \frac{T+\bar{T}'}{2}; \tag{14$'$}$$

$$\bar{U}'\left(\frac{A-\bar{A}'}{2i}\right)U = \frac{T-\bar{T}'}{2i}. \tag{14$''$}$$

The matrix $(T+\bar{T}')/2$ is Hermitian with diagonal elements $\alpha_1, \cdots, \alpha_n$ where the notation is that of section **5**. If then $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic

* *Cf.* Parker, Duke Mathematical Journal, *op. cit.*, Theorem 2, p. 487.

roots of $(T+\bar{T}')/2$, and therefore of $(A+\bar{A}')/2$, we have by the lemma:

$$\rho_1 \geqq \alpha_r \geqq \rho_n, \qquad (r = 1, 2, \cdots, n).$$

Similarly, the diagonal elements of $(T-\bar{T}')/2i$ are $\beta_1, \cdots, \beta_n$. If the characteristic roots in this case are $\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_n$, then

$$\mu_1 \geqq \beta_r \geqq \mu_n.$$

Thus we have established Theorems II (c) and III.

Moreover from $\bar{U}'A\bar{A}'U = T\bar{T}'$, we have, since the element in the first row and the first column of $T\bar{T}'$ is $\lambda_1\bar{\lambda}_1$,

$$R_1 \geqq \lambda_1\bar{\lambda}_1 \geqq R_n,$$

where $R_1 \geqq R_2 \geqq \cdots \geqq R_n \geqq 0$ are the characteristic roots of $A\bar{A}'$. Since in the triangular form $T$ any one of the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ may be taken as $\lambda_1$, this furnishes an alternate proof of Theorem VII.

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## ANALYTIC TREATMENT OF SOME ORTHOPOLE THEOREMS

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**1. Introduction.** In recent years a considerable number of writers have made effective use of complex coördinates in proving geometrical theorems. Last year J. R. Musselman published in this MONTHLY* an article using this method in treating the orthopole of a line with respect to a triangle.† It is the object of the present paper to extend the discussion concerning the orthopole by this analytic method, particularly to give results obtained by other methods by Neuberg, Lemoyne, Lienard, and Gallatly.

We may first note that F. D. Murnaghan gave a treatment of the orthopole theorem and related properties by analytic methods in *Mathesis* in 1927 (pp. 24–28), and that R. Bouvaist gave extensions to the isopole in the same volume (pp. 97–101; 211–217; 355–359) by the same methods.

**2. Orthopoles with respect to parallel lines.** Let $A_1A_2A_3$ be a triangle inscribed in the base-circle $\Gamma$, having as center $O$, the *turns*‡ corresponding to the vertices being $t_1$, $t_2$, $t_3$ and the unit-point $\Omega$ corresponding to the turn 1; the symmetric functions of $t_1$, $t_2$, $t_3$ are

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* On the line of images, vol. 45, 1938, pp. 421–430. For an extended treatment of orthopole-geometry, see my paper in the Tôhoku Mathematical Journal, 1926–27, pp. 77–125.

† The orthopole of a line $\Delta$ with respect to a triangle $A_1A_2A_3$ may be described as follows. Let $B_i$ be the foot of the perpendicular from $A_i$ to $\Delta$. Through $B_i$ draw a line perpendicular to the side opposite $A_i$; the three lines thus obtained by taking $i=1, 2, 3$ meet in a point called the orthopole of $\Delta$ with respect to $A_1A_2A_3$.

‡ Complex numbers whose moduli are unity, see Frank and F. V. Morley, Inversive Geometry, 1933, p. 15.

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

When the equation of any straight line is written in its self-conjugate form

$$x/a + \bar{x}/\bar{a} = 1,$$

where $\bar{x}$ is the conjugate of $x$, and $\bar{a}$ the conjugate of $a$, the directed distance from the point $y$ to that line is

$$|a|(1 - y/a - \bar{y}/\bar{a})/2.$$

Let then $\Delta$ be a straight line perpendicular to $O\Omega$, the angle $(O\Omega, \Delta)$ being a right angle in the positive sense.

If $2\delta$ is the directed distance from $O$ to $\Delta$, the equation of $\Delta$ is

$$x + \bar{x} = 2\delta, \qquad (\delta \text{ real}).$$

The projection of $A_1$ on $\Delta$ is the point

$$\delta + (t_1^2 - 1)/2t_1,$$

and the perpendicular from this point on $A_2A_3$ is

$$2(xt_1 - \bar{x}s_3) = 2\delta t_1(1 - t_2t_3) + (t_1^2 - 1)(1 + t_2t_3).$$

This line and the two others obtained by starting with $A_2$ and $A_3$ instead of $A_1$ meet at the point

$$\omega = \delta + (s_1 + s_3)/2,$$

the orthopole of $\Delta$ as to the triangle $A_1A_2A_3$.

The directed distance from $\omega$ to $\Delta$ is

$$d = -(s_1 + s_3 + \bar{s}_1 + \bar{s}_3)/4.$$

*For all parallel lines the distance from these lines to their respective orthopoles is constant.*

Since $\bar{\omega} = \delta + (s_2 + 1)/2s_3$,

$$\omega - \bar{\omega} = (s_1s_3 + s_3^2 - s_2 - 1)/2s_3.$$

Hence we have a theorem due to Soons,* *the orthopoles of a system of parallel lines lie on a line perpendicular to the system.*

**3. The orthopole and Simson lines.** The Simson line of a point $\tau$ on $\Gamma$ is

$$x\tau - \bar{x}s_3 = (\tau^3 + s_1\tau^2 - s_2\tau - s_3)/2\tau,$$

and, since

$$\bar{\omega} = \delta + (s_2 + 1)/2s_3,$$

the condition for $\omega$ to be on the Simson line of $\tau$ may be easily transformed into

* *Mathesis*, 1896, p. 57.

$$(\tau - s_3)(\tau^2 - 2\delta\tau + 1) = 0.$$

When $\tau = s_3$, the Simson line is perpendicular to $\Delta$; the second factor shows that $\omega$ is at the intersection of the Simson lines of the points $\tau$ and $\tau'$ such that

$$\tau + \tau' = 2\delta, \qquad \tau\tau' = 1,$$

*i.e.*, at the intersection of the Simson lines of the points where $\Delta$ cuts $\Gamma$.

*The orthopole $\omega$ is at the intersection of the Simson line perpendicular to $\Delta$ with the Simson lines of the intersections of $\Delta$ with the circumcircle.*

4. **The power of the orthopole.** Call $\alpha$ a point on $\Delta$ and $\kappa$ the power of $\alpha$ with respect to $\Gamma$. The counter-point $\beta$ to $\alpha$ in the triangle $A_1A_2A_3$ is then such that*

$$-(1 - \bar{\alpha}\beta)\kappa = 1 - s_1\bar{\alpha} + s_2\bar{\alpha}^2 - s_3\bar{\alpha}^3,$$

and the center $Q$ of the common pedal circle $\gamma$ of $\alpha$ and $\beta$ is

$$-[s_1 - (s_2 + \alpha^2)\bar{\alpha} + s_3\bar{\alpha}^2]/2\kappa.$$

Hence, as $\alpha + \bar{\alpha} = 2\delta$,

$$\begin{aligned} 4s_3\kappa^2\overline{Q\omega}^2 &= [-s_1\alpha\bar{\alpha} + s_2\bar{\alpha} - s_3(\kappa + \bar{\alpha}^2) - 2\delta\kappa + \alpha^2\bar{\alpha}] \\ &\times [s_1\alpha - s_2\alpha\bar{\alpha} - s_3(2\delta\kappa - \alpha\bar{\alpha}^2) - \kappa - \alpha^2]. \end{aligned}$$

But the radius $\rho$ of $\gamma$ is given by†

$$4\rho^2 = (1 - \bar{\alpha}\beta)(1 - \alpha\bar{\beta})$$

or‡

$$4s_3\kappa^2\rho^2 = (1 - s_1\bar{\alpha} + s_2\bar{\alpha}^2 - s_3\bar{\alpha}^3)(s_3 - s_2\alpha + s_1\alpha^2 - \alpha^3).$$

Hence

$$\overline{Q\omega}^2 - \rho^2 = \delta(s_1s_3 + s_2 + s_3^2 + 1)/2s_3 = -2d\delta.$$

*The power of the orthopole $\omega$ for the pedal circle of any point on $\Delta$ is twice the product of the distance from $\omega$ to $\Delta$ by the distance from $\Delta$ to the circumcenter.*§

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* Frank and F. V. Morley, *loc. cit.*, p. 196; "counter-points" are also called "isogonal conjugate points" or "focal pairs."

† F. and F. V. Morley, *loc. cit.*, p. 198.

‡ The formula may also be written

$$4\kappa^2\rho^2 = \Pi(\alpha - t_i)(\bar{\alpha} - \bar{t}_i) = \lambda_1^2\lambda_2^2\lambda_3^2,$$

$\lambda_1$, $\lambda_2$, $\lambda_3$ being the distances from $\alpha$ to $A_1$, $A_2$, $A_3$ and we obtain a theorem given by Neuberg in Triangles et coniques combinés, Nouvelles Annales de Mathématiques, 1870, p. 65.

§ Gallatly, The Modern Geometry of the Triangle, 2nd ed., p. 51.

**5. Other theorems.** Let us determine the reciprocal transversal to

$$x + \bar{x} = h,$$

a parallel $\Delta'$ to $\Delta$. The intersection with $A_2A_3$ being

$$(t_2 + t_3 - ht_2t_3)/(1 - t_2t_3)$$

the image of this intersection in the mid-point of $A_2A_3$ is

$$\xi_1 = t_2t_3(t_2 + t_3 - h)/(t_2t_3 - 1),$$

and the straight line $\Delta_0'$ (reciprocal to $\Delta'$) passing through $\xi_1$ and the two similar points is

$$x(\bar{s}_1 - \bar{s}_3 - h) + \bar{x}(s_1 - s_3 - h) = s_1\bar{s}_1 - h(s_1 + \bar{s}_1) + h^2 - 1.$$

When $h = 2\delta$, *i.e.*, when $\Delta' \equiv \Delta$, the equation of $\Delta_0'$ is

$$x(\bar{s}_1 - \bar{s}_3 - 2\delta) + \bar{x}(s_1 - s_3 - 2\delta) = s_1\bar{s}_1 - 2\delta(s_1 + \bar{s}_1) + 4\delta^2 - 1,$$

a straight line perpendicular to

$$x(\bar{s}_1 - \bar{s}_3 - 2\delta) - \bar{x}(s_1 - s_3 - 2\delta) = s_2 - \bar{s}_2 - 2\delta(s_1 - \bar{s}_1),$$

the join of the orthocenter $s_1$ to the orthopole $\omega$.

*The orthopole of a line $\Delta$ lies on the perpendicular from the orthocenter on the reciprocal line to $\Delta$.**

When $h = -2\delta$ the equation of $\Delta_0'$ becomes

$$x(\bar{s}_1 - \bar{s}_3 + 2\delta) + \bar{x}(s_1 - s_3 + 2\delta) = s_1\bar{s}_1 + 2\delta(s_1 + \bar{s}_1) + 4\delta^2 - 1,$$

and $\Delta_0'$ passes through $\omega$.

*The orthopole of a line $\Delta$ lies on the reciprocal to the image of $\Delta$ in the circumcenter.*

When $\Delta$ is a Simson line, $\tau = -s_3$ and $\omega$ is the point

$$\omega = (3s_1 + \bar{s}_1 + s_3 - \bar{s}_3)/4;$$

as, in this case

$$\bar{\omega} = (s_1 + 3\bar{s}_1 - s_3 + \bar{s}_3)/4,$$

$\omega$ is on the line

$$x + \bar{x} = s_1 + \bar{s}_1,$$

*i.e.*, the parallel to $\Delta$ through the orthocenter $s_1$.

As the orthopole lies also on the Simson line, perpendicular to $\Delta$, we have the following theorem:

* R. Goormaghtigh, Nouvelles Annales de Mathématiques, 1919, p. 39; Opperman, Le quadrilatère complet, 2nd ed., p. 69; de Lepiney, Mathesis, 1922, p. 314; Servais, Mathesis, 1923, p. 152.

*The orthopole of the Simson line of a point M on the circumcircle is the projection of the orthocenter on the Simson line of the image of M in the circumcenter.**

When $\delta=0$, $\Delta$ is a circumdiameter and

$$2\omega = s_1 + s_3,$$

a relation which shows a very simple geometric meaning of the symmetric function $s_3$: *the point $s_3$ is the image of the orthocenter in the orthopole of the circumdiameter perpendicular to the base line.*

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## A SPECIAL PAIR OF ISOTOMIC CONJUGATES†

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**1. Introduction.** With any triangle $A_1A_2A_3$ there are associated numerous special points possessing interesting properties. In this paper we shall discuss two points $\alpha$ and $\alpha'$, defined as follows:

$$A_1\alpha_3 = A_2\alpha_1 = A_3\alpha_2, \qquad A_1\alpha_2' = A_2\alpha_3' = A_3\alpha_1',$$

where $\alpha_i$ ($\alpha_i'$) is the point of intersection of $A_i\alpha$ ($A_i\alpha'$) and the side opposite $A_i$.‡

**2. Elementary Theorems.**

THEOREM 1. *In any triangle there is one and only one pair of points $\alpha$ and $\alpha'$.*

*Proof.* Let

$$A_1\alpha_3 = A_2\alpha_1 = A_3\alpha_2 = x.$$

Then, by Ceva's Theorem,

$$x^3 = (a_1 - x)(a_2 - x)(a_3 - x) = S_3 - S_2x + S_1x^2 - x^3,$$

where

$$a_1 = A_2A_3, \qquad a_2 = A_3A_1, \qquad a_3 = A_1A_2.$$

$$S_1 = a_1 + a_2 + a_3, \qquad S_2 = a_1a_2 + a_2a_3 + a_3a_1, \qquad S_3 = a_1a_2a_3.$$

We see that

$$2x^3 - S_1x^2 + S_2x - S_3 = 0.$$

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* R. Goormaghtigh, Mathesis, 1914, p. 152. This is also a special case of one of the theorems in section 5, as any Simson line is perpendicular to its reciprocal.

† The theorems in this paper were developed in a course in Modern Geometry given by Prof H. F. MacNeish at Brooklyn College.

‡ R. A. Johnson, Modern Geometry, p. 157.

A somewhat similar situation is discussed by R. Goormaghtigh in the article entitled "Sur deux points du plan d'un triangle et sur une généralization des points de Brocard"; Nouvelles Annales de Mathématique, vol. 18 (1918), pp. 417–424. He discusses two points $B$ and $B'$, whose projections on the sides $A_2A_3$, $A_3A_1$, $A_1A_2$ of triangle $A_1A_2A_3$ are $B_1$, $B_2$, $B_3$ and $B_1'$, $B_2'$, $B_3'$, respectively, where $A_1B_3=B_3'A_2=A_2B_1=B_1'A_3=A_3B_2=B_2'A_1$.

If the roots of this equation be decreased by $S_1/6$, the resultant equation is

$$x^3 + (6S_2 - S_1^2)x/12 + (9S_1S_2 - S_1^3 - 54S_3)/108 = 0.$$

If the discriminant, $\Delta$, of this equation is negative, then there exists one and only one real root.* The equation $x^3+px+q=0$ has for its discriminant, $\Delta \equiv -4p^3-27q^2$. Therefore, if $p>0$, then $\Delta<0$. It remains to be shown that, $6S_2-S_1^2>0$.

In a triangle having sides $a_1, a_2, a_3$,

$$a_1^2 - 2a_1a_2 + a_2^2 < a_3^2,$$
$$a_2^2 - 2a_2a_3 + a_3^2 < a_1^2,$$
$$a_3^2 - 2a_3a_1 + a_1^2 < a_2^2.$$

Adding, we have

$$a_1^2 + a_2^2 + a_3^2 - 2S_2 < 0.$$

Using

$$a_1^2 + a_2^2 + a_3^2 = S_1^2 - 2S_2,$$

we obtain,

$$6S_2 - S_1^2 > 4S_2 - S_1^2 > 0.$$

To show that $\alpha$ lies within the triangle, we need only prove that the real root is positive and less than the shortest side of the triangle.

Let

$$f(x) = 2x^3 - S_1x^2 + S_2x - S_3 = x^3 + (x - a_1)(x - a_2)(x - a_3),$$

and let $a_k$ be the minimum of $a_1, a_2, a_3$. Since $f(x)<0$ when $x=0$, and $f(x)>0$ when $x=a_k$, it is evident that $\alpha$ lies within the triangle.

If we let

$$A_1\alpha_2' = A_2\alpha_3' = A_3\alpha_1' = y,$$

the resultant equation,

$$2y^3 - S_1y^2 + S_2y - S_3 = 0,$$

makes it obvious that $\alpha'$ also lies within the triangle, and the theorem is established.

THEOREM 2. *The points $\alpha$ and $\alpha'$ are isotomic conjugates.*

*Proof.* The two equations,

$$2x^3 - S_1x^2 + S_2x - S_3 = 0,$$
$$2y^3 - S_1y^2 + S_2y - S_3 = 0,$$

are identical, so that $x=y$, and $\alpha$ and $\alpha'$ are isotomic conjugates.†

* L. E. Dickson, Elementary Theory of Equations, p. 34.

† A simple rational case arises for the triangle with sides 9, 14, 15, for which $x=y=6$.

THEOREM 3. *The locus of a point $P$ such that $A_1P_3=A_2P_1$ is a conic, where $P_i$ is the intersection of $A_iP$ and $A_jA_k$, $i\neq j\neq k\neq i$.*

*Proof.* Using trilinear coördinates, with triangle $A_1A_2A_3$ as the triangle of reference, and with the center of the inscribed circle as the unit point, we see that the line $A_1P$ passes through the points $A_1(1, 0, 0)$ and $P_1(0, [a_1-c]a_3, ca_2)$, and the equation of $A_1P$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ 0 & (a_1-c)a_3 & ca_2 \end{vmatrix} = 0, \quad \text{where} \quad A_1P_3 = A_2P_1 = c,$$

or

$$a_2x_2c + a_3x_3c - a_1a_3x_3 = 0.$$

Similarly, the equation of $A_3P$ is found to be

$$a_1x_1c + a_2x_2c - a_3a_2x_2 = 0.$$

Eliminating $c$, we find that the locus of the point $P$ is the conic

$$a_1x_3(a_1x_1 + a_2x_2) = a_2x_2(a_2x_2 + a_3x_3).$$

Similarly, the locus of a point $Q$, such that $A_2Q_1=A_3Q_2$, where $Q_i$ is the intersection of $A_iQ$ and $A_jA_k$, $i\neq j$, $j\neq k$, $k\neq i$, is found to be the conic

$$a_2x_1(a_2x_2 + a_3x_3) = a_3x_3(a_3x_3 + a_1x_1);$$

also, the locus of a point $R$, determined similarly, where $A_3R_2=A_1R_3$, is the conic

$$a_3x_2(a_3x_3 + a_1x_1) = a_1x_1(a_1x_1 + a_2x_2).$$

THEOREM 4. *The conic which is the locus of $P$ passes through $A_3$ and is tangent to the line $x_3=0$ at $A_1$.*

This is evident by direct substitution in the equation for the locus of $P$.

Similarly, the locus of $Q$ passes through $A_1$ and is tangent to $x_1=0$ at $A_2$; also, the locus of $R$ passes through $A_2$ and is tangent to $x_2=0$ at $A_3$.

THEOREM 5. *The three conics which are the loci of $P$, $Q$, and $R$, are concurrent at $\alpha$.*

Since, by definition,

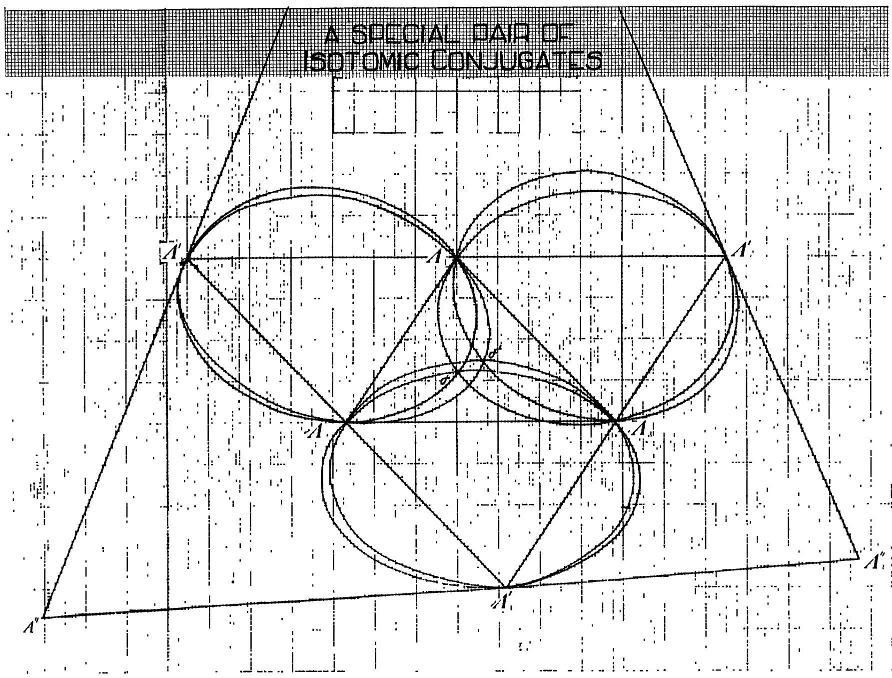
$$A_1\alpha_3 = A_2\alpha_1 = A_3\alpha_2,$$

$\alpha$ must be common to the loci of $P$, $Q$, and $R$.

If we consider the loci of $P'$, $Q'$, and $R'$, such that

$$A_2P_3' = A_3P_1', \qquad A_3Q_1' = A_1Q_2', \qquad A_1R_2' = A_2R_3',$$

respectively, then we arrive at three more conics which will have the point $\alpha'$ in common. The loci of $P'$, $Q'$, and $R'$ pass through the points $A_1$, $A_2$, and $A_3$, respectively, and are tangent to the lines $x_1=0$ at $A_3$, $x_2=0$ at $A_1$, and $x_3=0$ at $A_2$, respectively.




THEOREM 6. *At least two of the three conics which are the loci of $P$, $Q$, and $R$, ($P'$, $Q'$, and $R'$) are ellipses.*

To determine what type of conic the locus of $P$ is, we may solve simultaneously its equation,

$$a_1x_3(a_1x_1 + a_2x_2) = a_2x_2(a_2x_2 + a_3x_3),$$

with the equation of the line at infinity,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

We see that* if $a_3 < 4a_1$, the locus of $P$ is an ellipse;
if $a_3 = 4a_1$, the locus of $P$ is a parabola;
if $a_3 > 4a_1$, the locus of $P$ is a hyperbola.

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* S. L. Loney, Elements of Coördinate Geometry, Part II, p. 64.

In like manner, if $a_1<4a_2$, the locus of $Q$ is an ellipse, and if $a_2<4a_3$, the locus of $R$ is an ellipse.

It is evident that at most one of the three relations, $a_1\geqq 4a_2$, $a_2\geqq 4a_3$, $a_3\geqq 4a_1$, can hold.

Similarly, at least two of the three conics which are the loci of $P'$, $Q'$, and $R'$ are ellipses.

THEOREM 7. *The loci of $P$, $Q$, and $R$ are tangent to the loci of $P'$, $Q'$, and $R'$, respectively, at the points* $A_2'(1/a_1,\ -1/a_2,\ 1/a_3)$, $A_3'(1/a_1,\ 1/a_2,\ -1/a_3)$, *and* $A_1'(-1/a_1,\ 1/a_2,\ 1/a_3)$, *respectively.*

The locus of $P$,

$$a_1x_3(a_1x_1 + a_2x_2) = a_2x_2(a_2x_2 + a_3x_3),$$

and the locus of $P'$,

$$a_3x_1(a_3x_3 + a_2x_2) = a_2x_2(a_2x_2 + a_1x_1),$$

both pass through the points $A_1$ and $A_3$. It can be shown that both conics are tangent to the line

$$a_1^2x_1 + a_2(a_1 + a_3)x_2 + a_3^2x_3 = 0, \quad \text{at} \quad A_2'(1/a_1,\ -1/a_2,\ 1/a_3).$$

Similarly, the tangent to the loci of $Q$ and $Q'$ at $A_3'(1/a_1,\ 1/a_2,\ -1/a_3)$ is the line

$$a_2^2x_2 + a_3(a_2 + a_1)x_3 + a_1^2x_1 = 0;$$

also, the tangent to the loci of $R$ and $R'$ at $A_1'(-1/a_1,\ 1/a_2,\ 1/a_3)$ is the line

$$a_3^2x_3 + a_1(a_3 + a_2)x_1 + a_2^2x_2 = 0.$$

The triangle formed by these three tangents is found to have the vertices $A_1''(-s/a_1,\ [s-a_3]/a_2,\ [s-a_2]/a_3)$, $A_2''([s-a_3]/a_1,\ -s/a_2,\ [s-a_1]/a_3)$, $A_3''([s-a_2]/a_1,\ [s-a_1]/a_2,\ -s/a_3)$, where $s=(a_1+a_2+a_3)/2$.

**3. Horn angles.** It is evident that the loci of $P$ and $P'$, $Q$ and $Q'$, $R$ and $R'$, form horn angles at $A_2'$, $A_3'$, $A_1'$, respectively. Using the ratios of the curvatures at the points of tangency as the measures of these horn angles, we find that these measures, which are projective invariants, are

$$K_P/K_{P'} = a_1/a_3, \qquad K_Q/K_{Q'} = a_2/a_1, \qquad K_R/K_{R'} = a_3/a_2.$$

It follows that

$$K_PK_QK_R = K_{P'}K_{Q'}K_{R'}.$$

**4. Perspective relations.** Further investigation discloses that $\alpha$ and $\alpha'$ are instrumental in producing several interesting perspectivities. It is found that triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ are both in perspective with triangle $A_1''A_2''A_3''$ and with each other.

THEOREM 8. *Triangle $A_1A_2A_3$ is in perspective with triangle $A_1''A_2''A_3''$ with $G$, the Gergonne* point, as the center of perspective.*

If $I_1$ is the point of tangency of the inscribed circle and $A_2A_3$, then

$$\frac{\Delta A_1A_2I_1}{\Delta A_1A_3I_1}=\frac{s-a_2}{s-a_3}=\frac{\Delta GA_2I_1}{\Delta GA_3I_1}=\frac{\Delta A_1A_2I_1-\Delta GA_2I_1}{\Delta A_1A_3I_1-\Delta GA_3I_1}=\frac{\Delta A_1A_2G}{\Delta A_1A_3G}=\frac{a_3x_3}{a_2x_2}=\frac{s-a_2}{s-a_3},$$

where $G$, the Gergonne point, has the coördinates $(x_1, x_2, x_3)$. Hence

$$x_2:x_3=\frac{1}{a_2(s-a_2)}:\frac{1}{a_3(s-a_3)};$$

and similarly,

$$x_1:x_2:x_3:=\frac{1}{a_1(s-a_1)}:\frac{1}{a_2(s-a_2)}:\frac{1}{a_3(s-a_3)},$$

so that the Gergonne point may be represented as $G(1/a_1[s-a_1], 1/a_2[s-a_2], 1/a_3[s-a_3])$.

To show that $A_1''$, $A_1$, and $G$ are collinear we need only prove that

$$\begin{vmatrix} 1 & 0 & 0 \\ -s/a_1 & (s-a_3)/a_2 & (s-a_2)/a_3 \\ 1/a_1(s-a_1) & 1/a_2(s-a_2) & 1/a_3(s-a_3) \end{vmatrix}=0.$$

In like manner it can be shown that $G$ lies on $A_2''A_2$ and $A_3''A_3$ and is therefore the center of perspective of triangles $A_1''A_2''A_3''$ and $A_1A_2A_3$. The axis of perspective is the line

$$a_1^2x_1+a_2^2x_2+a_3^2x_3=0.$$

THEOREM 9. *Triangle $A_1'A_2'A_3'$ is in perspective with triangle $A_1''A_2''A_3''$ with $N$, the Nagel† point, as the center of perspective.*

The proof is similar to the proof of Theorem 8, with

$$\frac{\Delta A_1A_2N_1}{\Delta A_1A_3N_1}=\frac{s-a_3}{s-a_2}=\frac{a_3x_3}{a_2x_2},$$

where $N_1$ is the point of contact of the escribed circle on $A_2A_3$, and the coördinates of $N$ are found to be $x_i=(s-a_i)/a_i$, $(i=1, 2, 3)$.

The axis of perspective is the line

$$a_1^2(a_2+a_3)x_1+a_2^2(a_3+a_1)x_2+a_3^2(a_1+a_2)x_3=0.$$

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* The Gergonne point is defined as the intersection of the lines joining each vertex of a triangle to the point of contact of the inscribed circle on the opposite side.

† The Nagel point is defined as the intersection of the lines joining each vertex of a triangle to the point of contact of the respective escribed circle on the opposite side.

THEOREM 10. *Triangle $A_1A_2A_3$ is in perspective with triangle $A_1'A_2'A_3'$ with $M$, the Median point, as the center of perspective.*

The proof is similar to the proof of Theorem 8, where the coördinates of $M$ are found to be $x_i = 1/a_i$, $(i=1, 2, 3)$.

The axis of perspective is the line at infinity

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

**5. Constructibility.** Since $A_1$, $A_2$, $A_3$, are given and $A_1'$, $A_2'$, $A_3'$, $M$, $N$, $G$, can be constructed with ruler and compass, it follows that we can find $A_1''$, $A_2''$, $A_3''$, with ruler and compass, for $A_iG$ and $A_i'N$ meet at $A_i''$, $(i=1, 2, 3)$. The conics are known and easily constructible pointwise.

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## GENERALIZATIONS ON PARTIAL FRACTIONS IN DETERMINANTAL FORMS

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1. **Arbitrary quadratic factors.** Professor Turnbull has shown* how partial fractions can be expressed in determinantal form for functions of the form $f(x)/Q(x)$ where $Q(x)$ consists of simple (non-repeated or repeated) factors $(x-a_r)$ and $f(x)$ is a polynomial of order lower than that of $Q(x)$. He suggested that the method could be extended to functions $Q(x)$ which included complex factors, and for arbitrary pairs of complex roots a straightforward generalization seems to be the simplest process. But if there are purely imaginary roots, that is, if $Q(x)$ has factors of the form $(x^2+a_r^2)$ a modified process leads to simpler results. Such a modification is suggested here.

To recall Turnbull's method we shall first describe briefly, by means of an example, the method to be used in the case of arbitrary quadratic factors. For simplicity we shall assume that $Q(x)$ has the form

$$Q(x) = (x^2 + a_1x + b_1)(x^2 + a_2x + b_2),$$

and suppose that the roots of $x^2+a_1x+b_1=0$ and of $x^2+a_2x+b_2=0$ are $s_1$, $t_1$ and $s_2$, $t_2$ respectively. Then Turnbull's formula for the partial fractions may be written down immediately:

$$(1)\qquad \frac{f(x)}{Q(x)} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ s_1 & t_1 & s_2 & t_2 \\ s_1^2 & t_1^2 & s_2^2 & t_2^2 \\ \dfrac{f(s_1)}{x-s_1} & \dfrac{f(t_1)}{x-t_1} & \dfrac{f(s_2)}{x-s_2} & \dfrac{f(t_2)}{x-t_2} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & 1 & 1 \\ s_1 & t_1 & s_2 & t_2 \\ s_1^2 & t_1^2 & s_2^2 & t_2^2 \\ s_1^3 & t_1^3 & s_2^3 & t_2^3 \end{vmatrix}.$$

* H. W. Turnbull, Note on Partial Fractions and Determinants, Proceedings of the Edinburgh Mathematical Society, Series 2, vol. 1, pp. 49–54, May, 1927.

Elementary transformations of the determinants enable us to write the ratio in the form

$$(2)\qquad \frac{f(x)}{Q(x)} = \frac{1}{\Delta}\begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & H_1^{(1)} & 1 & H_2^{(1)} \\ H_1^{(1)} & H_1^{(2)} & H_2^{(1)} & H_2^{(2)} \\ \dfrac{xF_1 - G_1}{x^2 + a_1x + b_1} & \dfrac{xK_1 - b_1F_1}{x^2 + a_1x + b_1} & \dfrac{xF_2 - G_2}{x^2 + a_2x + b_2} & \dfrac{xK_2 - b_2F_2}{x^2 + a_2x + b_2} \end{vmatrix},$$

where

$$F_k = \frac{f(s_k) - f(t_k)}{s_k - t_k}, \qquad G_k = \frac{t_kf(s_k) - s_kf(t_k)}{s_k - t_k}, \qquad K_k = \frac{s_kf(s_k) - t_kf(t_k)}{s_k - t_k},$$

and the quantities $H_k^{(r)}$ denote the elementary homogeneous expressions in $s_k$ and $t_k$ of the $r$th degree. Clearly these can be expressed in terms of the original coefficients of $Q$:

$$H_k^{(1)} = -a_k; \quad H_k^{(2)} = a_k^2 - b_k; \quad H_k^{(3)} = -a_k(a_k^2 - 2b_k); \text{ etc.}$$

The quantities $F_k$, $G_k$, $K_k$ can also be expressed in terms of the original coefficients of $f$ and $Q$.

The denominator $\Delta$ has the form

$$\Delta = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & H_1^{(1)} & 1 & H_2^{(1)} \\ H_1^{(1)} & H_1^{(2)} & H_2^{(1)} & H_2^{(2)} \\ H_1^{(2)} & H_1^{(3)} & H_2^{(2)} & H_2^{(3)} \end{vmatrix},$$

and its value is found to be

$$\Delta = (b_1 - b_2)^2 + (a_1 - a_2)(a_1b_2 - a_2b_1).$$

The form of the ratio $f/Q$ in (2) can obviously be extended to the case of any number of quadratic factors in $Q$, and it can also be extended to cover the case of repeated factors.

**2. Factors of the form $x^2+a^2$.** When $Q(x)$ has purely imaginary roots, *i.e.*,

$$Q(x) = (x^2 + a_1^2)^{r_1}(x^2 + a_2^2)^{r_2} \cdots (x^2 + a_n^2)^{r_n},$$

we may write $x^2 = y$ and express $f(x)$ as

$$f(x) = g(x^2) + xh(x^2) = g(y) + xh(y).$$

To each part of $f$ in this form the method for simple roots may be applied and

the partial fraction formula obtained. This is not possible, however, if $Q(x)$ also includes simple factors such as $(x-c)$. In this case the following method may be adopted.

Assume that

$$Q(x) = (x^2 + a_1^2) \cdots (x^2 + a_m^2)(x - \alpha_{m+1}) \cdots (x - \alpha_n).$$

Then we may write

$$\frac{f(x)}{Q(x)} = \frac{(x + \alpha_{m+1}) \cdots (x + \alpha_n) f(x)}{\prod_1^n (x^2 - \alpha_r^2)},$$

where

$$\alpha_r^2 = - a_r^2, \qquad r = 1, 2, \cdots, m.$$

The numerator may now be expressed in the form $F(x^2)+xG(x^2)$, and therefore

$$\frac{f(x)}{Q(x)} = \frac{F(y)}{\prod_1^n (y - \alpha_r^2)} + x \frac{G(y)}{\prod_1^n (y - \alpha_r^2)}.$$

Each of the terms on the right can be expressed as partial fractions in determinantal form, and combining the two parts thus obtained we get the formula

$$\frac{f(x)}{Q(x)} = \frac{1}{\Delta} \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \alpha_1^2 & \cdots & \alpha_m^2 & \alpha_{m+1}^2 & \cdots & \alpha_n^2 \\ \alpha_1^4 & \cdots & \alpha_m^4 & \alpha_{m+1}^4 & \cdots & \alpha_n^4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{2(n-2)} & \cdots & \alpha_m^{2(n-2)} & \alpha_{m+1}^{2(n-2)} & \cdots & \alpha_n^{2(n-2)} \\ \dfrac{F(\alpha_1^2) + xG(\alpha_1^2)}{x^2 - \alpha_1^2} & \cdots & \dfrac{F(\alpha_m^2) + xG(\alpha_m^2)}{x^2 - \alpha_m^2} & \dfrac{G(\alpha_{m+1}^2)}{x - \alpha_{m+1}} & \cdots & \dfrac{G(\alpha_n^2)}{x - \alpha_n} \end{vmatrix},$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \alpha_1^4 & \alpha_2^4 & \cdots & \alpha_n^4 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{2(n-1)} & \alpha_2^{2(n-1)} & \cdots & \alpha_n^{2(n-1)} \end{vmatrix} = \begin{aligned} (\alpha_2^2 - \alpha_1^2)(\alpha_3^2 - \alpha_1^2) \cdots (\alpha_n^2 - \alpha_1^2) \\ (\alpha_3^2 - \alpha_2^2) \cdots (\alpha_n^2 - \alpha_2^2) \\ \cdots\cdots\cdots \\ (\alpha_n^2 - \alpha_{n-1}^2). \end{aligned}$$

In the determinant in the numerator the element of the last row corresponding to $(x^2-\alpha_r^2)$ when $r>m$ has been written as $G(\alpha_r^2)/(x-\alpha_r)$. For, when $r>m$, $F(x^2)+xG(x^2)$ contains $x+\alpha_r$ as a factor, so that $F(\alpha_r^2)=\alpha_r G(\alpha_r^2)$ and $f(\alpha_r^2)+xG(\alpha_r^2)=(x+\alpha_r)G(\alpha_r^2)$.

Expansion of the determinant will lead to the partial fraction formula

$$\frac{f(x)}{Q(x)} = \sum_{r=1}^{m} \frac{F(-a_r^2) + xG(-a_r^2)}{A_r(x^2 + a_r^2)} + \sum_{r=m+1}^{n} \frac{G(\alpha_r^2)}{A_r(x - \alpha_r)},$$

where

$$A_r = (\alpha_r^2 - \alpha_1^2)(\alpha_r^2 - \alpha_2^2) \cdots (\alpha_r^2 - \alpha_{r-1}^2)(\alpha_r^2 - \alpha_{r+1}^2) \cdots (\alpha_r^2 - \alpha_n^2).$$

3. **Repeated factors.**. The case of repeated quadratic or linear factors can be dealt with in the same way as in Turnbull's original paper. An example will illustrate the process. Let

$$Q(x) = (x^2 + a_1^2)^2(x^2 + a_2^2)\,(x - b_1)^2(x - b_2)(x - b_3),$$

and write

$$a_1^2 = -\alpha_1, \qquad a_2^2 = -\alpha_2, \qquad b_1^2 = \alpha_3, \qquad b_2^2 = \alpha_4, \qquad b_3^2 = \alpha_5.$$

Then

$$\frac{f(x)}{Q(x)} = \begin{vmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \alpha_1 & 1 & \alpha_2 & \alpha_3 & 1 & \alpha_4 & \alpha_5 \\ \alpha_1^2 & 2\alpha_1 & \alpha_2^2 & \alpha_3^2 & 2\alpha_3 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & 3\alpha_1^2 & \alpha_2^3 & \alpha_3^3 & 3\alpha_3^2 & \alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & 4\alpha_1^3 & \alpha_2^4 & \alpha_3^4 & 4\alpha_3^3 & \alpha_4^4 & \alpha_5^4 \\ \alpha_1^5 & 5\alpha_1^4 & \alpha_2^5 & \alpha_3^5 & 5\alpha_3^4 & \alpha_4^5 & \alpha_5^5 \\ \dfrac{F+xG}{x^2-\alpha_1} & \left(\dfrac{F+xG}{x^2-\alpha_1}\right)' & \dfrac{F+xG}{x^2-\alpha_2} & \dfrac{G}{x-\sqrt{\alpha_3}} & \left(\dfrac{G}{x-\sqrt{\alpha_3}}\right)' & \dfrac{G}{x-\sqrt{\alpha_4}} & \dfrac{G}{x-\sqrt{\alpha_5}} \end{vmatrix}$$

$$\div \begin{vmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ \alpha_1 & 1 & \alpha_2 & \alpha_3 & 1 & \alpha_4 & \alpha_5 \\ \alpha_1^2 & 2\alpha_1 & \alpha_2^2 & \alpha_3^2 & 2\alpha_3 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & 3\alpha_1^2 & \alpha_2^3 & \alpha_3^3 & 3\alpha_3^2 & \alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & 4\alpha_1^3 & \alpha_2^4 & \alpha_3^4 & 4\alpha_3^3 & \alpha_4^4 & \alpha_5^4 \\ \alpha_1^5 & 5\alpha_1^4 & \alpha_2^5 & \alpha_3^5 & 5\alpha_3^4 & \alpha_4^5 & \alpha_5^5 \\ \alpha_1^6 & 6\alpha_1^5 & \alpha_2^6 & \alpha_3^6 & 6\alpha_3^5 & \alpha_4^6 & \alpha_5^6 \end{vmatrix},$$

where in the last row of the numerator the argument of $F$ and $G$ is always the particular $\alpha$ which appears in the denominator, and a prime denotes differentiation with respect to that $\alpha$. The value of the determinant in the denominator is

$$\Delta = (\alpha_2 - \alpha_1)^2(\alpha_3 - \alpha_1)^4(\alpha_4 - \alpha_1)^2(\alpha_5 - \alpha_1)^2(\alpha_3 - \alpha_2)^2(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2) \times (\alpha_4 - \alpha_3)^2(\alpha_5 - \alpha_3)^2(\alpha_5 - \alpha_4).$$

To find the coefficient of $(x^2+a_1^2)^{-2}$ we proceed as follows. By definition

$$\left(\frac{F(\alpha_1) + xG(\alpha_1)}{x^2 - \alpha_1}\right)' = \frac{F'(\alpha_1) + xG'(\alpha_1)}{x^2 - \alpha_1} + \frac{F(\alpha_1) + xG(\alpha_1)}{(x^2 - \alpha_1)^2}.$$

The cofactor of this term in the determinant in the numerator is

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & 1 & \alpha_4 & \alpha_5 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & 2\alpha_3 & \alpha_4^2 & \alpha_5^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & 3\alpha_3^2 & \alpha_4^3 & \alpha_5^3 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & 4\alpha_3^3 & \alpha_4^4 & \alpha_5^4 \\ \alpha_1^5 & \alpha_2^5 & \alpha_3^5 & 5\alpha_3^4 & \alpha_4^5 & \alpha_5^5 \end{vmatrix},$$

which has the value

$$(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)^2(\alpha_4 - \alpha_1)(\alpha_5 - \alpha_1)(\alpha_3 - \alpha_2)^2(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2)(\alpha_4 - \alpha_3)^2 \times (\alpha_5 - \alpha_3)^2(\alpha_5 - \alpha_4).$$

Thus the coefficient of $(x^2+a_1^2)^{-2}$ is

$$\frac{F(\alpha_1) + xG(\alpha_1)}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)^2(\alpha_4 - \alpha_1)(\alpha_5 - \alpha_1)}.$$

Evidently this method can be generalized to any combination of repeated factors.

# MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This department of the* MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

## THE DEADLY PARALLEL

A. J. KEMPNER, University of Colorado

Readers of the MONTHLY may be interested in two excerpts from the daily press:

I. (From the *Rocky Mountain News*, Denver, Friday, February 10, 1939). At the Annual Regional Study Conference of the Progressive Education Association, Denver, Colorado, Dr. Lois Hayden Meek, Professor of Education at Columbia University, is quoted as saying: "Intelligent supervision and encouragement of such group relationship at that time is of far greater importance in the future education of the youth than cut and dried history or mathematics.... It is my opinion we should not have so many specialists on our teaching staffs. ... A better method would be to have one teacher teach more subjects and have longer and more frequent contact with pupils, studying more carefully the individual problems of each. *The function of such a teacher would not hinge so much on knowing the subjects to be taught as on knowing how to arrange and present such subjects so they could be most readily learned.*" (Italics mine.)

II. (From the *Denver Post*, Saturday, February 11, 1939, reporting opinions of students at the same conference).

A high school senior from Northern Colorado: "Everything would be fine if we had good teachers. You've been talking a lot about dull subjects that should be dropped from the curriculum. Those subjects are dull because the teachers are dull. If all our teachers were alert and interesting, there would be no such thing as wasted courses. Even a course in Latin would be fun and would accomplish real education if the teacher presented it the right way."

A college student: "It's all a matter of the personality of the teacher. All of these proposed drastic changes in the method of education would be unnecessary if teachers generally had a real understanding of what young people want. We don't resist learning. As a matter of fact we're anxious to learn. All we ask is that there be some pleasure, some intellectual excitement in the process. . . . Every student knows there are some teachers in whose classes he would like to spend much more time than allotted. When the student fidgets and waits impatiently for the bell to ring, it is the fault of the teacher, not the fault of the subject."

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May it be that we have overlooked our best allies in our fight for a sound education based on fundamentals,—the students themselves?

# QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

*The department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

## A CORRECTION BY PROFESSOR CANDY

A. L. CANDY, University of Nebraska

For the benefit of those readers of the MONTHLY who have copies of my book *Magic Squares of an Even Order*, I wish to make the following correction: The last four Basic Squares on p. 91, *viz.*, Nos. 5, 6, 7, 8, should be stricken out, because they are duplicates, respectively, of Nos. 2, 1, 4, 3, on the same page, since they give the same "Groups of Four" when rows and columns are permuted so as to retain the same principal diagonals. This reduces the total number E. D. S. (p. 95) from 896 to 880, which agrees with the number obtained by D. H. Lehmer and others. Hence, wherever this number 896 appears elsewhere in this book, it should be changed to 880. See especially pp. 37, 174, and 175.

I am very glad to have this opportunity of making this public acknowledgement of my error.

## A CORRECTION BY PROFESSOR MUSSELMAN

J. R. MUSSELMAN, Western Reserve University

I wish to call attention to an error in my article, "On the line of images," this MONTHLY, vol. 45, pp. 421–430. On page 430, line 2 following should read: "of the Kiepert hyperbola and at the point whose line of images as to $A_1A_2A_3$ is the line joining the orthocenter and the symmedian point of $A_1A_2A_3$."

## A THEOREM ON GRAPHS, WITH AN APPLICATION TO A PROBLEM OF TRAFFIC CONTROL

H. E. ROBBINS, Institute for Advanced Study

It is the object of this note to answer a question* which is suggested by the problem of traffic control in a modern city: "When may the arcs of a graph be so oriented that one may pass from any vertex to any other, traversing arcs in the positive sense only?" Any graph with this property will be called orientable. The answer is provided by the following theorem:

THEOREM. *A graph is orientable if and only if it remains connected after the removal of any arc.*

Let us suppose that week-day traffic in our city is not particularly heavy, so that all streets are two-way, but that we wish to be able to repair any one street at a time and still detour traffic around it so that any point in the city may be reached from any other point. On week-ends no repairing is done, so

* Proposed by S. Ulam. For a simplification in the proof we are indebted to H. Whitney.

that all streets are available, but due to the heavy traffic (perhaps it is a college town with a noted football team) we wish to make all streets one-way and still be able to get from any point to any other without violating the law. Then the theorem states that if our street-system is suitable for week-day traffic it is also suitable for week-end traffic and conversely. We proceed to give a few definitions and a proof of the theorem.

By a graph $G$ we mean a (finite) set of objects $\langle p, q, \cdots \rangle$ called *vertices*, together with a (finite) set of objects called *arcs*, which join certain pairs of distinct vertices. We shall represent an arc joining $p$ and $q$ by $(pq)$, with the understanding that there may be more than one such arc. $G$ is *connected* if, given any two of its vertices $p$, $q$, there exists a chain of arcs joining $p$ and $q$ of the form

$$(pr_1)(r_1r_2) \cdots (r_sq). \tag{1}$$

$G$ is *oriented* if a direction is assigned to each arc, symbolized by choosing one of the two representations $(pq)$ and $(qp)$ of the unoriented arc, to be called the *admissible* representation. $G$ is *orientable* if it may be oriented in such a way that any two of its vertices may be joined by a chain (1) of arcs, each in its admissible representation. The definitions of *subgraph*, and of the special subgraph obtained from $G$ by removing one of its arcs, are obvious.

Now referring to the theorem, we see that the necessity of the condition is clear, for if $G$ is disconnected by the removal of an arc $(pq)$, then no matter which representation of $(pq)$ we call admissible in an orientation of $G$, passage either from $p$ to $q$ or from $q$ to $p$ by a chain of admissible arcs will be impossible, so that $G$ is not orientable. It remains to prove the sufficiency of the condition. Suppose $G$ remains connected after the removal of any arc. (Then *a fortiori* $G$ is connected.) Choose a vertex $p$ of $G$ and consider the class of all subgraphs of $G$ which include $p$ and which are orientable. This class is non-void, since the subgraph consisting of $p$ alone satisfies the conditions. Let $G'$ be a sub-graph in this class with maximal number of vertices. We may assume that $G'$ contains all arcs of $G$ whose vertices are in $G'$, since otherwise any such arcs which did not belong to $G'$ could be added, oriented arbitrarily, and $G'$ would remain orientable. Then $G'$ must be identical with $G$. For suppose $p^*$ is a vertex of $G$ not in $G'$. Join $p^*$ to $p$ by a chain

$$(p_1p_2)(p_2p_3) \cdots (p_{n-1}p_n), \qquad [p_1 = p^*;\ p_n = p], \tag{2}$$

and order the arcs of (2) from left to right. Let $(p_kp_{k+1})$ be the last arc of (2) which is not in $G'$. Then $p_k$ does not belong to $G'$, while $p_{k+1}$ does. By hypothesis, there exists a chain of arcs

$$(\bar{p}_1\bar{p}_2)(\bar{p}_2\bar{p}_3) \cdots (\bar{p}_{m-1}\bar{p}_m), \qquad [\bar{p}_1 = p_k;\ \bar{p}_m = p_{k+1}], \tag{3}$$

which does not include $(p_kp_{k+1})$. Let $(\bar{p}_s\bar{p}_{s+1})$ be the last arc of (3) which does not belong to $G'$. Then the subgraph $G^*$ consisting of $G'$ plus $(p_{k+1}p_k)$ and the chain

$$(\bar{p}_1\bar{p}_2)(\bar{p}_2\bar{p}_3) \cdots (\bar{p}_s\bar{p}_{s+1}), \tag{4}$$

where these arcs are to be oriented as written, is clearly orientable, contains $p$, and has more vertices than $G'$, which is a contradiction and completes the proof.

## EVEN PERMUTATIONS AS PRODUCTS OF CYCLES

LEONARD MILLER, Brooklyn, New York

THEOREM. *Every even permutation $P$ on $n$ distinct letters can be written as the product of an even number of cycles on these $n$ letters, each cycle being of order $m$, where $m=2, 3, \cdots$, or $n$.*

*Proof.* By definition, any even permutation can be written as the product of an even number of transpositions. It is also known that any even permutaton can be written as the product of cycles of three letters each. The theorem is therefore true for $m=2, 3$. We will show that any even permutation written as the product of an even number of cycles of $r$ letters each, where $2 \leqq r < n$, can be written as the product of the same number of cycles of $r+1$ letters each. The theorem will then follow by induction. Since the permutation group is of degree $n$, the theorem loses significance for $r \geqq n$.

Suppose that the permutation is written as the product of an even number of cycles each of order $r$. Let us group the cycles into pairs. Any given pair of cycles will fall into one of the following two types.

I. Pairs of cycles in which all of the letters of the first cycle are repeated in some order in the second.

II. Pairs of cycles in which at least one letter of the first cycle is not repeated in the second.

We will now show that every pair of cycles of type I and type II can be written as a pair of cycles of order $r+1$.

Take any pair of cycles of type I,

$$(a_1 a_2 \cdots a_r)(a_1' a_2' \cdots a_r'). \tag{1}$$

Since the letters of the first cycle are the same as the letters in the second, we may suppose that $a_1 = a_1'$. Since $r < n$, there is an $a_{r+1}$. If we form

$$(a_1 a_2 \cdots a_r a_{r+1})(a_1 a_{r+1} a_2' \cdots a_r'), \tag{2}$$

it is easy to verify that $(1) = (2)$.

Take any pair of cycles of type II,

$$(a_1 a_2 \cdots a_i \cdots a_r)(a_1' a_2' \cdots a_i' \cdots a_r'). \tag{3}$$

Then let $a_i$ be the last letter in the first cycle which is not found in the second, and let $a_j'$ be the last letter in the second which is not found in the first. If we form

$$(a_1 a_2 \cdots a_j' a_i \cdots a_r)(a_1' a_2' \cdots a_j' a_i \cdots a_r'), \tag{4}$$

it is easy to verify that $(3) = (4)$, and so the theorem follows.

## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Mathematical Adventures.* By Fletcher Durell, Boston, Bruce Humphries, 1939, 157 pages. $2.00.

*A Course in General Mathematics.* By Clinton Harvey Currier, Emery Ernest Watson, and James Sutherland Frama. Revised Edition. New York, Macmillan, 1939. 10+382 pages. $3.00.

*Your Chance to Win: The Laws of Chance and Probability.* By Horace C. Levinson. New York and Toronto, Farrar and Rinehart, 1938. 343 pages. $2.50.

*Lezioni di Analisi Matematica, Parte Prima.* By Franscesco Tricomi. Fourth Edition. Padova, Cedam, 1939. 8+328 pages. Lire 70.

### REVIEWS

*Formelsammlung zur praktischen Mathematik.* By Günther Schulz. (Sammlung Göschen, 1110.) Berlin, de Gruyter, 1937. 147 pp.

This little book of the Göschen series gives the principal methods and formulas of numerical mathematics. Its seven chapters cover aids in computation, adjustment of observations, interpolation, solution of numerical equations, numerical integration and summation, representation of functions by approximating formulas, and the numerical solution of differential equations. The inherent errors or remainder terms of the processes and formulas are given in many instances.

Illustrative examples are worked out under some of the more important topics. References to some of the literature (mostly German) are given at frequent intervals, and a longer list of references (some of little value) are given at the end of the book.

A valuable feature of the book is the section on the numerical solution of partial differential equations (12 pages). The equations treated are second-order equations of the elliptic, hyperbolic, and parabolic types.

J. B. SCARBOROUGH

*Medieval Number Symbolism.* Its Sources, Meaning and Influence on Thought and Expressions. By Vincent Foster Hopper. New York, Columbia University Press, 1938. 12+241 pp.

The author states in the "Preface" that "it is the purpose of this study to reveal how deeply rooted in medieval thought was the consciousness of numbers, not as mathematical tools, nor yet as the counters in a game, but as fundamental realities, alive with memories and eloquent with meaning."

The author presents at the outset the origins of medieval number philosophy in three phases which are designated "elementary," astrological, and Pythagorean. The first of these is concerned with certain groups of objects which we all possess such as 5 fingers, easily identified with "hand"; the second goes back to the science of astrology among the Babylonians, a science which involves numbers associated with the heavenly bodies, among which appears the 7, the number of the then known planets; the third "fixed the relationship of the numbers to one another," as in particular, three numbers which are the sides of a right triangle.

These introductory chapters are followed by penetrating studies of "The Gnostics," "The Early Christian Writers," "Medieval Number Philosophy." As these topics indicate, the work is wholly a study in the belief held for centuries that number is the first principle of all things. The illustration from Ramus is well known—that God perfected the world in 6 days because 6 is a perfect number, a case in which mathematical reasoning gave added justification for the number. "Medieval philosophers and theologians understood the purely mathematical aspects of number to be of divine origin." An outstanding development of this belief is found in the science (not exact) of gematria to which present day numerology is a distant kin.

The mark of this belief in the power of number is found to a certain degree in the sciences and pseudosciences of the Middle Ages although its chief outlet was theology. The purely scientific attitude today repudiates any such belief.

The work culminates in a chapter entitled "The Beauty of Order—Dante" and uses the phrase "Dante's sharp and studied usage of significant numbers." Here a commentary is offered by the author as a suggestive method rather than a final statement of at least one "gate and key" to the Dantesque vision of the universe. The reviewer is no judge of the value of these interpretations. To a specialist in the field of Dante's works, they hold fascinating possibilities.

An appendix is devoted to "Number Symbols of Northern Paganism."

The book is of interest to anyone in the mathematical as well as any other field to whom any study on concepts in his field widens his horizon of thought and knowledge.

LAO G. SIMONS

*Leitfaden der Stereometrie.* By W. Benz. Zürich and Leipzig, Orell Füssli. 1938. 215 pages, 122 figures. Fr. 3.80.

The present text is designed to follow a similar one for plane geometry written by Gonseth and Marti, published by the same firm. The elements of plane analytic geometry, harmonic properties, and plane affinity are presupposed known; sines and cosines are used.

The approach to the subject is very different from that familiar to American students. No exercises are given, and little emphasis is laid on mensuration. On the other hand, great stress is laid on construction; sufficient details are provided to enable the pupil to make any construction in the book.

The first chapter begins with the principles of skew perspective, without axioms, and with only a few definitions; intuition is freely drawn upon. This is followed by the concept of a plane, defined as the locus of points determined by a linear range of points and one other point. After having established the usual descriptive properties, perpendicularity is introduced. Parallels are first brought in on page 50. Motion plays an important part. Infinite elements are first introduced intuitively, but are firmly established later in the chapter.

The second chapter considers problems in drawing and plane representation—practically identical with the ordinary $h$, $v$ projections of descriptive geometry, applied to rectilinear figures. The third chapter applies these principles to bodies and surfaces of revolution; the fourth is similarly applied to solid angles. Prisms and cylinders, pyramids and cones are now treated rapidly, but when the little book is completed, the reader will have met all the properties ordinarily discussed in our texts, and also acquired a comprehensive knowledge of drawing and of mapping.

The text is clear, the style simple and pleasing; the proof-reading has been well done, and the figures are excellent. American text-book writers of solid geometry could profit from a complete mastery of this book.

VIRGIL SNYDER

*College Algebra.* By Louis J. Rouse. New York, John Wiley and Sons; London, Chapman and Hall. Second Edition, 1939. 13+462 pages. $2.25.

The first edition of this book appeared in 1931; it was reviewed in this MONTHLY, vol. 39, 1932, pp. 23–24, by Mildred Waters Dean. In 1934 a reprinting appeared, with a chapter on logarithms added.

The present edition retains the essential features of the earlier one, but the exercises are new, and several chapters have been added, including one on compound interest and annuities, one on probability and one on partial fractions. The chapter on determinants has been completely re-written, based on induction rather than inversions. The printing is excellent, and the proof-reading has been well done.

VIRGIL SNYDER

*Advanced Algebra.* Volume I by C. V. Durell. 8+193+22 pages. 4s. 6d. Volumes II and III by C. V. Durell and A. Robson. London, G. Bell and Sons, Ltd. 1938. 11+315+27 pages. 12s. 6d.

Volume I was originally published in 1932. The material covered corresponds roughly to that found in the last half of the conventional college algebra, but the treatment is more difficult. For example, calculus methods are used extensively in Chapter VI on "Logarithmic and Exponential Functions" and in Chapter VII on "Rational Functions."

Some idea of the scope of Volumes II and III may be obtained from the following chapter titles, of which the first five are from Volume II: Finite Series, Difference Equations, Factors and Partial Fractions, Theory of Equations, Sequences, Inequalities, Determinants, Matrices, Choice and Chance, Theory of

Numbers. A wealth of material is here presented for advanced undergraduates and graduates.

Developing theory by means of brief but well chosen paragraphs, the authors include more topics in each chapter than might be expected from the actual number of pages devoted to exposition.

As one has learned to expect from English textbooks, the emphasis is on exercises. Volume I contains 35 sets of Test Papers. Volumes II and III contain about 1500 exercises graded into three sections. The authors include more than the usual number of illustrative examples. Answers are given to all exercises, and books of hints are available for the last two volumes.

It is not likely that many teachers would be willing to give up the traditional courses in theory of equations and higher algebra. However, for supplementary material, reference, and as a source book for exercises, the authors have added a three volume work of unquestionable merit to their list of successful textbooks.

F. A. LEWIS

*Magische Quadrate und Magische Parkette.* By Gerhard Kowalewski. Leipzig, K. F. Koehlers Antiquarium, 1937. 78 pp. Rm. 2.

This is the second number in the series *Scientia Delectans* which has for its purpose the presentation of popular essays on various topics which in English we usually class under the term, Mathematical Recreations.

The present little book is devoted to the study of certain classes of magic squares of order four. These classes arise naturally from what the author calls the "cross condition" (Kreuzbedingung). Let the elements of the first row of the magic square be denoted by $a_1$, $a_2$, $a_3$, $a_4$, and the elements of the second, third and fourth rows in a similar way with the letters, $b$, $c$ and $d$. Then if $a_2+d_3=b_4+c_1=s$, the square is said to satisfy the cross condition. If this holds, it is found that the relation $a_3+d_2=b_1+c_4=s'$ is also satisfied. Now if the difference $s-s'$ is zero, the square is said to be a Dürer square, since it is of the type which appears in Albrecht Dürer's engraving "Melancholy." All squares of this type are determined, and are shown to be identical with what are here called antisymmetric squares, but which have long been known in the literature as symmetric.

If the above difference $s-s'$, instead of having its least value zero, has its greatest possible value we have a new class of squares which the author names Sun squares. All squares of this type are determined and are found to be sixteen in number.

Certain, but not all, of the Sun squares are pandiagonal. This leads the author to a discussion of pandiagonal squares and the so-called magic parquetry or mosaic. The most interesting part of the book in connection with the pandiagonal squares, however, is their representation by means of a four-dimensional cube. Such a cube has sixteen vertices, and a fourth order magic square has sixteen elements. It is shown how these elements may be distributed over the

sixteen corners of the four dimensional cube in such a way that the elements at the corners of each face of the bounding three dimensional cubes give the magic sum. These face squares may be arranged in three systems, or nets, from each one of which may be written down a fundamental pandiagonal magic square. Instead of the four dimensional cube we may deal with its three dimensional representation, namely, a cube with a square hole through it. This three dimensional figure may then be used to write down, by a very simple rule, all pandiagonal magic squares of order four.

The book is extremely well written. The explanations are very clear and presuppose absolutely no previous knowledge of the subject. Besides the main topics noted above, there are many details which we feel sure will delight anyone interested in magic squares.

The reviewer feels that it would have been worth-while, on page 31, to have pointed out that the relation satisfied by the four central elements and the four corner elements of a Dürer square holds for all magic squares of order four. A similar remark applies to the statement on page 68 concerning the sum of the four elements at the ends of the two middle columns (or rows). In regard to the term "Sun square" it seems that its choice may have been ill-advised, since Cornelius Agrippa, in 1531, in his *De Occulta Philsophia* associated a square of the *sixth* order with the sun.

G. E. RAYNOR

*Der Keplersche Körper und Andere Bauspiele.* By Gerhard Kowalewski. Leipzig, K. F. Koehlers Antiquarium, 1938. 65 pp. Rm. 2.

This is the third number in the series, *Scientia Delectans.*

Professor Kowalewski has succeeded in packing into the space of 65 pages an amazing amount of interesting and instructive material. The central and unifying theme of the book is the historically important Kepler solid. This is a convex solid with thirty congruent rhombic faces, the shorter diagonals of the rhombs forming the edges of a regular dodecahedron, and the longer diagonals the edges of a regular icosahedron. The ratio of these two diagonals is that of the well-known Golden Section, namely $\frac{1}{2}(\sqrt{5}-1)$ to 1.

It is shown how the Kepler solid may be built up from two sets, of ten each, of vari-colored parallelopipedal blocks, in such a way that two neighboring blocks touch along like colored faces. Other problems arise by distributing certain sets of markers over the thirty faces of the Kepler surface. For this purpose plane representation of the surface is used, one vertex being the point at infinity. Some of the distribution problems described are by no means simple, and are not likely to be solved by anyone not familiar with the theory explained by the author. The connection of the Kepler solid with a six-dimensional cube is another interesting high-light of the book.

It is impossible in a short review of this sort to give an adequate idea of the many interesting details. The reviewer thoroughly enjoyed reading the book and heartily recommends it to anyone interested in things mathematical.

G. E. RAYNOR

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N.J.*

NOTICE TO ALL CLUBS. The spring semi-annual letter to all mathematics clubs should have reached you by the middle of May. This department would appreciate it if the secretary of each organization would see to it that the year's report is mailed to this office before he leaves his campus if it is possible. If there are any clubs who failed to receive this letter through some error, will they please communicate with this department as soon as possible, so that the yearly report will not be too long delayed.

## CLUB TOPICS

### 42. SYMBOLIC LOGIC

SAUNDERS MAC LANE, Harvard University

Symbolic logic, born to be a universal language and a sort of calculus of all reason, bred in a turbulent atmosphere with controversies and paradoxes galore, now approaching maturity with the attainment of subtle methods and profound results, is a subject which has many attractions for mathematician and philosopher, for student, specialist, and layman. The logical analysis of demonstrations and of mathematical reasoning has an immediate interest to anyone who has dealt with mathematical proofs. Current research in logic is necessarily precise and technical, and often involves a heavy symbolic formalism, but the fundamental questions considered are generally direct and simple, and can be well understood by the uninitiate. Consider, for instance, Gödel's results which state (roughly speaking) that some theorems are true but not provable by any standard methods, and the intuitionists' contention that some proven theorems are not true! Again, raise the question whether there are mathematical problems which are insoluble. Questions like these are sure to provoke good discussion.

To assist mathematics clubs in discussing logic we give here a list of possible topics and a bibliography. The topics vary in difficulty and are more or less independent of each other. The bibliography is intended to give a fair representation of readable books and articles useful as an introduction to symbolic logic; we hope that it may also serve those who wish to study the subject independently. For complete bibliography we refer to Church [**15**].*

**1. Paradoxes.** A presentation of some of the standard paradoxes offers great opportunities. What happens to the beard of the barber who obligingly shaves all the men in town who do not shave themselves? This paradox and many others can be readily formulated in elementary terms; they are sure to produce discussion in a meeting. An excellent and readable discussion of the various standard paradoxes may be found in Fraenkel [**24**, §13]. In chapter 2 of the introduction of *Principia Mathematica* [**59**] there is (on pp. 60–61) a compact statement of a number of paradoxes, which may be read without a knowledge of the rest of this introduction. The same paradoxes are also stated in section 1 of [**49**]. Paradoxes are also stated by Menger [**44**], Bell [**2**, pp. 575–576], Black [**9**, pp. 97–101], Poincaré [**46**, *Science and Method*, chapter 5, parts 5 and 10]. Among the more sophisticated paradoxes is that due to Richard, which indicates that any system of symbolic logic is in some sense inadequate (Church [**14**]).

The classical solution of the paradoxes depends on the theory of types. To avoid

* Numbers enclosed in brackets refer to the corresponding items of the bibliography below.

paradoxical statements which refer to themselves in a circular fashion this theory separates the objects of discourse into various levels. In general outline, this idea can be presented as a natural one.* The original form of the type theory involve a difficult axiom of reducibility.† Another reasonable method of resolving the paradoxes appears in Behmann [**1**].

**2. Calculus of propositions.** An elementary and useful branch of logic is the calculus of propositions. This calculus offers methods of symbolizing the connectives "and," "or," "not," "if · · · then · · · ", and their various properties. Its methods can be readily explained to beginners. They are best found in some of the systematic treatments.‡ A modern treatment of this calculus in a form especially easy to use for mathematical proofs is given by Bernays's mimeographed lecture notes [**5**, pp. 7–47]. There are several very useful such sets of up-to-date lecture notes available.§ Brief treatments of the calculus of propositions may be found, for instance, in Menger [**44**] or in Black [**9**, pp. 41–48]. The newest text on symbolic logic is Bennett-Baylis [**3**].

**3. Propositional functions.** More fundamental than the calculus of propositions is that of propositional functions, which deals with statements involving a variable $x$ and with "quantifications" of such statements by such phrases as "so-and-so holds for every $x$" or "there exists an $x$ such that so and so." The elementary properties of these quantifiers are illuminating, especially in connection with the definition of limits, of uniform continuity, and of similar mathematical notions. An excellent and simple introduction is given by Tarski [**55**].‖ Other discussions of propositional functions are given by¶ Bernays [**5**, pp. 47–56] and Eaton [**22**, chapter 2].

**4. Formalization of mathematics.** An elementary but important aspect of symbolic logic is the possibility of using logical symbols to express in perfectly precise and completely symbolic form any mathematical theorem (or, for that matter, any proof). An interesting topic for club discussions would be the development of just enough logical symbolism to show how this could be done, with examples of mathematical proofs and definitions in precise form. The original ideas of Peano in this direction are discussed in chapter 1 of Lewis-Langford [**42**]. In Bernays [**5**, p. 2] is an excellent elementary logical treatment of groups, in Tarski [**55**] a similar discussion of real numbers. Carnap's Abriss [**11**] gives illustrations of the logical analysis of topology, point-set theory and projective geometry (§§31–35). More elementary examples of mathematical formalism appear in Hilbert-Bernays [**34**, §1], which gives careful and detailed explanations of the

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* Eaton [**22**, chapter 3, §4]; Menger [**44**]; Lewis-Langford [**42**, chapter 13].

† Principia Mathematica [**59**, chapter 3 of the introduction]; Jørgensen [**39**, volume 3, chapter 12], Black [**9**, pp. 97–104].

‡ Such as Hilbert-Ackermann [**33**, chapter 1]; Eaton [**22**, chapter 1]; Principia Mathematica [**59**, part 1, Section A (which can be read independently of the long introduction)]. A concise, efficient presentation of the Whitehead-Russell calculus is given in Carnap [**11**, §§1–5].

§ Rosser [**47**], Cooley [**17**]. The latter contains a large collection of verbal problems in symbolic logic.

‖ For those who can read German this is probably the best elementary introduction to symbolic logic.

¶ And also in most of the systematic treatises mentioned above: Lewis-Langford [**42**, chapter 5]; Lewis [**41**, chapter 4]. Cf. also Black [**9**, pp. 41–68].

development involved. A classic American example of mathematics in logical form is E. H. Moore's *General Analysis*.* These symbolic techniques have even found use in the science of biology (Woodger, [**60**]).

**5. Definitions of number.** One of the earliest and most controversial applications of logic consists of an answer to the old problem, "What is number?" The logistic definition of number in terms of correspondence can be presented in a simple fashion, as in the first chapters of Russell [**50**]. These notions go back to Cantor, whose introduction of transfinite numbers by similar methods led to many violent controversies. The ideas behind these disputes are engagingly described by Bell [**2**, pp. 564–568]. Other approaches to the number concept are the Peano postulate method;† the historically significant method of Dedekind [**18**], and the radical approach of the intuitionists‡ (see below).

**6. Logic of real numbers.** Of fundamental importance for calculus and analysis is the application of logic to the definition of the real numbers. The most attractive, simple, and precise method today is that by means of postulates. Using elementary logic, Tarski presents in [**55**] two sets of postulates for the real numbers. Another approach§ uses the Dedekind cut idea to define numbers in terms of the logical properties of classes; it is brilliantly described in Russell [**50**, chapter 8]. This approach, however, involves all the difficulties of set-theory; a demonstration of its consistency (see below) is probably even more difficult than in the postulational approach.

In any club discussion, a good starting point would be the discussion of the paradoxes of Zeno.‖ The puzzling argument of the hare and the tortoise shows very clearly how necessary it is to think carefully and logically about the continuum of all points on the line.

The absolutely general notion of a "class" or "set" which appears in these investigations of numbers is the source of many difficulties in symbolic logic.¶ It offers a prime elementary example of mathematics in its most abstract and general form.

**7. Boolean algebra.** A different and mathematically stimulating approach to logic is that through Boolean algebra. It can be readily explained how "algebra" can be done with classes instead of numbers as objects. The laws of this algebra, which agree but in part with those of ordinary algebra, furnish a fine illustration of the possibility of new mathematical disciplines.** The foundations of Boolean algebra have been a favorite problem in postulate-theory. Innumerable systems of postulates for this algebra have

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* General Analysis, Part I, by Eliakim H. Moore; Memoirs of the American Philosophical Society, Philadelphia, 1935.

† Hilbert-Bernays [**34**, p. 219 ff.]; mathematical developments of the postulates may be found in E. Landau, Grundlagen der Analysis, Leipzig, 1930.

‡ Weyl [**57**, §6]; Poincaré [**46**, various sections on complete induction].

§ Still another procedure appears in Huntington [**35**].

‖ Cf. the profound treatment in Russell [**48**, chapters 33, 34, and 35].

¶ Discussions of classes in Russell [**50**, chapters 15 and 17]; Huntington [**35**, chapter 2], Russell [**48**, chapters 6 and 7], Weyl [**57**, §8].

** Treatments of Boolean algebra are given in Couturat [**16**]; Eaton [**22**, chapter 3]; Lewis-Langford [**42**, chapter 2]; Lewis [**41**, chapter 2]. The algebra in its historical setting appears in Bell [**2**, chapter 23]. Chapman-Henle [**13**, chapter 11], gives an elementary account.

been set up.* In [**38**], Huntington discusses the methods involved. For club talks the use of this Boolean algebra to solve explicitly complicated verbal problems is an attractive possibility.† Consider, for instance, its use in the insurance problems with which actuaries contend! (Berkeley [**4**]).

Important recent developments have been made in Boolean algebra, by applying to it the powerful methods of modern abstract algebra (Stone [**53**]). Another startling extension lies in using more "truth values" than the ordinary "true" and "false." If ordinary Boolean algebra is considered as an algebra of statements (or propositions), every statement‡ is "equal" either to 0 (falsity) or to 1 (truth). It is possible to construct similar algebras using more than two such truth values.§ Hence the suggested topic: How can algebra treat statements which are neither true nor false (but are part way between)?

**8. Intuitionism.** A radical and startling attack on the methods of symbolic logic and of classical mathematics is offered by Brouwer and the intuitionists. They propound problems which are real posers. Consider the statement "somewhere in the decimal development of $\pi$ appears a sequence of digits 1 2 3 4 5 6 7 8 9." Is it false?—Nobody could calculate $\pi$ all the way to infinity just to disprove it. Is it true?—No one has yet found such a sequence. Perhaps it is neither true nor false! The writings of the intuitionist abound in many such neat dilemmas, sure to catch the eye of an inquisitive student.

The constructive ideas of the intuitionists are often pontifical and obscure, but some of their positions are clear.‖ Thus "$A$ implies the absurdity of the absurdity of $A$, but not vice versa." Fortunately the ideas of Brouwer have been interpreted in English (Dresden [**19**], [**20**]). The position of the intuitionists is radical enough to demand the revision of a large part of mathematics.¶

Poincaré was an earlier and livelier intuitionist, who emphasized the importance of mathematical induction as an extra-logical element in the foundation of mathematics.** Weyl gives in [**57**] (cf. also [**58**]) a profound and stimulating discussion of these ideas. He emphasizes rightly the importance of constructive proofs for existence theorems. If, on this view, the proof of an existence statement can be accomplished only by exhibiting the object supposed to exist, then it follows that some existence statements are neither true nor false.

There are available many semi-popular discussions of intuitionism as compared with other schools of thought in the foundations of mathematics.†† One warning on these discussions is necessary: It was once fashionable to discuss just three schools; intuitionism, formalism, logistic. This neat trichotomy is no longer adequate. It fails to take into ac-

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* Stone [**52**], Huntington [**36**], [**37**], Bernstein [**7**], and other references given in these papers. Sheffer [**51**] is important.

† Lewis-Langford [**42,** chapter 3, part 2].

‡ Lewis-Langford [**42,** chapter 4].

§ Frink [**25**], Lewis-Langford [**42,** chapter 7].

‖ Easier than most is Brouwer [**10**]; cf. also the bibliography and general discussion in Heyting [**31**].

¶ Considerable progress on this revision has been made. Cf. Heyting [**31,** part 1].

** [**46,** Science and Hypothesis, chapter 11; Science and Method, chapters 3, 4, and 5.]

†† Fraenkel [**24,** §§14, 15, 18]; Hedrick [**30**]; Bell [**2,** pp. 575–579]; Black [**9,** pp. 9–11, 169–211]; Frink [**25**]; Menger [**44**]; Dubislav [**21**].

count the interaction of intuitionism and formalism and the recent work of Church, Curry, Carnap, Quine and others (Church [**15**]). An up-to-date discussion appears in Gentzen [**26**], [**27**].

**9. Geometry and logic.** An amusing geometrical interpretation of intuitionism has recently been found.* It has long been customary to treat Boolean algebra by Venn diagrams.† A class $A$ is pictured as the set of points inside a circle; the complement of the class $A$ (the negation of the corresponding proposition) then consists of the points outside the circle. If the boundary is neglected, $A$ and not $-A$ together constitute the whole plane—but if the boundary is considered there are points which are neither $A$ nor not $-A$, exactly as in intuitionist logic! This has been developed technically for very general spaces, but the underlying ideas could be presented to a club in a simple, attractive form, using ordinary Euclidean geometry.

Quantum mechanics has also led to the introduction of novel logics (Birkhoff-von Neumann [**8**]).

**10. Consistency proofs.** A fundamental problem in the foundation of mathematics may be stated thus: Can the postulates for all the ordinary parts of mathematics be chosen so that all the ordinary paradoxes are avoided and so that we can also prove that no new contradictions or paradoxes could arise? Hilbert has attempted to use symbolic logic to furnish such a "consistency proof." An early discussion of his ideas appears in [**32**]. Only for limited parts of mathematics has such a proof been obtained.‡ One especially important idea has developed from these investigations. One wishes to consider a possible proof which leads to an inconsistency within a system of mathematics; to do this one must go outside the given system and talk *about* it. This "outside" metamathematics or syntax is discussed in Bernays [**6**], Carnap [**12**], Hilbert-Bernays [**34**]. A clear understanding of this "syntax" is necessary in any modern system of logic.

**11. Gödel's theorem.** The most influential result in modern symbolic logic is the Theorem of Gödel concerning statements which can be neither proven nor disproven in a given logical system. It raises enormous difficulties for the Hilbert program, for one consequence is the impossibility of proving the consistency of most systems by the methods of proof used in those systems.§ The basic idea of Gödel's proof can be presented to a club that is not too immature, for it depends on a suitable use of one of the paradoxes. A good outline is given in the introduction to Gödel's original paper [**28**]; the details may be found there or in Gödel [**29**] or Carnap [**12**, §§35 and 36]. There is a general discussion of the Theorem in Menger [**44**]. The difficulty in understanding this Theorem is well repaid by the importance of the result.

**12. Miscellaneous topics.** *History of logic*: Jørgensen [**39**, volume 1]; Lewis [**41**, chapter 1]; Lewis-Langford [**42**, chapter 1]; Bell [**2**]. *Transfinite numbers*: Russell [**50**]; Russell [**48**, chapters 11, 13, 43]; Bell [**2**]. *Strict implication*: Lewis-Langford [**42**, chap-

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* Stone [**54**], Tarski [**56**].

† Lewis-Langford [**42**, chapter 3]; Eaton [**22**, chapter 3]; Lewis [**41**, chapter 3, part 1].

‡ Details in Bernays [**5**, pp. 56–100]; Hilbert-Bernays [**34**]; Summaries in Mac Lane [**43**]; Heyting [**31**, part 2]; Black [**9**, pp. 147–169]; Weyl [**57**, §10]; Gentzen [**27**].

§ See, however, the discussion in Gentzen [**26**], [**27**].

ter 6]; Lewis [**41**]; Huntington [**37**]; Feys [**23**]. The role of "strict implication" is subjected to a penetrating analysis in Carnap [**12**, §§69–70]. *Applications of logic to philosophical problems* Carnap [**12**]. *Calculus of Relations*: Whitehead-Russell [**59**]. *Nature of Deductive Systems*: [**13**, chapters 9 and 10].

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## CLUB REPORT

### *Pi Mu Epsilon, New York University*

The chapter held regular meetings during the college year. Papers were presented on Euclid's parallel postulate; Bishop Berkeley's criticism of differentiation; Dimension theory; Function spaces. The guest speaker of the year was Professor W. V. Quine of Harvard University who spoke in February on The theory of logical types. The fifth annual Pi Mu Epsilon Interscholastic Mathematics Contest was held on April 30, 1938. There were 114 high schools represented, with a total of 430 contestants. The highest score was made by the Abraham Lincoln High School of Brooklyn, N. Y., which was awarded the large cup. The team placing second was from Boys High School, Brooklyn, N. Y. Four smaller cups were awarded to the teams making the highest score in their section. Medals were awarded to the three contestants who made the highest individual scores. The chapter also conducted two mathematics contests for undergraduates, one for freshmen and the other for upper-classmen. In each case an examination was given, and a prize of $20 was offered for the best paper in each contest.

Director, Professor F. W. John; Vice-Director, H. I. Zagor; Secretary, Frieda Agin; Treasurer, Naomi Rosenstein, Permanent Secretary, J. H. Moss.

## "THE MATHEMATICAL SAGA OF LINNIE R. E. QUASHUN."

Below are words which may be used to fill in spaces in the story printed last month, pp. 234–235.

| | | | | |
|---|---|---|---|---|
| 1. regions | 20. curves | 39. circles | 58. pi | 77. problem |
| 2. sum | 21. mean | 40. limit | 59. hypotenuse | 78. solution |
| 3. planes | 22. integrate | 41. inclined | 60. table | 79. irrational |
| 4. operations | 23. difference | 42. perspective | 61. ratio | 80. continuity |
| 5. slipstick | 24. root | 43. whole | 62. locus | 81. secant |
| 6. radicals | 25. table | 44. system | 63. one | 82. terminate |
| 7. elliptical | 26. unknown | 45. rational | 64. logarithm | 83. square |
| 8. high | 27. proof | 46. index | 65. root | 84. rate |
| 9. quotient | 28. tangent | 47. exponent | 66. power | 85. properties |
| 10. line | 29. less | 48. positive | 67. compute | 86. center |
| 11. proportion | 30. plane | 49. multiply | 68. mechanics | 87. complementary |
| 12. figure | 31. indeterminate | 50. parallel | 69. differential | 88. minimum |
| 13. slope | 32. height | 51. spaces | 70. reduced | 89. revolved |
| 14. circumference | 33. satisfied | 52. lemniscate | 71. rhombus | 90. coincided |
| 15. infinitesimal | 34. increment | 53. times | 72. series | 91. loci |
| 16. foot | 35. added | 54. demonstration | 73. maximum | 92. remainder |
| 17. nine | 36. function | 55. period | 74. group | 93. vertical |
| 18. thirty | 37. normal | 56. coördinate | 75. base | 94. horizontal |
| 19. six | 38. subnormal | 57. greater | 76. altitude | 95. exercise. |

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Connecticut State College, Storrs, Connecticut.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 378. *Proposed by J. L. Brenner, University of Minnesota.*

Find the number of integral values of $B$ which make $B^2+m$ a perfect square, for any given, fixed, integer $m$.

E 379. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Find a trapezoid whose sides, altitude, diagonals and area are rational.

E 380. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

If the radius of a circle is any odd prime, $p$, there are just two different primitive Pythagorean triangles circumscriptible about that circle. Show that, for each such pair of triangles:

(A) their shortest sides differ by one;
(B) their hypothenuses exceed their corresponding longer legs by one and by two respectively;
(C) the sum of their perimeters is six times a perfect square;
(D) as $p$ increases without limit, the ratio of their least angles approaches the limit 2;
(E) as $p$ increases without limit, the ratio of their areas approaches the limit 2; and finally,
(F) the smaller triangle can always be placed inside the larger, so as not to touch it.

E 381. *Proposed by W. B. Clarke, San Jose, California.*

Show how to construct a square with one corner on each of four generally placed straight lines in a plane. How many solutions are there in general? What constitute special cases? What happens if the lines are placed askew in space?

E 382. *Proposed by Virgil Claudian, Bucharest, Roumania.*

$A$, $B$ and $C$ are three fixed points in space. $(S)$ is a sphere fixed in space, bearing $A$ and $B$, but not $C$, on its surface. $P$ is a point which moves over the surface of $(S)$. $(Q)$ is the sphere determined by $A$, $B$, $C$ and $P$. $T$ is the plane tangent to $Q$ at $C$. $M$ is the plane $ABP$. Planes $M$ and $T$ intersect in the line $l$. Determine the locus of $l$.

## SOLUTIONS

E 314 [1938, 47]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Inst.*

Find the locus of the center of a variable sphere which cuts two (or three) fixed planes in circles of constant size.

*Solution by G. A. Yanosik, New York University.*

Let the two fixed planes be $y = \pm mx$. Let the center of the variable sphere be $(X, Y, Z)$. Let the radii of the circles of section with the two fixed planes be $q$ and $r$ respectively.

Since all radii of the sphere at any instant are equal, we have

$$q^2 + (mX - Y)^2/(1 + m^2) = r^2 + (mX + Y)^2/(1 + m^2),$$

which reduces to $4mXY = (q^2 - r^2)(1 + m^2)$, a hyperbolic cylinder whose generators are parallel to the line of intersection of the two fixed planes.

If three fixed planes are given, the locus of $(X, Y, Z)$ is obviously a space curve, the common curve of intersection of three hyperbolic cylinders with elements respectively parallel to the three lines of intersection of the fixed planes taken in pairs.

Also solved by L. E. Malvern and F. L. Wilmer.

E 338 [1938, 320]. *Proposed by C. W. Trigg, Los Angeles City College.*

There are only two numbers which are permutations of the ten digits, which are also the products of three consecutive integers. Find them and show that there are no others.

*Solution by W. E. Buker, Pittsburgh Public Schools.*

Let the consecutive integers be $n-1$, $n$ and $n+1$, so that their product is $n^3 - n$. It is readily seen that $n$ lies between 1000 and 2147, and differs by not more than 1 from a multiple of 9. Using a table of cubes to seek values of $n$ which satisfy these two conditions, we reject cubes showing any duplication of digits in the first six digits at the left. We subtract $n$ from the remaining eligible cubes and again reject all cases where any digit appears more than once. The only two cases remaining are

$$1267 \cdot 1268 \cdot 1269 = 2{,}038{,}719{,}564$$

and

$$1332 \cdot 1333 \cdot 1334 = 2{,}368{,}591{,}704.$$

Also solved by the proposer.

E 339 [1938, 478]. *Proposed by V. Thébault, Le Mans, France.*

Consider all triangles inscribed in a given circle on a fixed chord as base, and determine the locus of the feet of the interior and exterior base-angle bisectors. Thus show that if two interior angle bisectors of a triangle are equal, the triangle is isosceles.

*Solution by the proposer.*

If side $BC$ and angle $A$ of triangle $ABC$ remain constant, the feet of the bisectors of angles $B$ and $C$ generate two strophoids which are symmetrically placed with respect to the perpendicular bisector of $BC$. This follows at once from the definition of a strophoid. Hence if triangle $ABC$ has the two interior angle bisectors, $BD$ and $CE$, equal, the points $D$ and $E$ are symmetrically placed with respect to the perpendicular bisector of $BC$, $DE$ is parallel to $BC$, and triangle $ABC$ is isosceles.

E 340 [1938, 478]. *Proposed by E. C. Kennedy, Texas College of Arts and Industries.*

If $B$ is a positive integer, what rational values may be assumed by

$$R=\sqrt{\left(\frac{B}{7}\right)^2+5038}.$$

*Solution by E. P. Starke, Rutgers University.*

We have at once $(7R)^2=B^2+49\cdot 5038$. Since $7R$ is to be rational and has an integral square, it is an integer. Now $7R-B$ and $7R+B$ are of like parity, and since their product is even, they must both be even. But that requires their product to be a multiple of four, which it is not. Hence we have a contradiction, and there is no integer value of $B$ which makes $R$ rational.

However, if we remove the requirement that $B$ be an integer, there are infinitely many rational values for $B$ and $R$, since we may let $7R-B$ equal any rational value, $K$, and let $7R+B=49\cdot 5038/K$. Then $B=(49\cdot 5038-K^2)/2K$ and $R=(49\cdot 5038+K^2)/14K$.

*Editorial Note.* As originally proposed in manuscript, this problem had 503 instead of 5038, which led by the methods shown above to the three solutions, $B=227$, 1757 and 12323, and $R=276/7$, 252 and 12324/7. Although the error in transcription has not been traced to its origin, appropriate apologies are hereby tendered to all concerned.

Also solved by J. L. Brenner, Fred Discepoli, F. Olson, C. W. Trigg and the proposer.

E 341 [1938, 478]. *Proposed by J. H. Edmonston, Washington, D. C.*

A triangular octahedron (one with all faces triangular) may be regarded as a space analogue of a plane quadrilateral. On this basis, state and prove a space-analogue of the theorem that the midpoints of the sides of any plane quadrilateral are the vertices of a parallelogram.

*Solution by C. W. Trigg, Los Angeles City College.*

THEOREM. *The centroids of the faces of a triangular octahedron are the vertices of a parallelopiped.*

Let the edges issuing from a vertex, $V$, of a triangular octahedron be $VA$, $VB$, $VC$ and $VD$. From $B$ and $D$, and in the faces determined by these four

edges, draw the medians to $VA$ and $VC$. On these medians locate the centroids, $E$, $F$, $G$ and $H$, of the faces. Draw $BD$, $EF$, $FG$ and $HE$. Then $EH=BD/3=FG$, $EH$ and $FG$ are parallel to $BD$, so that $EFGH$ is a parallelogram. Repeating this process at the other five vertices, we show the faces of the figure determined by the eight face-centroids to be all parallelograms, so that the figure is a parallelopiped.

Also solved by W. E. Buker, E. P. Starke and the proposer.

E 342 [1938, 478]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Show how to draw three circles with radii $a$, $b$ and $c$, and common tangents $d$, $e$ and $f$.

*Solution by W. B. Clarke, San Jose, California.*

Since the problem does not specify whether the tangents are internal or external, both cases must be considered. The distance between the centers of two circles of radii $b$ and $c$, with a tangent of length $d$, is obviously

$$\sqrt{d^2 + (b \pm c)^2}$$

where the alternative signs correspond to $d$ being an internal or external tangent respectively. Since such a length is readily constructible, the centers of the desired circles become the vertices of a triangle with known sides, and since the respective radii are also known, the construction is obvious.

While there are in general eight solutions, correspodning to the different possible combinations of internal and external tangents, any number of these may disappear in special cases, when the corresponding triangles become outlawed through having the length of one side exceed the sum of the lengths of the other two.

Also solved by E. R. Heineman, D. L. MacKay, Joseph Milkman and the proposer.

E 343 [1938, 478]. *Proposed by H. E. Stelson, Kent State University.*

Prove that $ax^2+2hxy+by^2+2gx+2fy+c$ is the product of two factors, linear and rational in $x$ and $y$, provided $[h+(h^2-ab)^{1/2}][f+(f^2-bc)^{1/2}]-b[g+(g^2-ac)^{1/2}]=0$, and that in this case, $abc+2fgh-af^2-bg^2-ch^2=0$.

*Solution by W. E. Buker, Pittsburgh Public Schools.*

Assume factors $c(lx+my+1)(rx+sy+1)$, whose product is $c[lrx^2+(ls+mr)xy+msy^2+(l+r)x+(m+s)y+1]$. Equating the coefficients to those in the given expression, we get

(A) $clr=a$, $cms=b$, $c(ls+mr)=2h$, $c(l+r)=2g$, $c(m+s)=2f$.

Solving these equations for $l$ gives $l=a/cr=(2g-cr)/c=(2h-cmr)/cs$. This enables us to eliminate $l$ from (A), getting

(B) $cms=b$, $2gr-cr^2=a$, $2hr-cmr^2=as$, $cm+cs=2f$.

This time we solve for $m$ where possible, getting $m=b/cs=(2hr-as)/cr^2=(2f-cs)/c$, which lets us eliminate $m$, getting

(C) $cr^2-2gr+a=0$, $as^2-2hrs+br^2=0$, $cs^2-2fs+b=0$.

The next step is to solve the first and third of equations (C) for $r$ and $s$, and substitute in the second, getting

(D) $a[f\pm(f^2-bc)^{1/2}]^2-2h[f\pm(f^2-bc)^{1/2}][g\pm(g^2-ac)^{1/2}]+b[g\pm(g^2-ac)^{1/2}]^2$ $=0.$

For all the ambiguous signs we use only the plus, and solve the above equation as a quadratic in $[g+(g^2-ac)^{1/2}]$, arriving at

(E) $[g+(g^2-ac)^{1/2}]=[h+(h^2-ab)^{1/2}][f+(f^2-bc)^{1/2}]/b$

which is the required relationship.

To verify the latter part of the problem, it is sufficient to notice that the values for $a$, $b$, $f$, $g$ and $h$, as given in (A), reduce $abc+2fgh-af^2-bg^2-ch^2=0$ to an identity.

Also solved by E. P. Starke and the proposer.

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3848 [1937, 667]. *Proposed by P. Erdös, Budapest, Hungary.*

(*Correction.*) Let $O$ be an arbitrary point in the interior of the triangle $ABC$, and let $A'$, $B'$, $C'$ be the points in which $AO$ cuts $BC$, etc. If $AA'\geqq BB'$ and $AA'\geqq CC'$, then $AA'\geqq OA'+OB'+OC'$, where the equality holds only if $AA'=BB'=CC'$.

3913. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

If $f(x)=e^{kx}\tan x$, $k\geqq 0$, discuss the effect of the value of $k$ on the location of maximum, minimum, and inflection points.

3914. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

A dog is tied to a rope of length $L$, which is fastened on the other side of a smooth topped fence of height $h$ at a point $a$ units from the top, $L>h+a$. Discuss the shape and dimensions of the region over which he can roam; and find its area.

3915. *Proposed by V. Thébault, Le Mans, France.*

Two numbers have each eight figures the last of which is seven in each one; and the first eight figures of their product are identical as well as the last eight. Find the two numbers.

3916. *Proposed by V. Thébault, Le Mans, France.*

Find a sequence of three figures such that if the sequence is regarded first

as a number in a system with a certain unknown base, and then as a number in the duodecimal system (base 12), the value of the number in the second case is one half of that in the first. Show that there are only two such unknown bases for the given problem.

3917. *Proposed by R. A. Johnson, Brooklyn College.*

Determine the values of $n$ for which the sum of the squares of $n$ positive integers in arithmetic progression is the square of an integer, and show how to determine such progressions.

**SOLUTIONS**

3822 [1937, 179]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

In a triangle, the inscribed conics passing through the intersections of the circum-and conjugate circles are tangent to the nine-point circle.

*Solution by Otto J. Ramler, Catholic University of America.*

In an article entitled "A geometrical proof of Professor Morley's extension of Feuerbach's Theorem" (Proceedings Edinburgh Mathematical Society, 1920, vol. 38, p. 2) H. W. Richmond develops the following theorems upon which we base our solution:

1) If the three unknown common tangents of two line cubics which touch the six sides of the complete quadrangle $OPQR$ form a triangle $XYZ$, the seven points $OPQRXYZ$ lie on a conic and conversely,

2) If the vertices of a triangle $XYZ$ lie on a conic with the points $OPQR$ then all the line cubics touching $OP$, $OQ$, $OR$, $QR$, $RP$, $PQ$ and two sides of $XYZ$ also touch the third side.

3) If $L$, $M$, $N$ are the points of contact of $YZ$, $ZX$, $XY$ with a line cubic touching the six lines mentioned above, then $XZ$, $ZX$, $XY$ are also tangent to a conic at $L$, $M$, $N$ and hence $XL$, $YM$, $ZN$ are concurrent.

4) If the positions of $ZX$, $ZY$, $L$ and $M$ are known, those of $XY$ and $N$ are readily found.

5) For the system $\Gamma_3$ of line cubics which touch the six lines $OP$, $OQ$, $OR$, $QR$, $RP$, $PQ$ and which pass through two given points $L$ and $M$ there is a relation between the tangents $LZ$ and $MZ$. Hence there is a locus for $Z$.

6) For the system of conics through $OPQR$ and conjugate with respect to $L$ and $M$ the locus of $N$, the pole of $LM$ as to these conics is another conic through $L$ and $M$ and circumscribed to the diagonal triangle $ABC$ of $OPQR$.

7) When two adjacent $\Gamma_3$'s touching the six lines $OP$, $OQ$, $OR$, $QR$, $RP$, $PQ$ and passing through $L$ and $M$ approach coincidence, their nine common tangents are ultimately

i) the six lines $OP$, $OQ$, $OR$, $QR$, $RP$, $PQ$

ii) the two lines $LYZ$, $MXZ$ through $L$ and $M$, the points of contact being $L$ and $M$;

iii) a line $XNY$ touching the conic $LMABC$ at $N$, the point of contact with the $\Gamma_3$ also being $N$.

The envelope of the cubics is thus seen to be the points $L$ and $M$ and the conic $LMABC$.

Now let $L$, $M$ be the intersections of the circumscribed circle $PQR$ and the conjugate circle of $PQR$. It is known that the nine-point circle also belongs to this pencil. It can also be readily proved that all conics through $O$, $P$, $Q$, $R$, where $O$ is the centroid of $PQR$, are conjugate to $L$ and $M$, and that the locus of $N$, the pole of $LM$ as to the pencil of conics is the nine-point circle. Hence the envelope of the system of line cubics touching $OP$, $OQ$, $OR$, $QR$, $RP$, $PQ$ and passing through $L$ and $M$ is the nine-point circle $ABC$ and the points $L$, $M$, where $ABC$ is the medial triangle of $PQR$. Now in the system of line cubics there are four degenerate cubics each consisting of a vertex of the quadrangle $OPQR$, and the line conic touching the other three lines not passing through that vertex. Hence there are sixteen conics through the intersections $L$, $M$ of the circumcircle and the conjugate circle and inscribed to the four triangles formed by the four points $O$, $P$, $Q$, $R$, and each of these conics touches the nine-point circle.

3823 [1937, 179]. *Proposed by J. R. Musselman, Western Reserve University.*

Given any triangle $T_1T_2T_3$, let us denote by $T_1'$, $T_2'$, $T_3'$ the reflections of $T_1$, $T_2$, and $T_3$ in any diameter of the circumcircle. The lines through $T_i$ perpendicular to $T_j'T_k'$ meet at a point $R_t$, similarly the lines through $T_i'$ perpendicular to $T_jT_k$ meet at a point $M_t$. It is known that the points $R_t$ and $M_t$ also lie on the circumcircle and are symmetric with respect to the above mentioned diameter. Given three sets of three points $T_i$, $V_i$, $W_i$ on the same circle, locate with reference to some diameter the points $R_t$, $R_v$, $R_w$. Similarly for the triangles $T_1V_1W_1$, $T_2V_2W_2$ and $T_3V_3W_3$ find $R_1$, $R_2$ and $R_3$. Show that the triangles $R_1R_2R_3$ and $R_tR_vR_w$ have the same $R$ point. A like theorem can be stated for the $M$ point.

*Solution by C. E. Springer, University of Oklahoma.*

Let $x^2+y^2=a^2$ be the common circle of the sets of points $T_i$, $V_i$, $W_i$ with the diameter of reflection along the $x$-axis. If the coordinates of $T_i$, $V_i$, $W_i$ are $(a\cos\alpha_i, a\sin\alpha_i)$, $(a\cos\beta_i, a\sin\beta_i)$ and $(a\cos\gamma_i, a\sin\gamma_i)$, respectively, then the line through $T_1$ perpendicular to $T_2'$, $T_3'$ is

$$x\sin\tfrac{1}{2}(\alpha_2+\alpha_3)+y\cos\tfrac{1}{2}(\alpha_2+\alpha_3)=a\sin\tfrac{1}{2}(2\alpha_1+\alpha_2+\alpha_3)$$

and the line through $T_2$ perpendicular to $T_3'T_1'$ is

$$x\sin\tfrac{1}{2}(\alpha_3+\alpha_1)+y\cos\tfrac{1}{2}(\alpha_3+\alpha_1)=a\sin\tfrac{1}{2}(2\alpha_2+\alpha_3+\alpha_1).$$

The point $R_t$ common to these lines has coördinates $[-a\cos(\alpha_1+\alpha_2+\alpha_3), a\sin(\alpha_1+\alpha_2+\alpha_3)]$. Similarly, we find $R_v[-a\cos(\beta_1+\beta_2+\beta_3), a\sin(\beta_1+\beta_2+\beta_3)]$ and $R_w[-a\cos(\gamma_1+\gamma_2+\gamma_3), a\sin(\gamma_1+\gamma_2+\gamma_3)]$. Likewise, the coördinates of the $R$ points $R_i(i=1, 2, 3)$ are given by $[-a\cos(\alpha_i+\beta_i+\gamma_i), a\sin(\alpha_i+\beta_i+\gamma_i)]$, $(i=1, 2, 3)$, and the $R$ point of the triangle $R_1R_2R_3$ has coördinates $(-a\cos\theta, a\sin\theta]$ where $\theta=\alpha_1+\beta_1+\gamma_1+\alpha_2+\beta_2+\gamma_2+\alpha_3+\beta_3+\gamma_3$. The $R$ point of $R_tR_vR_w$ is also $[-a\cos\theta, a\sin\theta]$. By symmetry a like result holds for the $M$ point.

*Editorial Note.* It is possible to avoid the trigonometric reductions. For the $T$ triangle the angles of its vertices are $\alpha_1$, $\alpha_2$, $\alpha_3$, as in the above solution; and we may take for the vertices of the $T'$ triangle the angles $-\alpha_1$, $-\alpha_2$, $-\alpha_3$. Let the perpendicular from $T_i$ to $T_j'T_k'$, where $i$, $j$, $k$ are distinct, cut the circle again in $R_i$ with the angle $\theta_i$. The perpendicular from $O$, the center of the circumcircle $(O)$, to $T_j'T_k'$ has the angle $-(\alpha_j+\alpha_k)/2+m_1\pi$, while the perpendicular from $O$ to $T_iR_i$ has the angle $(\theta_i+\alpha_i)/2+m_2\pi$, where $m_1$, $m_2$ are positive or negative integers. Since the above two perpendiculars are also perpendicular, we have $(\theta_i+\alpha_i+\alpha_j+\alpha_k)/2+(m_2-m_1)\pi=(2k+1)\pi/2$, or $\theta_i=(2n+1)\pi-\sigma_\alpha$, $\sigma_\alpha=\alpha_1+\alpha_2+\alpha_3$. It suffices to take $\theta_i=\pi-\sigma_\alpha$. This proves that the three perpendiculars for $i=1, 2, 3$, meet in the same point $R_t$ on $(O)$ given by the angle $\pi-\sigma_\alpha$. The theorem for $M_t$ is then obvious by reflection. We obtain special cases by making two vertices of the $T$ triangle coincide.

Now let there be nine points on the circle $(O)$ with the angles $\alpha_i$, $i=1, 2, \cdots, 9$, which we take as distinct although this is not necessary, Separate the nine points into three sets of three points. We then have three triangles $\alpha_{i_1}$, $\alpha_{i_2}$, $\alpha_{i_3}$; $\alpha_{j_1}$, $\alpha_{j_2}$, $\alpha_{j_3}$; $\alpha_{k_1}$, $\alpha_{k_2}$, $\alpha_{k_3}$. The first triangle gives the point $R_i$, $\theta_i=\pi-\sigma_i$; the second, $R_j$, $\theta_j=\pi-\sigma_j$; the third, $R_k$, $\theta_k=\pi-\sigma_k$. The triangle $R_iR_jR_k$ gives a point $R$ with the angle $\theta=\pi-(\pi-\sigma_i)-(\pi-\sigma_j)-(\pi-\sigma_k)=\Sigma\alpha_i-2\pi$. Hence we take $\theta=\Sigma\alpha_i$. This proves that the 280 such systems lead to the same point $R$. Similarly for the $M$ point.

3824 [1937, 251]. *Proposed by F. A. Lewis, University of Alabama.*

Determine the roots of the characteristic equation of the matrix

$$V = (v_{rc}) = (\epsilon^{(r-1)(c-1)}) \qquad \text{where} \qquad \epsilon = e^{2\pi i/n}.$$

*Solution by J. S. Frame, Brown University.*

If $V$ is the matrix $\epsilon^{(r-1)(c-1)}$, and $\epsilon=e^{2\pi i/n}$ then

$$V^2 = \left(\sum_{s=1}^{n}\epsilon^{(s-1)(r+c-2)}\right) = \begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & n & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & n & \cdots & 0 & 0 \\ 0 & n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The roots of $(V^2-\lambda^2I)=0$ are $\pm n$, the sum being $2n$ if $n$ is even, and $n$ if $n$ is odd. Hence for $[n/2]+1$ roots $\lambda^2=n$, and for the rest $\lambda^2=-n$. The roots of the given matrix $V$ are $\sqrt{n}$, $i\sqrt{n}$, $-\sqrt{n}$, $-i\sqrt{n}$, and if these occur with the respective multiplicities $m_0$, $m_1$, $m_2$, and $m_3$, then we have

$$m_0+m_1+m_2+m_3 = n, \qquad m_0+m_2 = [n/2]+1.$$

According to the theory of Gauss sums, the trace of the matrix $V$ has the

value $(1+i)\sqrt{n}$, $\sqrt{n}$, 0, $i\sqrt{n}$, according as $n\equiv 0$, 1, 2, 3 (mod 4). Since this is also $(m_0+im_1-m_2-im_3)\sqrt{n}$, we find

$$\begin{aligned} m_0 - m_2 &= 1 \quad \text{when} \quad n = 4k \text{ or } 4k+1 \\ &= 0 \quad \text{when} \quad n = 4k+2 \text{ or } 4k+3, \\ m_1 - m_3 &= 1 \quad \text{when} \quad n = 4k \text{ or } 4k+3 \\ &= 0 \quad \text{when} \quad n = 4k+1 \text{ or } 4k+2. \end{aligned}$$

It follows that the multiplicities are given in all cases by

$$m_0 = \left[\frac{n}{4}\right] + 1, \quad m_1 = \left[\frac{n+1}{4}\right], \quad m_2 = \left[\frac{n+2}{4}\right], \quad m_3 = \left[\frac{n+3}{4}\right] - 1.$$

3825 [1937, 251]. *Proposed by H. T. R. Aude, Colgate University.*

A number, written in the scale of 10, has the digit $d$ ($d=2, 3, \cdots, 9$) in the position on the extreme right. A second number is formed by moving the digit $d$ to the position on the extreme left. If the second number is $d$ times the first, find the least number of digits $n$ of the numbers for the various values of $d$. Also find these eight numbers, or obtain a formula for them.

*Solution by Foster Brooks, Kent State University, Kent, Ohio.*

The eight numbers required may be obtained by direct multiplication. For instance if $d=2$, a simple multiplication problem is formed with the multiplicand blank except for the final digit 2, and with multiplier 2. Thus 4 is the last digit in the product, and hence next-to-the-last in the multiplicand. As each digit of the product is found it is transferred to the multiplicand and the process is continued until the digit 2 appears in the product with nothing to carry.

The results are given in the following table:

| $d$ | $n$ | number |
|---|---|---|
| 2 | 18 | 105, 263, 157, 894, 736, 842. |
| 3 | 28 | 1, 034, 482, 758, 620, 689, 655, 172, 413, 793. |
| 4 | 6 | 102, 564. |
| 5 | 42 | 102, 040, 816, 326, 530, 612, 244, 897, 959, 183, 673, 469, 387, 755. |
| 6 | 58 | 1, 016, 949, 152, 542, 372, 881, 355, 932, 203, 389, 830, 508, 474, 576, 271, 186, 440, 677, 966. |
| 7 | 22 | 1, 014, 492, 753, 623, 188, 405, 797. |
| 8 | 13 | 1, 012, 658, 227, 848. |
| 9 | 44 | 10, 112, 359, 550, 561, 797, 752, 808, 988, 764, 044, 943, 820, 224, 719. |

*Note.* The existence of solutions of this problem does not depend upon the use of the decimal scale of notation, or upon the fact that the second number is as many times the first as the digit moved.

Suppose it is required to find a number which when written in the system with radix $r$ ends in the digit $d$, and such that if this digit is moved to the position on the extreme left the new number is $m$ ($m\leqq d$) times the original. If $n$

## NEWS AND NOTICES

*Readers are invided to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

Professor Karl Menger of Notre Dame University has announced "a little publication to be issued annually *Reports of a Mathematical Colloquium*, second series, issue 1, a continuation of the *Ergebnisse eines mathematischen Kolloquiums*" which he formerly published in Vienna.

On April 28 mathematicians of the University of Wisconsin were guests of the mathematics department of Northwestern University. Dr. Ernst Hellinger, recently appointed lecturer at the latter institution, gave an address on "Types of Integrals." At a dinner meeting the guests of honor were Dean C. S. Slichter, Wisconsin (B.S., Northwestern, 1885), and Dean T. F. Holgate, Northwestern (A.B., Toronto, 1884). By report "the home team won a tennis aftermath."

Dr. W. D. Cairns, professor of mathematics at Oberlin College, will retire in June 1939 after forty years of service there. His duties as secretary-treasurer of the Mathematical Association of America will continue as heretofore, no change of address being involved.

At Princeton University Salomon Bochner, H. F. Bohnenblust, and Alonzo Church, have been promoted from assistant professorships to associate professorships.

Dr. N. E. Steenrod of Princeton has been appointed to an assistant professorship at the University of Chicago.

Dr. R. W. Wagner of the University of Wisconsin has been appointed instructor in mathematics at Oberlin College for the next academic year.

Professor N. C. Grimes of Grove City College died November 12, 1938.

### THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The following results of the second annual William Lowell Putnam Mathematical Competition held March 4, 1939, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the reader only by number up to the time of this announcement.

The first prize, five hundred dollars, is awarded to the department of mathematics of Brooklyn College, Brooklyn, New York. The members of the team were Richard Bellman, Abraham Hillman, Bernard Sherman; to each of these is awarded a prize of fifty dollars.

The second prize, three hundred dollars, is awarded to the department of mathematics at Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were R. P. Feynman, J. W. Follin, Jr., H. E. Singleton; to each of these is awarded a prize of thirty dollars.

The third prize, two hundred dollars, is awarded to the department of mathematics of Mississippi Woman's College, Hattiesburg, Mississippi. The members of the team were Nina P. Byrd, Mary E. Fancher, Ethel L. Tate; to each of these is awarded a prize of twenty dollars.

The five persons ranking highest in the examination, arranged in alphabetical order, were R. P. Feynman, Massachusetts Institute of Technology; Abraham Hillman, Brooklyn College; E. L. Kaplan, Carnegie Institute of Technology; William Nierenberg, College of the City of New York; Bernard Sherman, Brooklyn College. Each of these will receive a prize of fifty dollars. The order of the names in this list has no relation to their rank in the examination.

The following teams won honorable mention: department of mathematics, College of the City of New York, members of the team being Herbert Mintzer, William Nierenberg, Harry Soodak; department of mathematics, Cooper Union Institute of Technology New York, members of the team being Theodore Berlin, Benjamin Lax, Samuel Manson; department of mathematics, University of California, Berkeley, members of the team being W. M. Kincaid, C. W. Lippmann, S. A. Schaaf.

Seven individuals are given honorable mention because of ties in the scores in this group. The seven names, arranged alphabetically, are: Richmond Albert, Brooklyn College; Richard Bellman, Brooklyn College; Theodore Berlin, Cooper Union Institute of Technology; Benjamin Lax, Cooper Union Institute of Technology; C. W. Lippmann, University of California; M. J. Norris, College of St. Thomas; T. S. Schreiber, Johns Hopkins University.

The order of the names in both lists for honorable mention has no relation to their rank in the examination.

The following is a list of all colleges and universities which entered teams in the Competition. (This list is arranged alphabetically, and the order of the names here has no relation to the rank of the teams in the examination.) Antioch College, Bowdoin College, Brooklyn College, Carleton College, College of St. Catherine, College of the City of New York, Columbia University, Cooper Union Institute of Technology, Cornell College (Mt. Vernon, Iowa), Harvard University, Heidelberg College, Iowa State College (Ames), Kent State University, Knox College, Lafayette College, Lehigh University, Massachusetts Institute of Technology, Michigan College of Mining and Technology, Middlebury College, Mississippi Woman's College, Mount Saint Scholastica College, Mundelein College, Washington Square College of New York University, Park College, Pomona College, Queen's University, Rutgers University, Union College (Schenectady, New York), the universities of British Columbia, Buffalo, California, California at Los Angeles, Tampa, Virginia, Washington, Wisconsin; Virginia Military Institute, Washington and Jefferson College, Webb Institute of Naval Architecture, Western Reserve University.

In addition to these forty-one teams, there were seventy-seven individual contestants from these and twenty-eight other institutions, making a total of two hundred individuals representing sixty-nine institutions.

W. D. CAIRNS, *Secretary-Treasurer*

# The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1938 to Professor G. T. Whyburn for his article "On the structure of continua" published in the *Bulletin of the American Mathematical Society,* volume 42, 1936, pp. 49-73.

As determined more recently by the Trustees, the prize is to be awarded for a noteworthy expository paper. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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# MONTHLY

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# The AMERICAN MATHEMATICAL MONTHLY

*EDITED BY*

ELTON JAMES MOULTON, Editor-in-Chief
WALTER BUCKINGHAM CARVER
AUBREY JOHN KEMPNER

*WITH THE CO-OPERATION OF*




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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.


## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY, Murray, April 28–29.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13.
MICHIGAN, Ann Arbor, March 18.
MINNESOTA, Northfield, May 13.
MISSOURI, Springfield, April 28.
NEBRASKA, Lincoln, May 5.
NORTHERN CALIFORNIA, San Francisco, January 28.
OHIO, Columbus, April 8.
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 2–3.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.


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## MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-two persons have been elected to membership in the Association on applications duly certified:

C. K. ALEXANDER, Ph.D.(Calif. Inst. of Tech.) Asst. Prof., Acting Chm. of Dept., Occidental Coll., Los Angeles, Calif.

R. C. BLACKWELL, A.M.(North Carolina) Grad. student, Univ. of North Carolina, Chapel Hill, N.C.

A. O. BOATMAN, A.M. (Indiana), Prof., Carthage College, Carthage, Ill.

W. C. BOYRER, M.E., M.M.E.(Cornell) Accounting Engr., Retired, Consolidated Edison Co., Charleston, S.C.

W. B. BROWN, Ph.D.(Ohio State) Prof., Mississippi Woman's Coll., Hattiesburg, Miss.

A. H. DIAMOND, Ph.D.(California). Prof., Head of Dept., Oklahoma A. and M. Coll., Stillwater, Okla.

R. T. DONNELL, A.M.(Vanderbilt) Head of Dept., Cumberland Univ., Lebanon, Tenn.

Rev. W. G. DOYLE, M.S.(Catholic Univ.) Teacher, Bishop England High School, Charleston, S.C.

H. T. FLEDDERMANN, M.S.(Louisiana State) Asst. Prof., Loyola Univ., New Orleans, La.

R. A. GOODPASTURE, B.S. in C.E.(Colo. State Coll.) Jr. Engr., U.S. Bureau of Reclamation, Denver, Colo.

P. W. HALLETT, B.A.(Sydney Univ.) Deputy Headmaster, and Math. Master, Sydney Boys High School, Moore Park, Surry Hills, N.S.W., Australia

E. E. HEMENOVER, M.S.(Wyoming) Prof., Engineering, Pueblo Jr. Coll., Pueblo, Colo.

WILFRED KAPLAN, A.M.(Harvard) Teaching fellow, Rice Inst., Houston, Texas

J. L. KELLEY, A.M.(U.C.L.A.) Instr., Univ. of Virginia, Charlottesville, Va.

GASPERINE MILO, B.S.(New River State Coll.) Substitute Teacher, New River State College, Montgomery, W. Va.

ABBA V. Newton, Ph.D.(Chicago) Prof., Hartwick Coll., Oneonta, N.Y.

Rev. C. W. O'HARA, M.S. Prof., Math. and Physics, Heythrop Coll., Chipping Norton, Oxon, England

C. C. OURSLER, A.B.(Indiana) Prin., High School, Lancaster, Ill.

W. T. SCOTT, Ph.D.(Rice) Instr., Armour Inst. of Tech., Chicago, Ill.

C. L. SEEBECK, JR., A.M.(Harvard) Fellow, Univ. of North Carolina, Chapel Hill, N.C.

HARRY SILLER, M.S.(New York Univ.) Jr. statistical clerk, Social Security Board, Washington, D.C.

C. U. WETZIG, A.M. (Texas) Instr., A. and M. Coll., Magnolia, Ark.

M. A. ZORN, Dr. res mat.(Hamburg) Asso. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

The Trustees have voted, 1, to hold the annual meeting of the Association in December 1940 at Louisiana State University, Baton Rouge, Louisiana, in conjunction with the annual meeting of the American Mathematical Society; 2, to appropriate the interest from the Chace Fund for a period of five years to the support of a proposed historical journal, a project which is in line with the interests of Chancellor A. B. Chace in his lifetime; 3, to guarantee $600 toward the expense of the International Congress of Mathematicians to be held at Cambridge, Massachusetts, in September 1940.

W. D. CAIRNS, *Secretary-Treasurer*

## THE SIXTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The sixteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Louisiana State University, Baton Rouge, March 3–4, 1939. Sessions were held on Friday afternoon and Saturday morning. On Friday evening a joint dinner with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was held. The chairman of the Section, Professor J. F. Thomson, presided at the Friday afternoon session and at the dinner. Vice-Chairman H. F. Schroeder presided at the Saturday morning session.

The attendance was about one hundred, including the following thirty-three members of the Association: A. A. Aucoin, W. G. Banks, Jr., E. T. Browne, H. E. Buchanan, D. S. Dearman, W. L. Duren, Jr., Virginia I. Felder, H. T. Fleddermann, Elizabeth Freas, F. C. Gentry, Charles Hopkins, H. T. Karnes, Dorothy McCoy, Janet McDonald, B. E. Mitchell, S. B. Murray, I. C. Nichols, Irene A. Nolan, Arthur Ollivier, R. L. O'Quinn, W. V. Parker, H. L. Quarles, F. A. Rickey, S. T. Sanders, S. T. Sanders Jr., H. F. Schroeder, P. C. Scott, C. D. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, Marelena White, M. C. Wicht.

The following officers were elected for the year 1939–40: Chairman, V. B. Temple, Louisiana College; Vice-Chairman for Louisiana, H. T. Fleddermann, Loyola University; Vice-Chairman for Mississippi, H. L. Quarles, University of Mississippi; Secretary, W. V. Parker, Louisiana State University. The meeting for 1940 is to be held at the University of Mississippi.

The Section was honored to have as guest speaker Professor E. T. Browne of the University of North Carolina. The two addresses which he gave contributed much to the value of the meeting. His first address on "Observations on the study and the teaching of mathematics" was given at the dinner Friday evening. At the Saturday morning session he spoke on "Quasi-$k$-commutative matrices."

The following are abstracts of Professor Browne's papers.

1. Professor Browne pointed out that in mathematics, probably even more than in other subjects, a capable teacher is necessary; and by a *capable* teacher is meant one who first of all has an adequate knowledge of his subject. Such knowledge is not gained merely by a course of study in school and college, but by constant reading on mathematics along with one's teaching. And further, if a teacher is to be successful he must see that his students really know how to study mathematics and that they learn how to make proper use of their memory. The speaker then mentioned three main criticisms that he had heard voiced concerning the study of mathematics: that mathematics is an exceptionally difficult subject except for the few who have a special talent for it; that mathematics is dead; and that it is of no use to the average man. He discussed these criticisms briefly, showing that none of them actually has any foundation in fact.

2. If $A$ and $B$ are two $n$-square matrices such that $AB-BA=0$, $A$ and $B$ are called *commutative*. If $AB-BA=C\neq 0$ where $CA-AC=0$, $CB-BC=0$, McCoy calls $A$ and $B$ *quasi-commutative*. This notion was generalized by Roth into what he calls mutually $k$-commutative matrices, which include the ordinary commutative matrices as mutually *one-commutative*, and McCoy's quasi-commutative matrices as mutually *two-commutative*. In this paper, Professor Browne suggested that a generalization be made in a somewhat different direction. Thus, let $AB-BA=C$ be *nilpotent*, and form the two sequences:

$$\begin{array}{ll} CA - AC = A_1, & CB - BC = B_1, \\ CA_1 - A_1C = A_2, & CB_1 - B_1C = B_2, \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ CA_{l-1} - A_{l-1}C = A_l, & CB_{k-1} - B_{k-1}C = B_k. \end{array}$$

Since $C$ is nilpotent, both of these sequences necessarily terminate. If $A_l$ and $B_k$, respectively, are the first matrices in these sequences which are zero, then if $k\geqq l$, we call $A$ and $B$ *quasi-k-commutative*. If $k=1$, it will be seen that $A$ and $B$ are quasi-commutative in the sense of McCoy, while if $C=0$, $A$ and $B$ are commutative. This paper is concerned with a study of pairs of quasi-$k$-commutative matrices $A$ and $B$.

The following program was given:

1. "Squares escribed and inscribed to a triangle" by Professor B. E. Mitchell, Millsaps College.

2. "A proof of the addition theorems of trigonometry" by Professor W. L. Duren, Jr., Tulane University.

3. "The summation of finite series" by Albert Farnell, Louisiana State University, introduced by the Secretary.

4. "The least super sphere of a set" by Professor H. T. Fleddermann, Loyola University.

5. "Cremona involutions determined by a pencil of surfaces," by Professor F. C. Gentry, Louisiana Polytechnic Institute.

6. "Equations with coefficients in a modular field" by Professor Charles Hopkins, Tulane University.

7. "Tchebycheff approximations for decreasing functions" by Professor C. D. Smith, Mississippi State College.

8. "A problem in minimum values" by Professor F. A. Rickey, Louisiana State University.

9. "Equation of heat in the fins of air-cooled engines" by Professor W. B. Brown, Mississippi Woman's College, introduced by the Secretary.

10. "A formula on installment buying" by Professor P. K. Smith, Louisiana Polytechnic Institute.

11. "Some unit and zero identities" by Professor V. B. Temple, Louisiana College.

12. "The cevian tetrahedron and some of its remarkable points" by M. C. Wicht, Louisiana State University.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles.

1. Defining an escribed square as one having two vertices on one side of a given triangle and one each on the other sides, Professor Mitchell proceeds to investigate the escribed squares to a triangle. In general there are six, associated one pair with each of the three sides of the triangle. The right triangle has only four distinct escribed squares, the two pairs associated with the legs having coalesced. If an altitude is equal to its associated side one of the escribed squares for that side becomes infinite. The relations of the squares to the triangle and to each other are many and varied: homothetic, harmonic, and quasi-harmonic. When the squares are considered in triplets the relationship leads to Brocard geometry.

2. The textbooks in trigonometry usually prove the addition theorems of $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ first for the case where $\alpha$ and $\beta$ are positive angles whose sum is acute. This is followed by induction to establish the formulas in general. In this paper Professor Duren gives a simple general proof without such an induction. The proof makes use of two coördinates systems and the general formula $A'B'=AB\cos\alpha$ for the projection of the directed line segment $AB$ upon a directed line.

3. Mr. Farnell gave a formula for the finite summation of series using repeated finite integration by which series whose terms are of the form $(ax+b)^{(m)}$, $\phi(x)$, $\alpha^x\phi(x)$, where $\phi(x)$ is a polynomial, can always be summed, and the forms $(ax+b)^{(-m)}$ and $U_x\phi(x)$ can be summed when the necessary number of integrations can be formed.

4. Professor Fleddermann showed there exists uniquely a least super sphere to every bounded set and derived some of the properties of this sphere.

5. A pencil of surfaces of order $N$ containing an $(N-2)$–fold line $d$ in its base can be made to determine a Cremona involution in space by putting its members in projective correspondence with the points of the line $d$. A point $P$ determines a member $S_N$ of the pencil which corresponds to a unique point $z$ of $d$. The line joining $P$ and $z$ meets $S_N$ again in $P'$ the inverse of $P$ in the involution. Professor Gentry discussed the particular features of this transformation which arise when the curve residual to $d$ in the base of the pencil becomes composite. It is found that, in addition to lines and conics, this curve may break up into two, three or four parts. Each conic reduces the order of the transformation by 1 and the number of fundamental curves of the second kind by 3. The only other effect of the degeneracy of the base curve is to change the configuration of the fundamental curves of the second kind.

6. Professor Hopkins discussed the equation $f(x)=0$ of degree $n$ with coefficients in a finite field $F_p$ and irreducible over $F_p$. The quotient-ring $F_p[x]/(f(x))$ is a field $K=F_p(\theta)$ of degree $n$ over $F_p$, in which not only $f(x)$, but every irreducible polynomial $\phi(x)$ in $F_p[x]$ of degree $n$, can be decomposed into the product of linear factors. For a fixed prime $p$ one can construct a field

$\Omega$—the 'smallest algebraically-closed extension of $F_p$—such that every equation, of arbitrary degree, with coefficients in $F_p$ has all its roots in $\Omega$.

7. Let $P_x$ be the probability that a variate taken at random from a distribution will deviate from the origin by an amount at least as great as $x$. Assume a frequency function $y=f(x)$ which increases from the origin to a point $[c\sigma, f(c\sigma)]$ and decreases beyond that point. Professor Smith showed that the corresponding probability function $y=P_x$ begins at (0, 1), is concave downward to the deviation $c\sigma$ and concave upward beyond that point. Using the chord as a means of approximating the value of $P_x$ beginning at (0, 1) and extending through $(c\sigma, P_{c\sigma})$ to a point near the distance $c\sigma/(1-P_{c\sigma})$, a former approximation $P_{2\sigma} \leqq .092$ is reduced to $P_{2\sigma} \leqq .056$.

8. Professor Rickey discussed the problem of determining the minimum time required to row a boat to shore and then walk to a designated point down the straight shore line. He showed that this time is not independent of the distance to the point as might be inferred from the results usually obtained in Calculus classes and discussed in detail the various cases involved.

9. Professor Brown computed the temperature distribution in the fins of an air-cooled engine, assumed to be losing heat to the air on two sides and one exposed end, the other end being attached to the engine cylinder. For a simple case, thin fin of rectangular section, this temperature is given by the equation

$$\theta = \theta_h \frac{\cosh a(x - w^1)}{\cosh aw^1}, \quad a = \sqrt{\frac{2q}{Kt}}$$

where $q$ = coefficient of heat transfer from the surface, $w$ = true width of fin, $t$ = fin thickness, $x$ = distance from the cylinder wall, $K$ = fin conductivity, $\theta$ = temperature of fin above the air, $\theta$ = temperature of cylinder wall above the air, $H$ = heat dissipated by the fin per unit time, $H_h$ = heat dissipated by equal area of wall surface per unit time, $w'$ = corrected fin width $= w + t/2$. If fin effectiveness is defined as the ratio of $H$ to $H_h$ then it was shown to be given by the equation $f = (\tanh aw')/aw'$. The function $(\tanh u)/u$ was discussed briefly.

10. The simple formula $M = 24C/(P(n+1))$ giving the approximate installment rate of interest is found in high school and college texts. Professor Smith showed this formula to be quite accurate, provided the number of monthly installments is not too great. He compared the simple formula to a formula for the installment interest rate derived in his paper. His formula is compact and easily applied. It is derived by finding the compound amounts of the overpaid interest for each month placing these portions of interest at compound interest at the nomial rate $j$ and compounded $m$ times annually.

11. Professor Temple referred to a Wronskian (see "Some functions analogous to the trigonometric and hyperbolic functions" by Temple, *The National Mathematics Magazine*, March, 1939) of order $n$ of solutions of $d^ny/dx^n + y = 0$ which is identically equal to unity, and which is analogous to the identity $\sin^2 x + \cos^2 x \equiv 1$ for $d^2y/dx^2 + y = 0$. He then stated some theorems of unit

identities composed of determinants of orders 2 and 3 where $n$ is a multiple of 2 or 3. He also gave theorems showing that there are zero identities which accompany these unit identities. Proofs of these theorems were not given.

12. Mr. Wicht presented an analytic method of poles and polars by which treatments of remarkable points of the tetrahedron and its cevian are simplified. The notion of linear dependence and one-dimensional homogeneous coordinates were used freely. By this method the theorems of Commandino and Mannheim were outlined. It was shown that the cross ratio of the corresponding vertices of any four consecutive cevians is $4/7$. Some of these results were proved synthetically by N. A. Court, this MONTHLY, 1936, p. 89.

W. V. PARKER, *Secretary*

## MEETING OF THE NORTHERN CALIFORNIA SECTION

A meeting to organize the Northern California Section of the Mathematical Association of America was held in the Galileo High School, San Francisco, on Saturday, January 28, 1939. A. L. McCarty of the San Francisco Junior College, acting chairman of the Section, presided. Sessions were held both in the morning and in the afternoon, and members and visitors lunched together during the recess.

The attendance at the two sessions was approximately sixty, including the following twelve members of the Association: H. M. Bacon, C. E. Corbin, G. C. Evans, Emma V. Hesse, R. D. James, Sophia H. Levy, A. L. McCarty, F. R. Morris, Falka G. Sturges, Gabriel Szegö, R. K. Wakerling, Harriet A. Welch, and four applicants for membership: T. J. Bass, Jr., Adeline M. Scandrett, Ethel Spearman, Ruth G. Sumner.

The program for the meeting and a proposed set of By-Laws had been prepared by a Joint Program and By-Laws Committee composed of G. C. Evans, University of California; A. L. McCarty, San Francisco Junior College; and Emma V. Hesse, University High School, Oakland. The proposed By-Laws were reported by Professor Evans and adopted by the Section, subject to the approval of the Trustees of the Association. Officers for the coming year were elected as follows: Chairman, A. L. McCarty, San Francisco Junior College; Vice-Chairman, Sophia H. Levy, University of California; Secretary-Treasurer, H. M. Bacon, Stanford University. On presentation of a request that the Section name a person to serve as associate editor of the California Journal of Secondary Education, the Secretary was instructed to cast a ballot for Mrs. Ruth G. Sumner, Oakland High School.

The main part of the program was given over to an address by Professor V. F. Lenzen of the department of physics, University of California. Four other papers were presented. The list of titles and speakers follows:

1. "Some adventures in teaching mathematics to freshmen" by Dr. M. J. Polissar, San Francisco Junior College.

2. "Physical geometry" by Professor V. F. Lenzen, University of California.

3. "Some qualitative properties of the solution of linear differential equations of the second order" by Professor Gabriel Szegö, Stanford University.

4. "Contemporary viewpoints in the teaching of plane geometry" by J. W. Hoge, University High School, Oakland.

5. "The place of mathematics in secondary education" by Adeline M. Scandrett, Mission High School, San Francisco.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Dr. Polissar sketched a course in problem-solving for freshmen in science courses, in which emphasis is placed on problems which are not stereotyped, but which require a certain amount of intuitive experimentation before a solution is found. Among other devices, puzzles are used, some strictly mathematical and and others requiring only logical analysis. Throughout the course consistent use is made of the slide-rule as a time-saver in solving the ordinary algebraic problems involving numerical data.

2. Professor Lenzen's paper appears in the present issue of this MONTHLY.

3. Professor Szegö called attention to two rather general methods which furnish information about the distribution of the zeros and magnitude of the extrema, respectively, of functions satisfying a linear and homogeneous differential equation of the second order. The first method is due to Sturm (1836), and the second to Sonin (1892). Both methods have been used recently rather frequently, sometimes with slight variations, though their possibilities have not been exhausted. A sketch of the underlying ideas was given and both methods were illustrated by the special case of Bessel's function $J_0(x)$.

4. Mr. Hoge enumerated some of the present day viewpoints of teachers of plane geometry concerning knowledge and use of basal propositions as the major objective of teaching plane geometry, a reduction in the number of required theorems, together with an increase in geometric originals and an increase in postulates, especially at the beginning of the course; integration of the mathematics of arithmetic, algebra, and trigonometry, with plane geometry; attention to patterns of teaching; practical applications used consistently and continuously; development of logical thinking; and teaching for transfer.

5. Miss Scandrett presented extracts from the Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The Place of Mathematics in Secondary Education." Education aims to develop well-rounded and adaptable individuals. The study of mathematics will develop in individuals the desirable characteristics that achieve these aims provided that the teaching methods are the best. After considering the mathematical needs of many people, the Commission presents a program for mathematics in grades seven to fourteen. To improve our teaching methods, to provide adequate guidance for our pupils, and to revise the curriculum, we must investigate and solve the problems now confronting us in the field of testing.

SOPHIA H. LEVY, *Secretary pro tem.*

# THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The nineteenth regular meeting of the Southern California Section of the Mathematical Association of America was held at Whittier College, Whittier, California, on Saturday, March 4, 1939. Professor W. M. Whyburn, chairman of the Section, presided.

The attendance was seventy, including the following thirty-four members of the Association: L. J. Adams, O. W. Albert, L. D. Ames, Harry Bateman, E. T. Bell, Jessie R. Campbell, P. H. Daus, D. C. Duncan, Harriet E. Glazier, C. G. Jaeger, G. R. Kaelin, G. R. Livingston, Ada McClellan, W. E. Mason, I. P. Maizlish, W. O. Mendenhall, A. D. Michal, P. M. Niersbach, Lena E. Reynolds, J. M. Robb, G. E. F. Sherwood, Marcus Skarstedt, C. E. Smith, D. V. Steed, T. Y. Thomas, W. I. Thompson, C. W. Trigg, S. E. Urner, Morgan Ward, L. E. Wear, D. E. Whelan, Jr., Mabel G. Whiting, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the coming year: Chairman, G. R. Livingston, San Diego State College; Vice-Chairman, O. W. Albert, University of Redlands; Program Committee, Harry Bateman, Chairman, D. C. Duncan, and the Secretary. The next meeting was tentatively scheduled to be held on March 2, 1940, at Compton Junior College.

The following six papers were read.

1. "Mathematics and the integrated curriculum" by Professor Marcus Skarstedt, Whittier College.

2. "Lattice theory and its applications" by R. P. Dilworth, California Institute of Technology, introduced by Professor Ward.

3. "The application of group theory to the normal vibrations of a cubic crystal" by Professor W. V. Houston, Professor of Physics, California Institute of Technology, introduced by Professor Ward.

4. "A new approach to the teaching of algebra and geometry in high schools" by P. M. Niersbach, Bell High School, Huntington Park.

5. "The fundamental sufficiency theorem in the calculus of variations" by Dr. F. A. Valentine, University of California at Los Angeles, introduced by Professor Daus.

6. "A note on the equation $x^y=y^x$" by Dr. F. A. Butter, Jr., University of Southern California, introduced by Professor Steed.

Abstracts of the papers follow, numbered in accordance with their place on the program.

1. To counteract the tendency in most small liberal arts colleges to eliminate mathematics as a required subject for graduation, various general courses called survey courses or integration courses or appreciation courses have been set up, and a number of textbooks for such courses have been written. These range in presentation and subject matter from the very simple to the almost profound. Professor Skarstedt outlined a special plan of integrated courses now in operation at Whittier College, and discussed especially the part devoted to mathe-

matics which is not so much concerned with teaching mathematics as teaching about mathematics. It is required of all freshmen, but the ordinary well-recognized courses in mathematics are offered for those whose main interests lie in mathematics and the physical sciences.

2. Mr. Dilworth began with a short historical account of the development of lattice theory, emphasizing the manner in which the lattice-theoretic ideas arose in number theory, set theory, geometry, logic, *etc.* Lattices with a multiplication were described and a short account was given of their applications to algebra, particularly to the Noether decomposition theorems for the ideals of a commutative ring.

3. Professor Houston illustrated the use that can be made of group theoretical considerations in analyzing the normal vibrations of a crystal in the case of a simple cubic lattice. The group under which the system is invariant can be made finite by the use of periodic boundary conditions. The invariant sub-group that contains the translations only can be completely reduced on the basis of plane waves of arbitrary polarization. These can be divided into sets of 144 members, and each set gives rise to a representation of the whole group of the crystal. In a number of special cases these representations can be reduced with respect to various sub-groups of rotations to such an extent that it is possible to determine the directions of the normal vibrations. In such cases the frequencies can be directly expressed in terms of the force constants. The group theoretical treatment makes it easy to distinguish those properties of the elastic spectrum which depend only on the symmetry of the lattice from those which depend on special properties of the force constants.

4. Mr. Niersbach described the work of G. R. Mirick of the Lincoln School, Teachers' College, Columbia University, who during the last five years has been experimenting in a new method of teaching algebra and geometry suggested by Dr. Veblen in a talk on the modern approach to elementary geometry made at the Rice Institute in 1932. He is working on a course in which algebra and geometry are taught simultaneously. The analytic method, which is so important in other branches of mathematics and science, is introduced. While it is a complete change from the traditional methods of Euclid, this method incorporates an understanding of the use of postulational thinking, the development of the power of analysis, and an appreciation of the nature of proof.

5. The fundamental sufficiency theorem in the calculus of variations can often be used to obtain a minimum for a definite integral in a prescribed region. Dr. Valentine described a method of prescribing the region so that it could be simply covered by a field of extremals. The construction was made for the case when the Euler equation has a first integral.

6. Dr. Butter considered the equation $x^y=y^x$, $(x>0, y>0)$. It was shown that if $0<x\leqq 1$, or if $x=e$, then $y=x$. On the other hand, if $0<x<e$, or if $x>e$, then two values of $y$ are determined: the one $y=x$; the other $y>e$, or $1<y<e$, respectively. It was shown that if $y\neq x$, then $y\to 1+0$ as $x\to +\infty$, and $y'(x)<0$. The graph of the equation was also given.

P. H. DAUS, *Secretary*

## THE APRIL MEETING OF THE OHIO SECTION

The twenty-fourth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 8, 1939, with a morning session, a dinner, and an afternoon session. Professor C. O. Williamson, chairman of the Section presided at these sessions. As a happy addition to the program Professor G. A. Bliss of the University of Chicago was guest-speaker.

Seventy persons registered attendance, including the following forty-one members of the Association: G. E. Albert, W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, H. M. Beatty, H. A. Bender, G. A. Bliss, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, O. E. Brown, R. S. Burington, Rufus Crane, Wayne Dancer, O. L. Dustheimer, T. M. Focke, N. A. Gilbert, B. C. Glover, E. M. Justin, L. C. Knight, A. C. Ladner, Jesse Pierce, H. L. Pollard, D. W. Pugsley, Tibor Radó, C. E. Rhodes, Hortense Rickard, R. F. Rinehart, N. S. Risley, S. A. Rowland, G. W. Starcher, H. E. Stelson, Otto Szász, C. F. Thomas, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, C. R. Wylie, Jr., C. H. Yeaton.

The following officers were elected for the coming year: Chairman, Wayne Dancer, University of Toledo; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, J. H. Weaver, Ohio State University; Member of Program Committee, I. A. Barnett, University of Cincinnati. It is expected that the next meeting will be held at the Ohio State University, April 4 or 6, 1940.

The following eight papers were read:

1. "Use of conformal mapping in shaping wing profiles" by Professor R. S. Burington, Case School of Applied Science.

2. "The semi-regular solids" by Professor Wayne Dancer, University of Toledo.

3. "Some properties of pairs of circular cubics associated with four points in a plane" by Professor M. G. Boyce, Western Reserve University.

4. "Extrema of $k$-symmetric functions of $n$ variables which are connected by $n-k+1$ $k$-symmetric relations" by Dr. R. F. Rinehart, Case School of Applied Science.

5. "A certain minimum problem associated with parallelograms" by Professor I. A. Barnett, University of Cincinnati, and Dr. Paul Pepper, University of Notre Dame, presented by Professor Barnett.

6. "An arithmetical property of the cosine function" by Dr. Otto Szàsz, University of Cincinnati.

7. "Why I teach mathematics" by the chairman of the Section, Professor C. O. Williamson, College of Wooster.

8. "The Hamilton-Jacobi theory in the calculus of variations and its sources" by Professor G. A. Bliss, University of Chicago, by invitation of the committee.

Abstracts of these papers follow:

1. Professor Burington discussed the application of conformal mapping to the problem of shaping airplane wing profiles, using slides to assist in the presentation. The paper was especially timely due to the fact that the growing improvement in the efficiency of airplanes has renewed interest in the question of shaping their contours mathematically. A brief summary of the airfoil theories of Joukowski, Karman-Trefftz, and Mises was given, together with a short treatment of the family of airfoils recently advanced by Piercy. A discussion of the advantages and limitations of these theories was given. The treatment was confined to the ideal two-dimensional case.

2. After defining the semi-regular solids, Professor Dancer explained the relationship between these figures and the five regular polyhedra. The two types, the facially regular and the vertically regular, were used to illustrate the principle of duality. The speaker showed why there can be no more than thirteen polyhedra in each of the two classes. The star polyhedra were also defined in terms of the simpler solids. Models of all the solids discussed were presented.

3. Two types of loci of a variable point $P$ relative to four fixed points $A$, $B$, $C$, $D$ were discussed by Professor Boyce. The first locus was defined by the condition that the angles $APB$ and $CPD$ be equal (modulo $\pi$) and the second that the ratios $AP/BP$ and $CP/DP$ be equal. These loci, both being circular cubics of the type called focal curves, have long been known and were extensively studied by Van Rees, Teixeira, and others. In this paper, however, the use of complex coördinates with the Argand diagram as a reference system yielded shorter proofs for many of the known properties and some new results, especially with reference to the mutual relationships of the two kinds of loci.

4. Dr. Rinehart discussed an extension of two well-known algebraic theorems: I. The function $x_1x_2 \cdots x_n$, where the variables are subject to the condition $x_1+x_2+ \cdots +x_n=K>0$ has a relative maximum at $x_1=x_2= \cdots =x_n$. II. The function $x_1+x_2+ \cdots +x_n$, where the variables are subject to the condition $x_1x_2 \cdots x_n=K>0$, has a relative minimum at $x_1=x_2= \cdots =x_n$. It was shown that these results are special cases of a very general theorem, a loose statement of which is: A function $f(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)$ which is symmetric in $x_1, \cdots, x_{k+1}$, where the variables satisfy the $n-k$ functional equations $V^{(i)}(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)=0$, $(i=1, \cdots, n-k)$, where each $V^{(i)}$ is symmetric in $x_1, \cdots, x_{k+1}$, has a relative maximum or minimum at $x_1=x_2= \cdots =x_{k+1}$, according as a certain determinant is negative or positive.

5. If a parallelogram with fixed sides $a$ and $b$ and included angle $\theta$ is given, there are an infinite number of parallelograms with one side of length $c$ that may be inscribed in the given parallelogram. Professor Barnett presented the problem of finding those inscribed parallelograms which have minimum perimeter. The problem reduces to finding the shortest distance from a point to an ellipse. A discussion of the dependence of this minimum perimeter on the quantities $a$, $b$, $\theta$, and $c$ was given. The particular cases where the given paral-

lelogram is a rectangle or a rhombus are of interest since in these cases the minimum inscribed parallelogram may be constructed with straight edge and compases only.

6. In a joint paper by Professor Szàsz and Professor Barnett, there arose the problem of determining all real rational values of $x$ for which $\cos \pi x$ is rational, or is a quadratic irrationality. With the non-essential restriction $0<x<1$, we find that the only rational values of $\cos \pi x$ are $0$, $\pm 1/2$, with $1/2$, $1/3$, $2/3$ as the corresponding values of $x$. In the irrational case, $\cos \pi x$ must be $\pm\sqrt{2}/2$, $\pm\sqrt{3}/2$, $(1\pm\sqrt{5})/4$, $(-1\pm\sqrt{5})4$, where the corresponding values of $x$ are respectively $1/4$, $3/4$, $1/6$, $5/6$, $1/5$, $3/5$, $2/5$, $4/5$. It is noticeable that $\cos \pi x$ is algebraic whenever $x$ is rational, and it is transcendental if $x$ is an algebraic irrationality.

7. Archimedes said "Give me a place on which to put my fulcrum, and I can move the earth." It was the thesis of Professor Williamson's paper that mathematics, set in the solid concrete of observed facts, is the fulcrum which Archimedes needed, and that by using observed data as the power on one end of the lever we are able to turn over and pry out new facts, and thus move the world. Newton moved the world of thought onto a new plane by this process. Our students need to know the principles of this process.

8. Professor Bliss's paper was concerned with a chapter in the calculus of variations which is called the Hamilton-Jacobi theory. It received this name because in the first half of the last century these two fertile-minded mathematicians were the most active in the formulation and application of the theory, especially in problems of dynamics. But the sources of the chapter lie in the beginnings of the calculus of variations in the first years of the eighteenth century and even earlier. Since then the theory has become a permanent and important part of classical dynamics, with applications especially in the perturbation theories of celestial mechanics. More recently it has greatly influenced the development of the calculus of variations and its applications to geometry as well as mechanics. The purpose of the present paper is to show how this evolution occurred, especially as a result of the comparison of problems in optics and dynamics. The paper was suggested by studies of recent formulations of the Hamilton-Jacobi theory for parametric problems in the calculus of variations by Carathéodory and Teach.

RUFUS CRANE, *Secretary*

## THE FALL MEETING OF THE MINNESOTA SECTION

A second meeting in 1938 of the Minnesota Section of the Mathematical Association of America was held at the University of Minnesota, Minneapolis, Minnesota, on October 28. This was a joint meeting with the Mathematics Section of the Minnesota Education Association.

The general meeting was attended by over two hundred representatives of the colleges and secondary schools of the state, including the following thirty-five members of the Association: Jacob Bearman, C. J. Blackall, W. E. Brooke, L. E. Bush, W. H. Bussey, E. J. Camp, S. Elizabeth Carlson, Sister M. Claudette, H. H. Dalaker, J. H. Daoust, H. C. T. Eggers, Margaret C. Eide, Gladys Gibbens, C. H. Gingrich, W. L. Hart, W. N. Herr, J. S. Hickman, Dunham Jackson, R. E. Langer, J. D. Novak, Lois E. Pollard, Inez Rundstrom, J. M. Rysgaard, R. B. Saunders, M. G. Scherberg, Ole Schey, C. Grace Shover, A. J. Strane, F. J. Taylor, H. P. Thielman, Ella Thorp, Robert Tucker, A. L. Underhill, K. W. Wegner, Marion A. Wilder.

The meeting started with a luncheon attended by one hundred forty-five persons. Following the luncheon, Dr. W. E. Peik, Dean of the College of Education, University of Minnesota, gave an address. At the general meeting in the afternoon Mr. W. B. Gundlach, Rochester High School, and Professor W. H. Bussey, University of Minnesota, presided.

The program consisted of the following three papers:

1. "The use of National Council publications as reference material for mathematics teachers" by Edith Woolsey, Sanford Junior High School, Minneapolis.

2. "People and mathematics" by Professor Dunham Jackson, University of Minnesota.

3. "Josiah Willard Gibbs, an American mathematician" by Professor R. E. Langer, University of Wisconsin.

A. L. UNDERHILL, *Secretary*

## THE ANNUAL MEETING OF THE TEXAS SECTION

The 1939 annual meeting of the Texas Section of the Mathematical Association of America was held in Abilene on Friday afternoon, March 31, and Saturday morning, April 1. Abilene Christian College, Hardin-Simmons University, and McMurray College acted cooperatively as hosts for the meeting. The Section chairman, Professor H. J. Ettlinger, presided at both sessions.

Among the fifty-two people attending the meeting were the following nineteen members of the Association: H. E. Bray, J. E. Burnam, Nat Edmonson, Jr., H. J. Ettlinger, C. A. Gilley, E. H. Hanson, E. R. Heineman, G. B. Huff, H. A. Luther, B. C. Moore, E. D. Mouzon, Jr., M. E. Mullins, C. A. Murray C. R. Sherer, F. W. Sparks, Jennie L. Tate, Earl Thomas, F. E. Ulrich, H. E. Woodward.

Those attending the meeting were the guests of the host schools at a dinner on Friday evening. At the business session the following officers were elected for the coming year: Chairman, E. D. Mouzon, Jr., Southern Methodist University; Vice-Chairman, J. E. Burnam, Hardin-Simmons University. The 1940 meeting was awarded to Southern Methodist University, Dallas, Texas. The fixing of the date for the 1940 meeting was left to the Section officers.

The following papers were read:

1. "The problem of type for Riemann surfaces" by Dr. F. E. Ulrich, The Rice Institute.

2. "On the consequences of the continuum hypothesis" by Walter Jennings, A. and M. College of Texas, introduced by Professor Edmonson.

3. "The evaluation of a binomial determinant" by Professor E. R. Heineman, Texas Technological College.

4. "A new approach to the Hermite polynomials" by Robert Greenwood, University of Texas, introduced by Professor Ettlinger.

5. "The development of the problem of Bolza in the calculus of variations" by Dr. C. P. Brady, Texas Technological College, introduced by Professor Sparks.

6. "The Bartky method of solving linear matrix equations" by Professor H. J. Ettlinger, University of Texas.

7. "Regular curve families in the plane" by Wilfred Kaplan, The Rice Institute, introduced by the Secretary.

8. "Computations for three dimensional seismographic shooting" by Dr. J. T. Hurt, A. and M. College of Texas, introduced by Professor Edmonson.

9. "Reducing the number of failures in freshman mathematics" by Professor C. A. Murray, West Texas State Teachers College.

Abstracts of some of these papers follow, numbered in accordance with their place on the program:

1. Dr. Ulrich discussed the deficiency relation of Nevanlinna for meromorphic functions and indicated its significance in the study of the Riemann surface on which the smooth plane is mapped by the function. He gave the function cos $z$ as an example of a function with positive total deficiency and positive total ramification. He formulated the problem of type and stated the theorems of Picard, Bloch and Gross. Finally he gave the criterion of Ahlfors for the parabolic type, and for certain classes of surfaces showed how by a suitable choice of metric this criterion can be used to determine sufficient conditions for the parabolic case.

3. Professor Heineman indicated a convenient method of expansion for the determinant whose $(i+1)$th row is $|a^{nl_i}, a^{(n-1)l_i} a^{(n-2)l_i} b^{2l_i}, \cdots, b^{nl_i}|$, where $i=0, 1, 2, \cdots, n-1$, and $l_{i+1}>l_i$. The value of this determinant was expressed in terms of $(b-a)$, $ab$, and the symmetric functions $F_r=\sum_{i=0}^{r} a^{r-i}b^i$.

4. Mr. Greenwood's paper presented material taken from the paper of the same title by E. U. Condon and Robert Greenwood, *Philosophical Magazine*, Series 7, vol. XXIV, 1937, page 281.

5. Bliss formulated the problem of Bolza and discussed necessary conditions; Hestenes was the first to formulate a condition of Mayer by the use of which a sufficiency proof could be made; following Hestenes's sufficiency theorems Bliss, Reid, Morse, and Hestenes proposed other useful forms of the Mayer condition. Dr. Brady in his Chicago thesis made use of the theory of the problem of Bolza to study the problem of minimizing a function of integrals, which as formulated by Dr. Brady is readily transformed into a problem of Bolza. However, an application of the sufficiency theorems for the latter problem yields only a restricted strong relative minimum for the former because the definitions of a strong relative minimum for the two problems are not equivalent. A satisfactory theorem concerning strong relative minima was deduced by the use of supplementary methods similar to those used by Hestenes for isoperimetric problems. The fact that the theory for the problem of Bolza failed to yield the results desired for the isoperimetric problem and for Dr. Brady's problem led Hestenes last fall to a reformulation of the problem of Bolza.

6. Professor Ettlinger discussed the matrix exponential functions obtained by Bartky. These methods, though completely independent of Heaviside's work, are nevertheless analogous to the latter's operational methods. In general these matrix exponential functions do not satisfy the ordinary exponential laws. The principal result presented in this paper is that, if the group of exponentials belonging to matrices having the same multiplicities for their characteristic numbers be put in a canonical form, the resulting exponentials obey the ordinary laws of exponents.

7. Mr. Kaplan considered families of Jordan curves $x=x(t)$, $y=y(t)$ in the plane such that (a) through each point of the plane passes exactly one curve, (b) locally the curves behave like parallel lines. He then established that each curve is open and tends to infinity in both directions. On the basis of the manner in which each triple of curves of the family divides the plane, the family can be pictured as an abstract system with two triadic order relations, termed a normal bracket system. To every such abstract system corresponds a curve family which generates it. Two curve families have the same normal bracket system if and only if one is a one-to-one continuous image of the other.

8. Dr. Hurt obtained equations for three dimensional seismograph reflection shooting, and put them into a form suitable for routine use. The resulting graphs and tables are universal in scope because the solutions are exact, no approximations are made beyond the initial assumptions, and they apply in any area since no knowledge of the variation of velocity with depth is assumed.

NAT EDMONSON, Jr., *Secretary*

# PHYSICAL GEOMETRY

V. F. LENZEN, University of California, Berkeley

**1. Introduction.** Prior to Einstein a distinction was usually made between geometry and physics. Geometry was viewed as a rational science which is independent of sensory experience; physics was known to be an empirical science based upon observation and experiment. The sharp separation between mathematics and physics may be illustrated by the sciences of kinematics and dynamics. In his *Principles of Mechanics*, which was published in 1905, Slate says, "In the first two chapters we shall be occupied with conceptions—Velocity and Acceleration—that rest entirely upon a mathematical basis. . . . If mechanics is taken to include kinematics also, as it frequently is, that part of the science which is physical and not geometrical must be specially distinguished. It is designated as Dynamics. The point should be watched at which the transition is · · · made by introducing experimental results into the framework of our science." The ideas expressed by Slate are characteristic of older books on mechanics. In the study of motion there was recognized the progression: geometry, the science of space; kinematics, the science of motion which was based upon the addition of time to space; dynamics or mechanics, which explained the motions of the material bodies in the physical world. Geometry and kinematics were viewed as mathematical sciences, dynamics or mechanics as a physical science. In the present paper I shall show how geometry and physics have been united in the science of physical geometry.

**2. Historical sketch.** Our discussion of the relation of geometry to physics may well be prefaced by a description of its subject matter. Geometry is frequently defined as the science of space, but what is space? One of the best answers to this question is given in Carnap's early monograph, *Der Raum* [1]. In this work he distinguishes between formal space, intuitional space, and physical space. Formal space is a system of general ordinal relations. The formal properties of the terms and relations of such a structure are determined by postulates. Formal or abstract space is the subject matter of abstract geometry. Intuitional space is the system of relations between spatial objects such as lines, surfaces, and volumes, the properties of which are apprehended in sense-perception or imagination. Intuitional space is especially considered in the Kantian philosophy of geometry. Physical space is the system of relations between the bodies and phenomena of the physical world and is the subject matter of physical geometry. It may be added that the distinction between topological, projective, and metrical properties applies equally to formal space, intuitional space, and physical space. The present discussion will find need only for abstract geometry and physical geometry.

The development of an understanding of the relation between geometry and physics may be credited principally to the theory of relativity. This theory initiated a program for the reduction of physics to geometry. The special theory

of relativity made it possible to express kinematics in terms of a four-dimensional space-time. In the general theory, space-time is viewed as a Riemannian continuum whose curvature is determined by matter. A free material particle describes a world line which is a geodesic of this continuum. The general theory of relativity thus reduces the physics of gravitation to geometry, and unified field theories have been constructed in order to reduce all physics to geometry. This geometrization of physics appears to have made it a branch of mathematics, to have freed it from dependence on experience. A unified mathematical representation of physical phenomena is offered, and this achievement has inspired Sir James Jeans to declare that God is to be conceived as a pure mathematician.

The reduction of physics to geometry requires, however, that geometry be exhibited as an empirical science. In so far as geometry can be applied to the physical world it is based upon observation and experiment. I shall represent geometry to be the most firmly established branch of physics. If physics is to be reduced to geometry, geometry must also be reduced to physics.

That the concept of physical geometry is a significant contribution may be shown by exhibiting historical philosophical interpretations of geometry. Geometry as a mathematical science was created by the ancient Greeks, but the raw materials for a geometry were fashioned by their predecessors, notably the Egyptians. The Egyptians had to make surveys of land in order to redetermine the marks of boundaries which had been washed away by the floods of the Nile. Hence they measured distances and lengths and discovered propositions that express the metrical relations of the elements of simple figures. The Egyptians thus discovered and used propositions of physical geometry. The Greeks organized such propositions into a deductive science; Euclid founded geometry upon axioms and postulates from which propositions may be derived as theorems. Euclidean geometry has furnished the classical model for science.

The Greeks created the deductive science of geometry and originated the view that geometry is a rational science which is independent of sensory experience. Thus Plato taught that the objects of science must be universal and permanent. The objects of perception are in a state of flux, and hence propositions about the world of experience are infected with uncertainty and relativity. He explained the possibility of rational science by the theory of a transcendent world of pure forms, or ideas, which can be known only by reason. Geometrical structures such as triangles and circles are pure forms which are to be distinguished from the crude perceptible triangles and circles in the world of sense-perception. Geometry is approximately applicable to experience because perceptible figures participate in the pure forms. The soul has direct knowledge of pure forms in a pre-earthly state of existence; perception through the senses stimulates recollection of the pure forms in which the objects of perception participate. In support of his theory that knowledge of geometrical figures is latent in the individual mind, Plato narrates how Socrates guides an uneducated slave boy step by step to the recognition of the truth of a proposition in ge-

ometry. Thus the Platonic philosophy of geometry interpreted the objects of geometry to be ideal entities which transcend ordinary experience.

Since the eighteenth century the theory of Kant has exerted a widespread influence. Kant started from the assumption that pure mathematics, which is exemplified by geometry, is *a priori* and therefore independent of experience. He propounded the question, how is pure mathematics possible? His answer as applied to geometry was that space is the *a priori* form of external intuition which is the condition of all perceptual experience. Geometrical figures are constructions in space and can be constructed in pure intuition independently of sensory experience. This theory provided a new foundation for the interpretation of geometry as the science of universal and necessary truths.

The Kantian theory dominated the philosophy of geometry during the nineteenth century. Geometrical figures were assumed to be constructed in pure intuition and analysis of such figures yielded the self-evident axioms of Euclidean geometry. In recent years the German philosopher Husserl has offered intuitions into the essence of geometrical structures as the foundation of geometry. Intuitional space which is referred to by Carnap, is an inheritance from Kant. During the nineteenth century, however, the non-Euclidean geometries were created and led to the development of new points of view. Helmholtz and others exhibited intuitive models of the non-Euclidean geometries, and thus shook the Kantian doctrine that intuition reveals physical space to be Euclidean. The study of foundations led to the abstract theory of geometry, according to which the propositions of geometry are blank forms devoid of empirical reference. The postulates of a geometry constitute an implicit definition of the fundamental concepts which express the properties of formal space. Geometrical theory is concerned with the deductive dependence of theorems upon postulates. Since postulates and theorems are devoid of empirical significance, the problem of their truth or falsity does not arise. A proposition in geometry becomes true or false only when a concrete interpretation is given to the concepts.

The criticism of the theory that pure intuition is the origin of geometry was accompanied by the development of the view that in so far as geometry can be used in physics, geometrical propositions express the positional relations of perceptible bodies. Gauss measured the angles of a physical triangle whose sides were light rays, in order to test whether or not the sum of the angles is equal to two right angles. Helmholtz [2] in an essay on the origin and significance of the axioms of geometry declared that these axioms describe the mechanical behavior of our most rigid bodies during motions. Riemann [3] in his famous essay on the hypotheses which constitute the foundations of geometry advanced the hypothesis that the metrical structure of physical space depends on the physical forces in it. Thus the question, is physical space Euclidean or non-Euclidean?, acquired significance. The significance of this question presupposes that the metrical structure of space is defined in terms of the positional relations of physical bodies or phenomena. The standpoints of Gauss, Helm-

holtz and Riemann eventually were realized in the contemporary concept of physical geometry which is exemplified in Einstein's relativistic theory of gravitation. Geometry, in so far as it is relevant to physics, is a physical science that is based upon observation and experiment.

**3. An operational theory. Synthetic treatment.** The function of physical geometry is to describe the properties of physical space. In preparation for an exposition of how physical geometry may be developed, it is desirable to set forth the elements of the problem. In agreement with Carnap, I distinguish data of experience, postulate of measure, and relational structure. Data of experience are the contact of two points at a specific time, the incidence of a point on a line, the inclusion of a body by a surface, and so forth. Perceptions of contact, or of coincidence, especially furnish the raw materials of geometry. But such data of experience are sufficient only for the topological structure of space. Projective properties require the determination of straight lines, and metrical properties require procedures for measuring length and angles.

Projective and metrical geometry are relative to definitions which are matters of convention. Carnap has clearly shown that it is possible to proceed in two ways. One may adopt a postulate of measure and then by observation determine the scheme of geometrical relations that describes the metrical structure of space. Experience determines whether physical space is Euclidean or non-Euclidean only if a standard of measure has been adopted. It has been traditional to adopt as standard of measure the distance between two points on a rigid body and to postulate that this distance is independent of position. As Carnap has pointed out, an alternative procedure is to postulate the scheme of geometrical relations and then determine from experience the standard of measure that is implied. The possibility of this procedure was especially emphasized by Poincaré, who declared that geometry is determined by conventional definitions. He contended that since Euclidean geometry is the simplest, convention will decree its continued employment for the description of physical phenomena. If light did not travel in straight lines, Euclidean geometry could still be used to formulate different laws of physics. The general theory of relativity, however, predicts a behavior of rigid bodies which makes it convenient to change the geometry rather than the standard of measure.

The foregoing discussion demonstrates that metrical physical geometry exemplifies the operational theory of physical concepts. This theory, which has been expounded notably by Bridgman [**4**], expresses the meaning of physical concepts in terms of operations. In order to measure a physical quantity it is necessary to control the conditions under which a quantity assumes a determinate value. The procedures of measurement require physical and mental operations that are performed in accordance with prescribed rules. The definition of a physical quantity is expressed by the description of the conditions and procedures of measurement. Consistent application of this operational theory leads to the interpretation of a physical quantity as a number assigned to a

physical property of bodies. Thus the definition of a physical quantity does not express an intuitive insight into an intrinsic essence of the quantity. Textbooks of physics have defined mass as the quantity of matter in a body, but this is only a verbal definition. A significant definition of mass must describe the procedure for measuring the mass of a body. The same point of view applies to physical geometry. Consider, for example, the concept of length. Some philosophers have declared that we have a direct perception of length which acquaints us with the meaning of the concept; this has been the basis for the concept of intuitional space. The operational theory, however, recognizes that length as a physical quantity depends on operations of measurement in terms of a standard. The operational nature of length is especially demonstrated in the special theory of relativity.

I have several times referred to a standard of measure as a basis for metrical physical geometry. This standard is based upon the properties of practically rigid bodies. I assume that we are acquainted with examples of such bodies: sticks, stones, and manufactured bodies, such as iron rods. In order to describe the properties of rigid bodies, let us suppose that two points have been made on such a body. A point will be a hole made by a pin or a dot with a pencil. The two points may be called a rigid point-pair and determine a stretch. Given two rigid point-pairs that may be placed alongside each other so that the points of one are in contact with the points of the other. If the rigid pairs are displaced together, the contacts are preserved. If one rigid pair is kept fixed and the other displaced and returned to its initial position, the contacts are restored. If a number of rigid point-pairs can be brought consecutively into contact with a specific pair, they can be brought into contact with one another. Stretches defined by rigid point-pairs in contact are said to be congruent. If it is postulated that the length of a stretch is independent of position, stretches at a distance may be defined to be congruent.

I shall now explain how metrical, physical geometry may be developed so as to describe the properties of physical space. Physical space may be defined as the system of positional relations of perceptible bodies and phenomena. Such positional relations may be investigated from the standpoint of topology, but I propose to study the metrical structure of space. For this purpose we adopt rigid point-pairs as standards of measure. Thus the metrical structure is determined by the positional relations of practically rigid bodies. Indeed, Einstein [5] has described space as the totality of possibilities of relative position of practically rigid bodies.

On investigating the properties of space it is necessary to specify a frame of reference relative to which rigid bodies are at rest or in motion. In elementary geometry a geometrical structure is ordinarily assumed to be at rest in a frame that is rigidly attached to the earth. As we shall see, however, the special theory of relativity has brought to light the relativity of space to a frame of reference.

The procedure in building physical geometry is exemplified by some ele-

mentary experiments which have been described by Carnap [**1,** p. 41]. Let us have given a standard body of which two points $A$, $B$ determine a standard stretch. Consider a physical surface such as the top of a desk.

(1) We discover that $A$ and $B$ and also $C$ and $D$ of the standard body may be brought simultaneously into contact with four points $A_1$, $B_1$, $C_1$, $D_1$, upon the surface. Repeated experiments demonstrate that whenever $A$, $B$, $C$ or $A$, $C$, $D$ or $B$, $C$, $D$ are in contact with their corresponding points, the fourth pair of points is in contact. The pair $A$, $B$ can be brought into contact with $B_1$ and $C_1$, with $C_1$ and $D_1$, and with $D_1$ and $B_1$. The conclusion is that with respect to the point-pair $(A, B)$ as a standard, $A_1$, $B_1$; $B_1$, $C_1$; $C_1$, $D_1$; $D_1$, $B_1$ are rigid point-pairs. From the first experiment one infers the rigidity of $C$, $D$ and further the rigidity of the set $A$, $B$, $C$, $D$.

(2) If $A$, $B$, $C$, $D$ are brought into contact with four other points of the surface, repeated experiments yield the same results as before, and therefore the other points constitute a rigid set of points. All sets of four points of the surface are demonstrated to be rigid, and hence the whole surface is rigid.

(3) In the first experiment it was found that the contact of three pairs chosen from $AA_1$, $BB_1$, $CC_1$, $DD_1$, involved that of the fourth, provided the fourth was not $CC_1$. We discover that while $A$, $B$, $D$ remain in contact with the corresponding points, an initial contact of $C$ with $C_1$ may be interrupted. We then declare that $A$, $B$, $C$, $D$ and $A_1$, $B_1$, $C_1$, $D_1$ have moved with respect to each other, and during the motion three pairs of points have remained in contact. This is the characteristic of a straight line. $A$, $B$, $D$ lie on a straight line and so do $A_1$, $B_1$, $D_1$. A straight line is thus defined by point-pairs that remain fixed with respect to a rigid frame during a rotation about the line.

(4) If we bring $A$ into contact with $A_1$ and simultaneously $B$ in contact with $B_1'$, $B_1''$, $\cdots$ one after another, it never occurs that $D$ is not in contact with a point of the surface $D_1'$, $D_1''$, $\cdots$. The points $A_1$, $B_1'$, $D_1'$, lie on a straight line, also $A_1$, $B_1''$, $D_1''$, and so forth.

(5) If the preceding experiment is performed with $A$ in contact with $A_2$, $A_3$, *etc.*, the same results are obtained. Thus from every point in the surface there extend straight lines in the surface in all directions, and hence the surface is judged to be a plane.

As a result of the preceding experiments we have learned how to recognize a straight line and a plane. In practice we test the straightness of a line by the physical law that light travels through a homogeneous medium in straight lines. Straight lines are exemplified by the edge of a solid, by a stretched cord, and by the path of a ray of light.

Our next task is to introduce the concept of distance or length. Suppose that we have given two stretches determined by rigid point-pairs $(A,B)$ and $(A_1,B_1)$ respectively, so that $A$ is in contact with $A_1$ and $B$ is in contact with $B_1$. As previously stated, the stretches are said to be congruent. The same length, or distance between their end points, is assigned to each of the congruent stretches. Congruence is directly tested when corresponding points are in con-

tact, but this test fails when the stretches are separated. However, we shall assign the same length, or distance, to the separated stretches. We thus adopt the fundamental postulate that the length of a stretch determined by a rigid point-pair, or the distance between the two points, is invariant in displacement. It is assumed, however, that the temperature remains constant. A standard stretch may be assigned the length one. The length of any straight line can then be determined with respect to our standard. We may measure the length of a line by counting the number of times that the standard can be laid off on the line, or by counting the number of equal stretches that may be placed end to end along the line.

The operational significance of the concept of length is especially exemplified by the special theory of relativity. I have already stated that space is associated with some frame of reference. A fundamental assumption of classical kinematics was that space is absolute, that is, the same for all frames of reference regardless of their state of motion. This means that the geometrical properties of figures were viewed as invariant under a transformation of the frame of reference. Thus the length of a rigid rod was postulated to be the same relative to frames of reference in relative motion with respect to one another. Indeed, it appears to be self evident that the length of a rod represents an intrinsic property which does not depend on the frame of reference. According to the operational theory, however, the concept of length is defined by the method of measurement, and in relativistic theory the result depends on the state of motion of the frame. If the frame is one in which the rod is at rest, an observer can measure the length of the rod in terms of a standard of length by placing a calibrated scale of length adjacent to the rod under investigation and observing the points on the scale that coincide with the end points of the rod. But in a frame relative to which the rod is moving, this procedure is not possible because of relative motion between the rod and the instrument of measurement. A possible procedure is to mark the simultaneous positions of the end points of the rod on the frame of reference. One may then at one's leisure measure the distance between the two points on the frame of reference with a scale at rest. Simultaneity, however, is relative to the frame of reference, and hence the outcome of measuring length is relative. In general, in the theory of relativity the geometrical structure of a body is relative to the frame. A configuration which is described as a circle from a frame relative to which it is at rest is described as an ellipse from a frame relative to which it is moving.

Let us now return to the problem of constructing a physical geometry for structures at rest in a selected frame of reference. We have a standard of length and methods for the recognition of straight lines and planes. We may verify the proposition that a straight line is shorter than an adjacent line between the same points; this proposition may be used to define a straight line. We may construct figures out of straight lines. The properties of a plane triangle may be used to determine the curvature of the plane; the curvature is zero, negative, or positive according as the sum of the angles is equal to, less than, or greater

than two right angles. The curvature of three mutually perpendicular planes at a point determines the curvature of space at that point.

By such procedures we build up a concept of metrical physical space. The positional relations of rigid bodies which determine the metrical structure of space are described by a geometry which is a branch of physics. Applied to the physical world of experience, our procedures yield the result that to the first approximation, at least, actual physical space is Euclidean. The sum of the angles of large triangles, the sides of which are the paths of light rays, is two right angles. It is possible to construct a Cartesian coördinate system out of equal rods. This means that out of a set of rods, the corresponding end points of which coincide when the rods are placed adjacent to one another, it is possible to construct a cubical lattice which is the physical realization of a Cartesian coördinate system.

The propositions that characterize the positional properties of configurations of rigid bodies are only approximately verified by experience on account of lack of precision in observation. In the development of geometry, the fiction of a precise observation is adopted and the propositions are interpreted to express definite relations between definite properties. This procedure makes it possible to study the deductive relations between propositions, and Euclidean geometry may then be founded on axioms which express the properties of a set of terms and relations. We may then transform these axioms into a set of postulates which implicitly define the formal properties of the objects of geometry and thereby obtain an abstract geometry. The structures of physical geometry then exemplify approximately the formal properties defined by the postulates. *In the passage from physical to abstract geometry it does not appear to be necessary to interpolate a science that is founded on pure intuition.*

**4. Analytic treatment.** In the foregoing discussion I have employed the synthetic method of building geometry, but one may use the analytic method. A Euclidean space is characterized by the fact that it admits a cubical lattice which will serve as a Cartesian coördinate system. The metrical structure of space is described by the formula which expresses the differential element of distance between two points in terms of the differences in the coördinates of the points. Thus Euclidean space admits a Cartesian coördinate system for which

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Curvilinear coördinates may also be used, but the formula for the line element in such coördinates can always be transformed to the Cartesian form.

On a curved surface it is impossible to extend a Cartesian coördinate system over a finite region. Accordingly one introduces Gaussian, that is, curvilinear, coördinates. The coördinate lines may be labelled $u_1 = \text{constant}$ and $u_2 = \text{constant}$. The position of a point on the surface is specified by giving its Gaussian coördinates $u_1$ and $u_2$. The distance between two points $u_1$, $u_2$ and $u_1 + du_1$, $u_2 + du_2$ is expressed by the formula

$$ds^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2 .$$

The $g$'s are function of $u_1$ and $u_2$ and are called the components of the fundamental metrical tensor. The measure of curvature of the surface is a function of the $g$'s and their derivatives.

In the classical accounts of differential geometry the curved surface is viewed as imbedded in a three-dimensional Euclidean space. The $g$'s are expressed as functions of the derivatives of the Cartesian coördinates with respect to the Gaussian coördinates on the surface $u_1$, $u_2$,

$$g_{ik} = \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_k} + \frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_k} + \frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_k} \cdot$$

A physicist, however, prefers an exposition of the immediate physical significance of the $g$'s. The discussion presupposes that in an infinitesimal region the surface may be assumed plane. I assume that we have a standard of length which is invariant during displacements on the surface. $ds$ is the length of the element of arc between the two points relative to the standard. $du_1$ and $du_2$ are increments of coördinates and have no immediate metrical significance. As we pass from $u_1$, $u_2$ to $u_1+du_1$, $u_2$, the distance $ds$ is related to the coördinate increment $du_1$ by $ds^2=g_{11}du_1^2$. Then $ds=\sqrt{g_{11}}\,du_1$. Thus $\sqrt{g_{11}}$ is the ratio of distance advanced to increment in coördinate $u_1$. For example, if $ds=\frac{1}{2}$ for $du_1=1$, $\sqrt{g_{11}}=\frac{1}{2}$. This means that if a unit of length is placed on the coördinate line $u_2=$constant, one half of the unit extends from the line $u_1$ to $u_1+1$. A similar explanation is given for $\sqrt{g_{22}}$. If $\theta$ is the angle between the coördinate lines, $g_{12}=g_{21}=\cos\theta\sqrt{g_{11}}\sqrt{g_{22}}$. The metrical structure of the surface is known if we determine the $g$'s for every point of the surface. Let us now consider the application of the methods of analytic geometry to physics.

**5. A metric for a geometry for physics.** Classical physics was founded on the assumption that physical space is Euclidean. This means that a set of equal rigid rods can be fitted together to form a cubical lattice of finite extent. The lines of the lattice may be used as the lines of a Cartesian coördinate system. Cartesian coördinates directly express the distance of a point from a coördinate plane and hence have direct physical significance. If Cartesian coördinates are symbolized by $x_1$, $x_2$, $x_3$, the metric of Euclidean space is expressed by the formula $ds^2=dx_1^2+dx_2^2+dx_3^2$. To the first approximation, at least, physical space is Euclidean, and this fact explains the universal application of Euclidean geometry in classical mechanics.

The special theory of relativity provided a basis for a four-dimensional space-time relational structure of events. In addition to three spatial coördinates $x_1$, $x_2$, $x_3$ there was introduced a fourth coördinate, the value of which is directly related to the time indicated by a clock. If $t$ is time indicated by a clock, we may define $x_4=ict$. Then

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

expresses the metric of space-time. $ds$, the invariant interval between two events, is thus expressed in terms of differences of spatial coördinates and the time.

The general theory of relativity assumes that space-time is a continuum characterized by a Riemannian metric. In a gravitational field the positional relations of rigid bodies do not satisfy the propositions of Euclidean geometry. It is not possible to build finite Cartesian lattices out of equal rigid rods. The rate of clocks is affected by a gravitational field. Hence the metrical structure of a space-time region containing a gravitational field cannot be expressed by the formula for $ds$ used in the special theory. The more general Riemannian formula

$$ds^2 = \sum g_{ik} dx_i dx_k$$

is necessary. The $g_{ik}$ have a physical significance that may be defined by a procedure similar to the one for the two-dimensional surface.

The theory that physical space-time is Riemannian raises the problem of how the standard of measure for $ds$ is set up at a particular space-time point. In ordinary space this is accomplished by bringing a standard of length to the point. But the interval of space-time contains spatial and temporal factors. The metrical evaluation of $ds$ may be made with the aid of the special theory of relativity. In a relatively small space-time region it is possible to select a frame of reference relative to which there is no gravitational field. A gravitational field is relative to a frame of reference and will vanish relative to a suitable accelerated frame. For example, there is no gravitational field relative to an elevator which is falling freely towards the surface of the earth. Relative to the frame with respect to which there is no field and in which the coördinates of an event are $x_1$, $x_2$, $x_3$, $x_4$, the interval between two events may be expressed by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2;$$

and $dx_1$, $dx_2$, $dx_3$, $dx_4$ may be determined by rigid rods and clocks as in special relativity, and hence the value of the corresponding $ds$ can be calculated.

The geometrical significance of the $g_{ik}$ is part of their physical significance. The $g_{ik}$ also have a dynamical significance, for they are the potentials of the gravitational field. The law of gravitation expresses a condition on the $g$'s and their derivatives. The fundamental law of motion is that a free particle describes a geodesic in curved space-time. In this sense physics is reduced to geometry, but geometry is a branch of physics.

**6. Summary.** This paper may be summarized by a restatement of the relation between physical geometry and abstract geometry. Typical propositions of Euclidean geometry may be formulated as generalizations from experiences of practically rigid bodies. Such laws are expressed in terms of quantities which may be determined within limits of precision. The next step is to assume that the propositions hold exactly for a set of objects, such as ideal rigid bodies. Propositions with a precisely defined content may be reduced to a set of axioms

from which theorems can be deduced. The status in reality of ideal objects is uncertain. Historically the attempt has been made to give them reality in a transcendent realm or to view them as constructions in pure intuition. The problem of the ontological status of the objects of geometry is avoided by eliminating the empirical reference of the concepts. The axioms then become postulates which implicitly define the formal properties of the objects of the concepts. Thus generalizations from experience become transformed into definitions. The self-evidence which has been attributed to the axioms of Euclidean geometry is founded on their status as definitions. The proposition that a straight line is the shortest distance between two points is self-evident in the sense that it may be used as the definition of a straight line.

Once we have the concept of abstract geometry, it is possible to create new abstract geometries and then seek physical interpretations of them. The interest in differential geometry stimulated by the general theory of relativity has resulted in the invention of non-Riemannian geometries. The geometry of Weyl, for example, is based upon the assumption that the standard of distance is a function of position. But such developments are beyond the scope of this paper.

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# UPPER LIMITS TO THE REAL ROOTS OF A REAL ALGEBRAIC EQUATION

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**1. Introduction.** There are in the literature a number of theorems concerning upper limits to the real roots of an algebraic equation with real coefficients. The more popular of these theorems are those which can be easily remembered and are easy to apply. In many cases, however, they give very high limits.

We shall give three theorems, the first due to Lagrange, the second due to Jean J. Bret, and the third to Newton. For proofs of these theorems the reader may refer to Dickson's *First Course in the Theory of Equations* for the first two theorems, and to Burnside and Panton's *Theory of Equations* for the third.

THEOREM I. *If in a real equation (an equation with real coefficients)*

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad (a_0 > 0),$$

*the first negative coefficient is preceded by k coefficients which are positive or zero,*

*and if $G$ denotes the greatest of the numerical values of the negative coefficients, then each real root is less than* $1+\sqrt[k]{G/a_0}$.

THEOREM II. *If, in a real algebraic equation in which the coefficient of the highest power of the unknown is positive, the numerical value of each negative coefficient be divided by the sum of all the positive coefficients which precede it, the greatest quotient so obtained increased by unity is an upper limit to the roots.*

THEOREM III. *If the real function*

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

*and its first $n$ derivatives evaluated for $x$ equal to a positive number $h$ are all positive or zero, then $h$ is an upper limit to the real roots of the equation $f(x)=0$.*

It is the object of this paper to present two other theorems, believed to be new, which give lower limits for the roots than are given by Theorems I and II.

**2. Two theorems on limits of roots.**

THEOREM IV. *If, in an equation with real coefficients,*

$$\begin{aligned} f(x) \equiv\ & a_0x^n + a_1x^{n-1} + \cdots + a_{k-2}x^{n-(k-2)} + a_{k-1}x^{n-(k-1)} + a_kx^{n-k} \\ & + a_{k+1}x^{n-(k+1)} + \cdots + a_{n-1}x + a_n = 0, \qquad (a_0 > 0), \end{aligned} \tag{1}$$

*the first negative coefficient is preceded by $k$ coefficients which are positive or zero, and if $G$ denotes the greatest of the numerical values of the negative coefficients, then each real root greater than $h$ is less than $L_1$, where*

$$L_1 = 1 + \sqrt[k]{\Big(G + a_0(h-1)^k - (h-1)\sum_{s=0}^{k-1} a_sh^{(k-1)-s}\Big)/a_0}, \tag{2}$$

*and where $h$ satisfies the inequality*

$$G - (h-1)\sum_{s=0}^{k-1} a_sh^{(k-1)-s} > 0. \tag{3}$$

THEOREM V. *With the same hypotheses as in Theorem IV, for $x>2$ an upper limit to the real roots of $f(x)=0$ is $L_2$, where*

$$L_2 = 1 + \sqrt[k]{\Big(G - \sum_{s=0}^{k-1} a_s\Big)/a_0}, \tag{4}$$

*whenever the expression under the radical is greater than one.*

**3. Proof of Theorems IV and V.** For $x$ positive, $f(x)$ will remain unchanged or be reduced in value if we replace each coefficient after $a_{k-1}$ by $-G$. Hence

$$f(x) \geqq a_0x^n + a_1x^{n-1} + \cdots + a_{k-2}x^{n-(k-2)} + a_{k-1}x^{n-(k-1)} - G(x^{n-k} + \cdots + 1).$$

If $x \neq 1$, then

$$f(x) \geqq a_0x^n + a_1x^{n-1} + \cdots + a_{k-2}x^{n-(k-2)} + a_{k-1}x^{n-(k-1)} - G\,\frac{x^{n-(k-1)} - 1}{x - 1}\cdot$$

Then, dropping positive $G$ from the numerator, we have

$$f(x) > \frac{x^{n-(k-1)}\left[(a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-2}x + a_{k-1})(x - 1) - G\right]}{(x - 1)}\cdot$$

For $x > 1$, $f(x)$ will be greater than zero, and hence $x$ will not be a root if

$$(5) \qquad (a_0x^{k-1} + a_1x^{k-2} + \cdots + a_{k-2}x + a_{k-1})(x - 1) - G > 0.$$

From the identity

$$x^{k-1} \equiv x^{k-1} - (x - 1)^{k-1} + (x - 1)^{k-1}$$

it follows, for $x \geqq h \geqq 1$, that

$$(6) \qquad x^{k-1} \geqq h^{k-1} - (h - 1)^{k-1} + (x - 1)^{k-1};$$

and hence (5) will be verified if the following inequality holds:

$$(7) \qquad \begin{aligned} &(a_0h^{k-1} - a_0(h - 1)^{k-1} + a_0(x - 1)^{k-1} + a_1x^{k-2} + \cdots \\ &\qquad + a_{k-2}x + a_{k-1})(x - 1) - G > 0. \end{aligned}$$

For $x > h$, this will be true if we have

$$(8) \qquad \begin{aligned} &[(a_0h^{k-1} + a_1h^{k-2} + \cdots + a_{k-2}h + a_{k-1}) - a_0(h - 1)^{k-1}](h - 1) \\ &\qquad + a_0(x - 1)^k - G > 0. \end{aligned}$$

Thus, if inequality (8) holds, then inequality (5) holds, and finally $f(x)$ will be greater than zero. Hence an $x$ for which inequality (8) holds cannot be a real root.

Taking $h = 1$ in the inequality (8) gives Theorem I.

From inequality (8) it also follows that $x < L_1$, where $L_1$ is defined in (2). Since, in deriving inequality (6), $x$ was assumed to satisfy $x \geqq h \geqq 1$, $h$ must be chosen so that (3) is satisfied. Thus Theorem IV is established.

To prove Theorem V, put $h = 1$ in the bracket of inequality (8), and $h = 2$ in its multiplier, $h - 1$. This gives $x < L_2$, where $L_2$ is defined in (4), provided $x > 2$. The expression under the radical must be greater than one. This completes the proof of Theorem V.

Since inequality (3) is the reverse of (5), with a change of variable, it would appear at first glance that little has been accomplished. However, if the numbers $a_0, a_1, \cdots, a_{k-1}$, are such that a considerable difference exists between

$$a_0(h - 1)^k \qquad \text{and} \qquad (h - 1)\sum_{s=0}^{k-1} a_s h^{(k-1)-s},$$

then an $h$ may be readily chosen which will materially decrease the upper limit given by Theorem IV as compared with the limit given by Theorem I.

For example, an upper limit to the roots of the equation

$$x^5 + 10x^4 - 61x^3 + 1 = 0$$

is, by Theorem I,

$$1 + \sqrt[2]{\frac{61}{1}} = 1 + 7.8 = 8.8.$$

By Theorem IV, an upper limit is

$$1 + \sqrt[2]{\frac{61 + (h-1)^2 - (h-1)(h+10)}{1}}$$

For $h=5$, $61 > (h-1)(h+10)$; hence an upper limit is

$$1 + \sqrt[2]{1 + 4^2} = 1 + 4.1 = 5.1.$$

**4. Improvements on the limits given by Theorems II and III.** When $k=1$, Theorems I and II often give very poor limits for roots. Thus, for the equation

$$x^5 - x^4 - x^3 - 2x^2 - 3x - 256 = 0,$$

Theorem I gives for an upper limit

$$1 + \sqrt[1]{256} = 257,$$

and Theorem II gives

$$1 + \frac{256}{1} = 257.$$

To apply Theorem II more effectively put $x = my$ and apply Theorem II to the resulting equation in $y$. For this example we get the upper limit

$$x = m\left(1 + \frac{256}{m^5}\right) = m + \frac{256}{m^4}\cdot$$

We may in general choose by inspection an $m$ which will give a satisfactory upper limit. In this particular case we may apply the theory of maxima and minima to obtain a low limit; thus

$$\frac{dx}{dm} = 1 - \frac{1024}{m^5} = 0,$$

from which, $m=4$. Hence we obtain the limit

$$x = 4 + \frac{256}{256} = 5.$$

Finally, to improve Theorem III, when $k=1$ or $k=2$, and $a_1$ or $a_2$ is large and negative, we proceed from the well-known rule that the remainders obtained by

dividing $f(x)$ by $(x-h)$, the quotient thus obtained by $(x-h)$, and so on, are respectively $f'(h)/1, \cdots, f^n(h)/n!$ that is, they are the coefficients of $(x-h)^0$, $(x-h)'$, $\cdots$, $(x-h)^n$ when $f(x)$ is expanded in a Taylor Series in powers of $(x-h)$. Hence by Theorem III if these remainders are all positive or zero, then $h$ is an upper limit to the real roots of $f(x)=0$.

If $a_1$ is negative* then, if $a_0$ is positive, we try $h \geqq -a_1/n$, since, in repeated synthetic division by $x-h$, this will make the lowest entry in the second column (which equals $f^{(n-1)}(h)/(n-1)!$) positive or zero.

Thus for the equation

$$x^3 - 20x^2 + 164x - 400 = 0$$

we should choose $h \geqq 20/3$ or $h=7$. When repeated synthetic division by $x-7$ is carried out, we obtain the successive remainders 111, 31, 1, 1. Hence 7 is an upper limit to the roots. Note that Theorems I and II give 21 as an upper limit.

According to the remark made above, if all the remainders obtained by dividing $f(x)$ and the successive quotients by $x-h$, where $h \geqq -a_1/n$, are not positive, then certainly in the new equation

$$\frac{f^n(h)}{n!}(x-h)^n + \frac{f^{n-1}(h)}{(n-1)!}(x-h)^{n-1} + \cdots + \frac{f'(h)}{1}(x-h) + f(h) = 0 \tag{9}$$

it will be true that $k \geqq 2$.

Theorems II and IV applied to this equation may give relatively better limits than when applied to $f(x)=0$. Since the roots of the original equation have been diminished by $h$, the upper limit to the roots of $f(x)=0$ will be $h$ plus the upper limit to the roots of equation (9).

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## ON SOME GENERALIZATIONS OF CAUCHY'S CONDENSATION AND INTEGRAL TESTS

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Cauchy's celebrated condensation test (C-Test) in its most general form runs as follows:—

C-TEST. *Let $\phi(x)$ be a positive, single-valued function of $x$ which steadily decreases to zero as $x$ increases to infinity and $F(x) \equiv a^x\phi(a^x)$, $a$ being a real number greater than unity; then the two series $\sum_1^\infty \phi(n)$ and $\sum_1^\infty F(n)$ converge and diverge together.*

The proofs of the test which are usually given hold for integral values of $a$. Chrystal [1] gives an incomplete proof of the general case. Borel [2] and Brom-

* If $a_1$ is positive, but $a_2$ is negative, then $h$ should be taken to satisfy the relation

$$h \geqq \sqrt[2]{\frac{(a_2)\,(2!)\,(n-2!)}{a_0(n!)}}.$$

wich [3] dismiss it with the remark that the proof can be easily extended by taking the integral part of $a^n$ when $a>1$. The test in its general form was first proved by Kohn [4] and afterwards by Hill [5]. Recently Krishnaswamy Ayyangar [6] and Moritz [7] have given two independent proofs of this test. Yet another can be added in the lines of the proof of the test below.

Cauchy's Integral test may be stated as follows:—

I-TEST. *Under the conditions imposed on $\phi(x)$ in the C-Test, the series $\sum^\infty \phi(n)$ and the integral $\int^\infty \phi(x)dx$ converge and diverge together.*

A generalization of the C-Test is also known (see, *e.g.*, Knopp [8] and Fort [9], which in a more extended form can be stated as follows:—

*If $\phi(x)$ satisfies the restrictions of the C-Test, and $g_n$ is a positive increasing sequence tending to infinity with $n$, then*

(i) *$\sum_1^\infty \phi(n)$ converges if $\sum_1^\infty (g_{n+1}-g_n)\phi(g_n)$ converges,*

(ii) *$\sum_1^\infty \phi(n)$ diverges if $\sum_1^\infty (g_{n+1}-g_n)\phi(g_{n+1})$ diverges,*

(iii) *$\sum_1^\infty \phi(n)$, $\sum_1^\infty (g_{n+1}-g_n)\phi(g_n)$, $\sum_1^\infty (g_{n+1}-g_n)\phi(g_{n+1})$ converge and diverge together if the sequence $(g_{n+1}-g_n)$ is bounded, or if for every positive integer $k$, $g_{k+1}-g_k<\lambda(g_k-g_{k-1})$, where $\lambda$ is constant.*

*Proof:* Since $\phi(g_k)>\phi(x)>\phi(g_{k+1})$ for $g_{k+1}>x>g_k$ we have

$$(g_{k+1}-g_k)\phi(g_k) \geqq \int_{g_k}^{g_{k+1}} \phi(x)dx \geqq (g_{k+1}-g_k)\phi(g_{k+1}). \tag{1}$$

Adding these inequalities for $k=1, 2, \cdots, n$, we get

$$\sum_1^n (g_{k+1}-g_k)\phi(g_k) \geqq \int_{g_1}^{g_{n+1}} \phi(x)dx \geqq \sum_1^n (g_{k+1}-g_k)\phi(g_{k+1}).$$

From this, with the help of the I-test, follow the results (i) and (ii).

To prove (iii), suppose $g_n=g_{n-1}<\mu$ for every $n$, where $\mu$ is constant; then

$$\begin{aligned} 0 &< \sum_2^n (g_k-g_{k-1})\phi(g_{k-1}) - \sum_2^n (g_k-g_{k-1})\phi(g_k) \\ &\leqq \mu\sum_2^n \{\phi(g_{k-1})-\phi(g_k)\} \\ &= \mu\{\phi(g_1)-\phi(g_n)\} < \mu\phi(g_1). \end{aligned}$$

Thus the two series $\sum^\infty (g_n-g_{n-1})\phi(g_n)$ and $\sum^\infty (g_n-g_{n-1})\phi(g_{n-1})$ converge and diverge together. The result with the I-test proves the first part of (iii).

To prove the second part of (iii), we note that if for every $k$,

$$g_{k+1}-g_k \leqq \lambda(g_k-g_{k-1}),$$

the inequality (1) can be replaced by either of the following:

$$\lambda(g_k - g_{k-1})\phi(g_k) \geqq \int_{g_k}^{g_{k+1}} \phi(x)dx \geqq (g_{k+1} - g_k)\phi(g_{k+1}),$$

$$(g_{k+1} - g_k)\phi(g_k) \geqq \int_{g_k}^{g_{k+1}} \phi(x)dx \geqq \frac{1}{\lambda}(g_{k+2} - g_{k+1})\phi(g_{k+1}).$$

Adding these inequalities for $k=2, 3, \cdots, n$ and $k=1, 2, \cdots, n$, respectively, we get

$$\lambda\sum_{2}^{n} (g_k - g_{k-1})\phi(g_k) \geqq \int_{g_2}^{g_{n+1}} \phi(x)dx \geqq \sum_{3}^{n+1} (g_k - g_{k-1})\phi(g_k), \tag{2}$$

$$\sum_{1}^{n} (g_{k+1} - g_k)\phi(g_k) \geqq \int_{g_1}^{g_{n+1}} \phi(x)dx \geqq \frac{1}{\lambda}\sum_{2}^{n+1} (g_{k+1} - g_k)\phi(g_k). \tag{3}$$

These inequalities taken with the I-Test prove the second part of (iii).

Let $g(n)=a^n$ where $a>1$, the inequality (1) can be written as

$$(a - 1)a^k\phi(a^k) \geqq \int_{a^k}^{a^{k+1}} \phi(x)dx \geqq \frac{a-1}{a} a^{k+1}\phi(a^{k+1}); \tag{4}$$

whence by adding these inequalities for $k=1, 2, \cdots$, we get Cauchy's C-Test.

A generalization of the I-Test will now be deduced. Transform the integral in (4) by the substitution $x=a^t$, and obtain

$$(a - 1)a^k\phi(a^k) \geqq \int_{k}^{k+1} a^t\phi(a^t) \log a dt \geqq \frac{a-1}{a} a^{k+1}\phi(a^{k+1}).$$

The addition of these inequalities for $k=1, 2, \cdots$, proves the following generalization of the I-Test.

*If $\phi(x)$ satisfies the restrictions set forth in the C-Test and $F(x)=a^x\phi(a^x)$, then $\sum^\infty F(n)$ and $\int^\infty F(x)dx$ converge and diverge together.*

It may be pointed out that $F(x)$ does not necessarily satisfy the restrictions of the I-Test. The function $F(x)$ is not necessarily a decreasing or even monotonic function. Thus Cauchy's Integral test holds under less stringent conditions than are usually inposed on it.

Similarly from (2) and (3) the following result can be deduced:

*If (i) $\phi(x)$ satisfies the restrictions set forth in the C-Test, (ii) $g(n)$ is a positive increasing sequence tending to infinity with n, (iii) the sequence $(g_n-g_{n-1})$ is either bounded or $(g_{n-1}-g_n)\leqq\lambda(g_n-g_{n-1})$ for every n, where $\lambda$ is constant, (iv) $h(x)$ is a monotonic increasing function of x defined in the interval $(1, \infty)$ such that $h(n)=g(n)$ for all values of n and $h'(x)$ exists, then*

$$\sum^{\infty} (g_{n+1} - g_n)\phi(g_n), \quad \sum^{\infty} (g_{n+1} - g_n)\phi(g_{n+1}), \quad \int^{\infty} \phi(h(x))h'(x)dx$$

*converge and diverge together.*

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# ON LOCI ASSOCIATED WITH OSCULANTS AND PENOSCULANTS OF A PLANE CURVE

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**1. A general principle.** It is an elementary fact, well known for a long time, that the center of the circle of curvature of a plane curve lies on the envelope of the normal to the curve. Cesàro* showed that the center of the osculating conic lies on the envelope of the axis of aberrancy. It can be shown directly, by the method used in §2, that the focus of the osculating parabola lies on the envelope of the circle whose diameter is that half of the radius of curvature terminating at the point of contact. For the purpose of this paper, it is significant that the normal is the locus of centers of penosculating circles, the axis of aberrancy is the locus of centers of penosculating conics, and the circle mentioned in the previous sentence is the locus of foci of penosculating parabolas. A statement and proof of the general principle which the above examples illustrate follows.

Let $C$ be a plane curve possessing values for $\rho, \rho', \cdots, \rho^{(n-3)}$ at all of its points, where $\rho$ indicates the radius of curvature of $C$, and the superscripts indicate differentiation with respect to arc length $s$ measured along $C$. $C$ will be referred to as the base curve. Let $\phi$ denote a class of curves whose cartesian equation contains $n$ independent parameters. Suppose there exists at each point $M$ of $C$ a unique member of $\phi$, denoted by $\Gamma$, which has $n$-point contact with $C$ there, and a one parameter family $g$, consisting of those members of $\phi$ which have $(n-1)$-point contact with $C$ there. ($\Gamma$ is called the osculating member of $\phi$ and $g$ the family of penosculating members of $\phi$.) Let $P$ be a point associated with $\Gamma$, whose coördinates $X$ and $Y$ with respect to the reference frame consisting of the tangent and normal to $\Gamma$ at any point are functions of $\bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-3)}$, where these symbols represent the radius of curvature of $\Gamma$ and its successive derivatives as to arc length $\bar{s}$ on $\Gamma$, all of which symbols are functions of $\bar{s}$. The existence of the quantities just mentioned is postulated. Let points corresponding to $P$ for all members of $g$ be defined as those points whose coördinates are the same functions of $\bar{\rho}_i, \bar{\rho}_i', \cdots, \bar{\rho}_i^{(n-3)}$ as $X$ and $Y$ are of $\bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-3)}$,

* Nouvelles Annales de Mathématiques, vol. 9, 1890, p. 150.

where the $\bar{\rho}$'s with subscript $i$ are formed for the $i$th member of $g$, and the reference frame consists of the tangent and normal to this $i$th member of $g$. Suppose the points corresponding to $P$ form a locus $L$, whose cartesian equation referred to the tangent and normal to $C$ at $M$ is analytic, of the form

$$F(x, y, \rho, \rho', \cdots, \rho^{(n-4)}) = 0.$$

Then as $M$ moves along $C$, $P$ will ordinarily trace out a curve associated with $C$, and $L$ will determine an envelope. Under the conditions stated, we have the following theorem:

THEOREM: *$P$ will lie on the envelope of $L$.*

*Proof*: To show that $P$ lies on the envelope of $L$, we must show that $X$ and $Y$ satisfy both $F=0$ and

$$\frac{\delta F}{ds} \equiv \frac{\partial F}{\partial x}\left(\frac{y}{\rho} - 1\right) + \frac{\partial F}{\partial y}\left(-\frac{x}{\rho}\right) + \frac{\partial F}{\partial s} = 0$$

at a typical point $M$ on $C$, the reference frame being the tangent and normal common to $C$ and $\Gamma$ at $M$. At $M$, due to the fact that $\Gamma$ osculates $C$,

$$\bar{\rho} = \rho, \bar{\rho}' = \rho', \cdots, \bar{\rho}^{(n-3)} = \rho^{(n-3)}. \tag{1}$$

Furthermore, let us take $M$ as the origin for measuring both $s$ and $\bar{s}$. If $\Gamma$ were taken as a base curve, the equation of $L$ would be

$$\bar{F} \equiv F(x, y, \bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-4)}) = 0.$$

Since $X$ and $Y$ satisfy this equation for every value of $\bar{s}$, we see that

$$F(X, Y, \bar{\rho}, \bar{\rho}', \cdots, \bar{\rho}^{(n-4)}) = 0$$

is an identity in $\bar{s}$. Hence the derivative of the left member with respect to $\bar{s}$ will be zero for all values of $\bar{s}$. Therefore, $P$ being a fixed point with respect to a changing $\bar{s}$ so that

$$\frac{dX}{d\bar{s}} = \frac{Y}{\bar{\rho}} - 1 \quad \text{and} \quad \frac{dY}{d\bar{s}} = -\frac{X}{\bar{\rho}},$$

we have

$$\frac{\partial \bar{F}}{\partial X}\left(\frac{Y}{\bar{\rho}} - 1\right) + \frac{\partial \bar{F}}{\partial Y}\left(-\frac{X}{\bar{\rho}}\right) + \frac{\partial \bar{F}}{\partial \bar{s}} = 0$$

But at $M$, using (1) and the fact that $\bar{s}=s$, this last equation reduces to

$$\left[\frac{\partial F}{\partial x}\left(\frac{y}{\rho} - 1\right) + \frac{\partial F}{\partial y}\left(-\frac{x}{\rho}\right) + \frac{\partial F}{\partial s}\right]_{\substack{x=X\\y=Y}} = 0.$$

Hence at $M$, the coördinates of $P$ satisfy both $F=0$ and $\dfrac{\delta F}{ds}=0$. But $M$ is any point of $C$. Hence the theorem is proved.

**2. Illustrations.** The direct investigation of envelopes arising in two families of penosculants will show the significance of this principle in the theory of osculants and penosculants of a plane curve. Throughout, the coördinate reference frame will consist of the tangent and normal to the base curve at the point of contact.

The locus of the centers of the members of the family of penosculating equilateral hyperbolas defined at a point of a plane curve is a circle, tangent to the curve on its convex side, with a diameter equal to the radius of curvature of the base curve.* The equation of this circle is

$$x^2 + y^2 + \rho y = 0. \tag{2}$$

To get its envelope as the point of contact moves along the base curve, differentiate (2) as to $s$ and use the relations

$$\frac{dx}{ds} = \frac{y}{\rho} - 1 \qquad \text{and} \qquad \frac{dy}{ds} = -\frac{x}{\rho}\cdot$$

This leads to $3x - \rho' y = 0$. Solving this equation simultaneously with (2), we get $x = 0$, $y = 0$ and

$$x = -\frac{3\rho\rho'}{\rho'^2 + 9} \qquad y = -\frac{9\rho}{\rho'^2 + 9}\cdot$$

The first pair of values correspond to the base curve, and the second to the center of the osculating equilateral hyperbola. Thus, the envelope consists of the base curve and *the locus of the center of the osculating equilateral hyperbola*, illustrating the principle stated in §1.

The locus of the vertices of penosculating parabolas defined at a point of a plane curve is a tacnodal, circular quartic whose equation is

$$2(x^2 + y^2)^2 - 5\rho x^2 y + 4\rho y^3 + 2\rho^2 y^2 = 0.$$

A parametric representation—more convenient for our purposes—is

$$x = -\frac{\rho t(2 + t^2)}{2(1 + t^2)^2}, \qquad y = \frac{\rho t^2}{2(1 + t^2)^2}\cdot \tag{3}$$

To get the envelope of this quartic, we employ the following method. Compute the indicated derivatives from (3), and substitute in

$$\begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} - \dfrac{y}{\rho} + 1 \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial s} + \dfrac{x}{\rho} \end{vmatrix} = 0. \tag{4}$$

This leads to the equation

$$3t^7 - \rho' t^6 - 3t^5 + \rho' t^4 - 6t^3 + 2\rho' t^2 = 0.$$

* J. W. Lasley, Jr., Penosculating Conics of a Plane Curve, Bulletin of the American Mathematical Society, vol. 37, 1931, p. 77.

The roots of this equation are 0, 0, $\pm i$, $\pm\sqrt{2}$ and $\rho'/3$. These values when substituted for $t$ in (3) give the parametric equations of the various branches of the envelope of the quartic. The value $\rho'/3$ gives

$$x = -\frac{3\rho\rho'(\rho'^2 + 18)}{2(\rho'^2 + 9)^2}, \qquad y = \frac{9\rho\rho'^2}{2(\rho'^2 + 9)^2}.$$

But these are parametric equations of *the locus of the vertex of the osculating parabola.**

**3. A dual principle.** The general principle given in §1 can easily be dualized by a few appropriate changes in wording. The meaning of this dual will be clear from the following investigation of some geometry concerning families of penosculating equilateral hyperbolas, penosculating parabolas, and penosculating conics.

The asymptotes of the family of penosculating equilateral hyperbolas envelop a hypocycloid of three cusps formed by a circle of radius $\rho/2$ rolling in a circle of radius $3\rho/2$. A parametric representation for the hypocycloid is

$$x = 2\rho \sin 2\alpha \sin^2 \alpha, \qquad y = -\,2\rho \cos 2\alpha \cos^2 \alpha. \tag{5}$$

Applying to (5) the technique used in investigating the envelope of the locus of vertices of penosculating parabolas, we get as roots of the equation obtained from (4) 0°, 60°, 120°, 90°, and $\frac{1}{2}$ arc cot $(-\rho'/3)$. The first three correspond to the cusps, and hence must not be used in getting the envelope; the fourth value corresponds to the origin; and the last value gives

$$x = \frac{3\rho(\pm\sqrt{\rho'^2 + 9} + \rho')}{\rho'^2 + 9}, \qquad y = \frac{\rho\rho'(\pm\sqrt{\rho'^2 + 9} - \rho')}{\rho'^2 + 9}.$$

These can be shown to be the parametric equations of the envelopes of the asymptotes of the osculating equilateral hyperbola. That is, *a part of the envelope of* (5) *consists of the envelopes of the asymptotes of the osculating equilateral hyperbola.* The previous statement embodies an example of the dualized form of the principle of §1.

The tangents at the vertices of penosculating parabolas envelope a hypocycloid of three cusps formed by a circle of radius $\rho/8$ rolling in a circle of radius $3\rho/8$. Applying the method just used, we obtain as the parametric equations of one branch of the envelope of this hypocycloid

$$x = -\frac{27\rho\rho'}{(\rho'^2 + 9)^2}, \qquad y = \frac{\rho\rho'^2(9 - \rho'^2)}{2(\rho'^2 + 9)^2}.$$

This curve can be shown to be *the envelope of the tangent at the vertex of the osculating parabola.*

As a final illustration of the dual principle, the envelope of the axes of penosculating conics defined at a point of a curve is a parabola, known as Tran-

* A. Colucci, Sulle coniche osculatrici ad una data curva, Giornale di Matematiche di Battaglini, vol. 71, 1933, p. 166.

son's parabola, which has some interesting relations to the osculants considered in this paper.* An application of the principle shows that as the point of contact moves along the base curve, *the axes of the osculating conic are always tangent to the envelope of Transon's parabola.*

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## QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

*The department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

### A NEW PAIR OF AMICABLE NUMBERS

B. H. BROWN, Dartmouth College

There are 68 known pairs of amicable numbers, of which 41 are (both) even, and 27 (both) odd.† With three exceptions these pairs have the property that the smallest prime factor occurs to the same power in each of the numbers of the pair. The exceptions are the two pairs

$$2^3\cdot 19\cdot 41,\quad 2^5\cdot 199 \qquad \text{and} \qquad 2^3\cdot 41\cdot 467,\quad 2^5\cdot 19\cdot 233,$$

both due to Euler, and the pair

$$2^5\cdot 37,\quad 2\cdot 5\cdot 11^2,$$

due to Paganini.

In an effort to produce an odd pair of exceptional type, many simple cases were investigated, among which was

$$3^3\cdot 5\cdot 7\cdot p,\quad 3^m\cdot 5\cdot 7\cdot q,$$

where $m<3$, and $p$ and $q$ are primes. This leads immediately to the necessary and sufficient conditions $m=1$, $p=13$, $q=139$. Hence

$$12285 = 3^3\cdot 5\cdot 7\cdot 13,\quad 14595 = 3\cdot 5\cdot 7\cdot 139$$

are amicable.

I have further verified that of all amicable number pairs, the smaller number of which is less than 15,000, the Euler list is complete, except for the Paganini pair, and the pair added in this paper.

### A HISTORICALLY INTERESTING FORMULA FOR THE AREA OF A QUADRILATERAL

J. L. COOLIDGE, Harvard University

The amount of information available in the literature of mathematics bearing on the quadrilateral, the general quadrilateral, the cyclic quadrilateral, the

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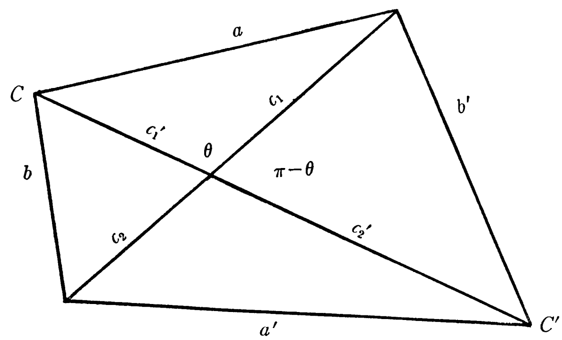
* See E. J. Wilczynski, Some Remarks on the Historical Development and the Future Prospects of the Differential Geometry of Plane Curves, Bulletin of the American Mathematical Society, vol. 22, 1916, p. 323, and J. W. Lasley, Jr., *loc. cit.*, p. 79.

† Dickson: *History of the Theory of Numbers*, 1919, vol. 1, chapter 1. A pair of numbers is said to be amicable if each is equal to the sum of the proper factors of the other.

complete quadrilateral, *etc.*, is discouragingly large. Yet there seems to be no part of the science so far from exhaustion as elementary geometry. It has seemed to me that there must be connecting links between different known formulas which were worth investigating, not only for their own sakes but also for historical reasons. Here is one which, so far as I can find out, is new, and which seems to me to come into that category.

Let the pairs of opposite sides of a quadrilateral be $aa'$, $bb'$ while the diagonals are $cc'$. The first diagonal shall be divided by the intersection into the parts




$c_1c_2$ while the second is divided into the parts $c_1' c_2'$. Let one angle of the diagonals be $\theta$, let the area be $A$, and finally let

$$2s = a + a' + b + b'.$$

Then

$$4A^2 = c^2c'^2 \sin^2 \theta,$$

and

$$a^2 = c_1^2 + c_1'^2 - 2c_1c_1' \cos \theta,$$

$$\frac{c_1c_1'}{2} = \frac{c_1^2 + c_1'^2 - a^2}{4 \cos \theta}.$$

Similarly,

$$\frac{c_2c_1'}{2} = -\frac{c_2^2 + c_1'^2 - b^2}{4 \cos \theta}$$

$$\frac{c_2c_2'}{2} = \frac{c_2^2 + c_2'^2 - a'^2}{4 \cos \theta},$$

$$\frac{c_1c_2'}{2} = -\frac{c_1^2 + c_2'^2 - b'^2}{4 \cos \theta}$$

Adding, we obtain

$$\frac{cc'}{2} = \frac{(b^2 + b'^2) - (a^2 + a'^2)}{4 \cos \theta}$$

Hence

$$A^2 = \frac{c^2c'^2}{4}[1 - \cos^2\theta]$$

$$= \frac{c^2c'^2}{4} - \frac{[(b^2+b'^2)-(a^2+a'^2)]^2}{16}$$

$$= -\frac{(a^4+a'^4+b^4+b'^4)+2(a^2+a'^2)(b^2+b'^2)-2(a^2a'^2+b^2b'^2)}{16} + \frac{c^2c'^2}{4}$$

$$= (s-a)(s-a')(s-b)(s-b') - \tfrac{1}{4}[a^2a'^2+b^2b'^2+2aa'bb'-c^2c'^2]\cdot$$

$$A^2 = (s-a)(s-a')(s-b)(s-b') - \tfrac{1}{4}[aa'+bb'+cc'][aa'+bb'-cc'].$$

Now a very famous theorem due to Ptolemy (*circa* 139 A.D.) tells us that a necessary and sufficient condition that the vertices of a quadrilateral should lie on a circle is that

$$aa' + bb' = cc'.$$

But in that case we have for the area of a cyclic quadrilateral

$$A = \sqrt{(s-a)(s-a')(s-b)(s-b')},$$

an almost equally famous formula due to Brahmagupta (*circa* 728 A.D.)

I should, perhaps, mention in this connection that in 1842 there appeared in Grunert, *Beiträge zur reinen und angewandten Mathematik*, vol. 2, two proofs by Bretschneider and Strehlke, both rather clumsy I think, of the formula

$$A^2 = (s-a)(s-a')(s-b)(s-b') - aa'bb'\cos^2\frac{C+C'}{2}\cdot$$

## ON THE GEOMETRY OF THE TRIANGLE

J. R. MUSSELMAN, Western Reserve University

In this note I wish to point out another application of some ideas presented in my article on The Line of Images (this MONTHLY, 1938, p. 421.) Using the same notation, let $T$ be any point on the circumcircle of triangle $A_1A_2A_3$. The line through $T$ parallel to its line of images is

$$Tx - \sigma_3\bar{x} = T^2 - \sigma_3/T. \tag{1}$$

This line cuts the side $A_2A_3$ in the point whose coördinate is

$$Z_1 = (T^2 - \sigma_3/T + t_1t_2 + t_1t_3)/(t_1 + T).$$

The line through $T'$, the diametrically opposite point of $T$, parallel to its line of images is

$$Tx + \sigma_3\bar{x} = -T^2 - \sigma_3/T. \tag{2}$$

This line cuts $A_2A_3$ in the point whose coördinate is

$$Z_2 = (T^2 + \sigma_3/T + t_1t_2 + t_1t_3)/(t_1 - T).$$

The midpoint of $Z_1Z_2$ has the coördinate

$$Z = \frac{(\sigma_2 + T^2)t_1}{t_1^2 - T^2}. \tag{3}$$

The coördinates of the midpoints of the segments cut on the sides $A_3A_1$ and $A_1A_2$ by the lines (1) and (2) may be written down from (3) by replacing $t_1$ by $t_2$ and $t_3$ respectively. These three points lie on a line whose equation is

$$(\sigma_3 + \sigma_1 T^2)x + \sigma_3(\sigma_2 + T^2)\bar{x} = 0. \tag{4}$$

The isogonal conjugate of the diameter $TOT'$ is the equilateral hyperbola

$$T^2x^2 + (\sigma_3 - \sigma_1T^2)x - \sigma_3^2\bar{x}^2 + (\sigma_2\sigma_3 - \sigma_3T^2)\bar{x} + \sigma_2T^2 - \sigma_1\sigma_3 = 0.$$

The tangent at the orthocenter $H$ of $A_1A_2A_3$ to the hyperbola is

$$(\sigma_3 + \sigma_1T^2)x - \sigma_3(\sigma_2 + T^2)\bar{x} = (\sigma_1^2 - \sigma_2)T^2 + \sigma_1\sigma_3 - \sigma_2^2.$$

Hence we have the theorem *that lines through the end points of any diameter $TOT'$ of the circumcircle of $A_1A_2A_3$ parallel to their respective lines of images will cut the sides of $A_1A_2A_3$ in segments whose midpoints lie on the line, passing through the circumcenter of $A_1A_2A_3$, which is perpendicular to the tangent, at the orthocenter of $A_1A_2A_3$, to the hyperbola isogonally conjugate to the diameter $TOT'$.*

The intersection of the lines (1) and (2) is the point

$$Z = -\sigma_3/T^2. \tag{5}$$

Hence *the lines through the endpoints of any diameter $TOT'$ of the circumcircle of $A_1A_2A_3$ parallel to their respective lines of images intersect on the circumcircle at that point which is the fourth intersection with the hyperbola isogonally conjugate to $TOT'$.*

The symmetric of (3) as to the midpoint of $A_2A_3$ is

$$d_1 = \frac{\sigma_3 + \sigma_1T^2}{T^2 - t_1^2}$$

Hence *the reciprocal transversal of the line* (4) *is the line $\Delta$ of the Droz-Farny Theorem.*

## NOTES ON CURVILINEAR MOTION IN THE PLANE

L. S. JOHNSTON, University of Detroit

Let $x=x(t)$, $y=y(t)$ describe the motion of a particle in the plane; let $v_x$, $v_y$, $v$, $\alpha_x$, $\alpha_y$, $\alpha$, $\alpha_t$, $\alpha_n$ have their conventional meanings; let $v_\rho$ and $\alpha_\rho$ be the radial components of $v$ and $\alpha$ respectively; let $v_\theta$ and $\alpha_\theta$ be the transverse components of $v$ and $\alpha$ respectively; let $C(x_c, y_c)$ be the center of curvature for the curve at any point $P(x, y)$, and let $R$ be the radius of curvature of the curve for any point

$P(x, y)$. The following formulas are easy to derive, easy to remember, and easy to apply:

$$\alpha_t = \frac{v_x\alpha_x + v_y\alpha_y}{v}, \qquad \alpha_n = \frac{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}{v},$$

$$v_\rho = \frac{xv_x + yv_y}{\rho}, \qquad v_\theta = \frac{\begin{vmatrix} x & y \\ v_x & v_y \end{vmatrix}}{\rho}, \qquad (\rho = \sqrt{x^2 + y^2})$$

$$\alpha_\rho = \frac{x\alpha_x + y\alpha_y}{\rho}, \qquad \alpha_\theta = \frac{\begin{vmatrix} x & y \\ \alpha_x & \alpha_y \end{vmatrix}}{\rho},$$

$$x_c = x - \frac{v^2 v_y}{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}, \qquad y_c = y - \frac{v^2 v_x}{\begin{vmatrix} v_y & v_x \\ \alpha_y & \alpha_x \end{vmatrix}},$$

$$R = \frac{v^3}{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}.$$

The reader will notice a certain symmetry running throughout this array which lends itself to easy memorization and use. The writer does not recall seeing these formulas set up in such manner in any text.

## A NOTE ON AN ALGEBRAIC IDENTITY

A. W. Boldyreff, University of Arizona

In his *Introduction to Mathematical Probability*, McGraw-Hill, 1937, p. 37, Professor J. V. Uspensky establishes indirectly by the principles of the theory of probability the following interesting identity for positive integers $a$ and $b$:

$$\text{(1)} \qquad \frac{a+b}{a} = 1 + \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots .$$

This identity can easily be shown to be a particular case of a much more general identity, from which an infinite number of other very interesting identities can be readily deduced.

Consider the obvious identity:

$$\text{(2)} \qquad \frac{b}{a} = \frac{b}{a+b+c}\left(1 + \frac{b+c}{a}\right).$$

Using (2) repeatedly, we deduce

$$
\begin{aligned}
\frac{a+b}{a} = 1 + \frac{b}{a} = 1 &+ \frac{b}{a+b+c} + \frac{b(b+c)}{(a+b+c)(a+b+2c)} + \cdots \\
&+ \frac{b(b+c)\cdots(b+\overline{n-1}\,c)}{(a+b+c)(a+b+2c)\cdots(a+b+nc)} \\
&+ \frac{b(b+c)\cdots(b+nc)}{a(a+b+c)\cdots(a+b+nc)},
\end{aligned} \tag{3}
$$

from which (1) is obtained by letting $c=-1$. The series (1) terminates of itself.

If in using (2) repeatedly we let $c$ take on successively the values $c_1, c_2-c_1, \cdots, c_n-c_{n-1}$, we obtain a more general result:

$$
\begin{aligned}
1 + \frac{b}{a} = 1 &+ \frac{b}{a+b+c_1} + \frac{b(b+c_1)}{(a+b+c_1)(a+b+c_2)} + \cdots \\
&+ \frac{b(b+c_1)\cdots(b+c_{n-1})}{(a+b+c_1)(a+b+c_2)\cdots(a+b+c_n)} \\
&+ \frac{b(b+c_1)\cdots(b+c_n)}{a(a+b+c_1)\cdots(a+b+c_n)}.
\end{aligned} \tag{4}
$$

(See J. A. Macdonald, "Note on the Summation of Finite Algebraic Series," *Mathematical Notes of the Edinburgh Mathematical Society*, no. 29, p. xiii, 1935, where a result equivalent to the above is established in a different way; and J. T. Hurt and L. R. Ford, "Polynomial expansions in the Borel region," *Proceedings of the Edinburgh Mathematical Society*, Series 2, vol. 5, part II, p. 82, 1937, where a similar procedure is used in expanding a rational fraction in a series.)

We note that by letting $c=0$ in (3) we deduce

$$
\frac{b}{a} = \frac{b}{a+b} + \frac{b^2}{(a+b)^2} + \frac{b^3}{(a+b)^3} + \cdots \tag{5}
$$

proving the well known fact that any rational fraction is the sum of an infinite geometric progression.

Again, by using two different sets of values for $c$'s in (4) we obtain:

$$
\begin{aligned}
1 &+ \frac{b}{a+b+c_1} + \frac{b(b+c_1)}{(a+b+c_1)(a+b+c_2)} + \cdots \\
&= 1 + \frac{b}{a+b+d_1} + \frac{b(b+d_1)}{(a+b+d_1)(a+b+d_2)} + \cdots
\end{aligned} \tag{6}
$$

It may be added that the above results can be generalized still further by using repeatedly instead of (2) the identity:

$$\frac{b}{a} = \frac{b}{f(a,\, b)}\left[1 + \frac{f(a,\, b) - a}{a}\right], \tag{7}$$

where either the same or a different function $f(a, b)$ is used in each successive application of the formula.

Finally, the identities (2) and (7) being valid for all values of $a$ and $b$, and not only for positive integral values, the results deduced from them will be, with proper restrictions, of very great generality.

Since the series in (3) is readily expressed as a hypergeometric series of the type $F(\alpha, 1, \gamma, 1)$, it can be summed using the formula:

$$F(\alpha, 1, \gamma, 1) = \frac{\gamma - 1}{\gamma - \alpha - 1}, \tag{8}$$

and other identities can be deduced from (3) using the well known transformations of hypergeometric series. Also (1), (3), (4), (5), and (6) can be established by the use of more intricate methods of summation of algebraic series (Chrystal, *Algebra*, vol. II, chapter XXXI, and Knopp, *Infinite Series*, English translation, chapter VIII).

In this case the most elementary and the most direct algebraic methods seem preferable.

*Note by the Editor.* Professor Boldyreff has not considered the question of convergence in case his series are infinite. It might be interesting to examine the series

$$\begin{aligned} &1 + \frac{b}{a + b + c_1} + \frac{b(b + c_1)}{(a + b + c_1)(a + b + c_2)} + \cdots \\ &+ \frac{b(b + c_1) \cdots (b + c_{n-1})}{(a + b + c_1)(a + b + c_2) \cdots (a + b + c_n)} + \cdots \end{aligned} \tag{4'}$$

for convergence. From (4) we see that the remainder in this expansion is

$$\begin{aligned} R_n &= \frac{b(b + c_1) \cdots (b + c_n)}{a(a + b + c_1) \cdots (a + b + c_n)} \\ &= \frac{b}{a} \cdot \frac{b + c_1}{a + b + c_1} \cdots \frac{b + c_n}{a + b + c_n}. \end{aligned}$$

Hence

$$R_n^{-1} = \frac{a}{b}\left(1 + \frac{a}{b + c_1}\right) \cdots \left(1 + \frac{a}{b + c_n}\right).$$

Hence (4′) will converge if and only if $R_n \to 0$; that is, if and only if $R_n^{-1} \to \infty$. If there is an $N$ such that $a/(b+c_n) \geqq 0$ for $n \geqq N$, a necessary and sufficient condition for this is the divergence of the series $\sum_1^\infty a/(b+c_n) = a\sum_1^\infty 1/(b+c_n)$. (Fine,

*College Algebra*, Ginn and Co., 1904, p. 566.) The series $\sum_1^\infty 1/(b+c_n)$ evidently diverges unless $|c_n|\to\infty$. If $|c_n|\to\infty$ there is an $M\geqq N$ such that $|c_n|>|b|$ for $n>M$. Then

$$2\left|\sum_M^\infty 1/c_n\right| = \left|\sum_M^\infty 1/(c_n/2)\right| > \left|\sum_M^\infty 1/(b+c_n)\right|$$

$$> \left|\sum_M^\infty 1/2c_n\right| = \frac{1}{2}\left|\sum_M^\infty 1/c_n\right|.$$

Therefore $\sum_1^\infty 1/(b+c_n)$ diverges if and only if $\sum_M^\infty 1/c_n$ diverges.

As a special case we find that the infinite series

$$(3') \qquad 1+\frac{b}{a+b+c}+\frac{b(b+c)}{(a+b+c)(a+b+2c)}+\cdots,$$

where $c_n=nc$, converges if $a$ and $c$ have the same sign.—R.J.W.

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## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 513 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

*Gewöhnliche Differentialgleichungen.* Dritte neubearbeitete Auflage. By Guido Hoheisel. Sammlung Göschen, Band 920, Berlin, Walter de Gruyter, 1938. 126 pages.

In addition to the usual treatment of elementary types of differential equations integrable by quadratures, this book treats such topics as the following: Existence and uniqueness theorems by the Picard method of successive approximations, integration of second order linear equations by series in the neighborhood of a regular singular point, and the Sturm-Liouville boundary value problem including a proof of the completeness of the systems of eigen functions and the absolute and uniform convergence of the corresponding series for functions having a derivative whose square is integrable over the interval in question.

The book is compact and reliable (save for a few obvious misprints) containing most of the standard material that an average student is likely to need. The proofs, though clear to the mature reader and in the simplest possible form, are perhaps too compact for the ready comprehension of a student meeting the subject for the first time. But accompanied by lectures and the companion volume (Sammlung Göschen, Band 1059) containing exercises, it should prove to be an excellent textbook.

D. C. LEWIS

Several of the chapters and parts of others are not only stimulating but a pleasure to read. This book should prove to be a very useful one.

B. W. JONES

*Mathematical Snapshots.* By H. Steinhaus. (Translated from Polish.) New York, G. E. Stechert and Company, 1938. 4+135 pages. $2.50.

*Mathematical Snapshots* is hardly to be classified under any of the usual categories. Indeed, the temptation is to make a new category—An Evening's Amusement—and to let this book be number one. The author uses pictures, figures, and models in addition to text to call attention to things of a mathematical or semi-mathematical character that are frequently overlooked or misunderstood. There is nothing like formal proof. The style is delightful, although there is an occasional lack of clarity, apparently introduced by translation. In fact, this book was more interesting to the reviewer than any book on so-called mathematical recreations that has ever come to his attention. Anyone, mathematician or what not, will enjoy turning its pages, looking at the pictures, and handling the models which are enclosed in a cover pocket, also reading the text. It is recommended to all, especially to those who are fond of puzzles and have a flair for mathematical recreations. The person who has never seriously studied mathematics, but who did well in high school mathematics and has a sneaking notion that he would have done well in more advanced work, will be delighted. Angle trisectors, circle squarers, and all such will find in it profitable channels for their abilities. Introduce them to it.

TOMLINSON FORT

*Trigonometry.* With Tables. By Howard K. Hughes and Glen T. Miller. New York, John Wiley and Sons, 1938. 8+189+79 pages. $2.00.

The first ninety-nine pages of this text are devoted to the solution of triangles and to those portions of the trigonometry which are required for this purpose. The trigonometric functions are defined first for the general angle, and the treatment of the reduction formulas is one which the student should have exceptionally little difficulty in mastering. The order of topics necessitates a geometric proof of the law of tangents, the usual proof being given later in the book. With the exception of one footnote, there is no mention of logarithms to bases other than ten.

Radian measure, trigonometric identities and equations, line values, graphs, and inverse functions occupy the next fifty-nine pages of the text. Noteworthy points in the section on identities are the emphasis on the restriction of the values of the unknown to those for which both members are defined, and the fact that the omission of the $\pm$ in the formula $\tan \frac{1}{2}x = \sin x/(1+\cos x)$ is justified.

There is no discussion of complex numbers, and the section on spherical trigonometry is very brief (19 pages). In the proof of the law of sines for spherical trigonometry, $AE$ is written for $CE$, and sin $a$ and sin $b$ are interchanged. There is one misprint in §110.

The smooth, cream-colored paper is exceptionally pleasing and free from glare, and the print throughout the book and the tables is excellent.

ETHEL MOODY

*Research and Statistical Methodology Books and Reviews of* 1933–1938. Edited by Oscar Krisen Buros. New Brunswick, Rutgers University Press (1938). 6+100 pp. $1.25.

This book is a reprint of a section of the 1938 *Mental Measurements Yearbook of the School of Education of Rutgers University*. It consists of excerpts from reviews of "practically all the research and statistical methodology books published between January 1, 1933 and November 15, 1938 and written in the English language," to quote the words of the editor. Certainly a wide range of topics is treated; the discussion includes such dissimilar items as a manual on how to write a thesis and Wilk's Lectures on Statistical Inference. For each book the usual bibliographical information is given, including both American and English prices.

The reviews quoted are taken from a wide variety of journals; for instance, Sasuly's *Trend Analysis of Statistics* is discussed by means of excerpts from reviews which appeared respectively in *Agricultural Economics Literature*, the *American Economic Review*, the *Catholic Charities Review*, the *Journal of the American Statistical Association*, the *Journal of Political Economy*, and the *Journal of the Royal Statistical Society*. No complete list of all publications consulted is given, although the book is otherwise well indexed. Usually in the case of a mathematical book, among the excerpts which are given are quotations from the reviews which appeared in the *Mathematical Gazette* and in the *Bulletin of the American Mathematical Society* or in this MONTHLY. It might be mentioned in this connection that in the discussion of the Sasuly book, a profitable reference could have been made to the reviews by A. E. Waugh in this MONTHLY, vol. 42, 1935, pp. 505–507, and by K. W. Halbert in the *Bulletin of the American Mathematical Society*, vol. 41, 1935, pp. 607–610.

But on the whole, the excerpts selected for each book seem to give a fairly accurate picture of the book. Especial emphasis is placed on evaluative statements which direct attention to modern methods in the literature. A stated aim of the book is to furnish a guide to beginners and teachers of introductory courses in statistics which will enable them to select textbooks which are abreast of the modern developments. It seems to the reviewer that the book should be quite valuable for this purpose. However, by limiting the scope to reviews only of books published in the English language, all reference to the important

recent French and German treatises on probability and statistics is omitted; and this may tend to curtail the usefulness of the book from the standpoint of the more advanced student. Perhaps it will be found possible in later supplements to give excerpts of reviews written in English of important books published in other languages. In any case, the job as it now stands is well done, and seems to be well worth doing.

J. H. CURTISS

*Statistical Methods*, 2d edition. By G. W. Snedecor. Ames, The Collegiate Press, 1938. 13+378 pages. $3.75.

A review of the first edition by this writer appeared on pages 614–615 of the November 1938 MONTHLY. In the second edition, occasional statements throughout the book have been modified and clarified. In the way of additions, there is some excellent advice in the use of ratios and percentages, contained in what are now sections 1.15, 5.9, 6.18, 10.16 and the last part of 12.7. The "missing plot technique" is now given in sections 11.5 and 11.6. A number of other new topics are introduced also in Chapter 11, and three sections on multiple regression with more than three variables in Chapter 13.

W. E. DEMING

*Geometrisieren im Bereiche wichtigster Kurvenformen*. By Dr. Louis Locher-Ernst. Zürich und Leipzig, Orell Füssli, 1938. 62 pages.

The material of this book, according to its preface, was first presented in a series of lectures to an audience of laymen at the Goetheanum, Freie Hochschule für Geisteswissenschaft, Dornach. With trivial exceptions, each paragraph can be classified as expository or interpretative.

The expository paragraphs contain an extremely lucid synthetic account of the curves traced by a point moving in such a way that its distances $(u, v)$ from two fixed points satisfy the equations: 1. $au+bv=\text{const}$, 2. $u^a v^b=\text{const}$, 3. $v=ae^{bu}$. In each case, by a process which amounts to using a parameter measuring the time, the author has generated the curves in a dynamic and vividly intuitive way, with close attention to the heuristic motivation of each stage. No mathematical knowledge is assumed beyond the rudiments of plane geometry. The results are tabulated in an appendix.

The interpretative paragraphs will interest anyone concerned with the pedagogy or philosophy of mathematics, but will perhaps be received most warmly by those most sympathetic to the doctrines of anthroposophy and eurythmics, which, by liberal quotation, the author associates with Rudolf Steiner. The reviewer is obliged to profess quasi-complete ignorance of these doctrines, and will merely attempt briefly to suggest his own understanding of the general significance attributed to mathematics by the author.*

* Many other speculative questions are mentioned in the course of the discussion, along with a number of biographical incidents.

The book is motivated by the impulse "to augment the light of thought by the warmth of experience, [and] to illuminate the warmth of immediate experience by the clarity of thought." This interaction is felt to occur most conspicuously in experience which is "übersinnlich," here translated "over-sensory"; the author's preference of this adjective to a more neutral one seems instructive. The meaning of "übersinnlich" is indicated by examples: "The assertion 'the sum of the interior angles of a triangle is two right angles' gives vocal expression to an oversensory fact." The essence of mathematics, of geometry in particular, is held to consist in the treatment of sensory facts as symbols for comprehensive oversensory facts; experience of the former may be presented directly to anyone, whereas experience of the latter can neither be given to one unable or unwilling to grasp it nor confiscated from one who already enjoys it.* Herein is felt to consist the happiness most appropriate to the reflective soul. For the encouragement of human aspiration to oversensory experience, mathematics is to furnish evidence of the possibility of such experience, along with a [low] standard of the clarity and intensity it may be hoped to have. The technical material is presented and interpreted in the light of this doctrine.

The interpretative portion of the book thus treats problems which have not yet been formulated with the sharpness to which mathematicians have become habituated. Appraisals of its value—in general and in detail—will therefore differ greatly among mathematicians. The views suggested (rather than formally maintained) have received little attention from mathematicians, and their full explanation would require a statement much more elaborate than the author has been permitted to give. In its technical sections, the book appears to have the merit of offering the layman an immediate example of mathematical "beauty" more dynamic than that of elementary geometry.

F. A. Ficken

* This sentence and the two following are very free translations of passages selected for their aptness in illustrating the general tenor of the book, in its interpretative portions.

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.*

## CLUB REPORTS 1937–38

### *Mathematics Club, University of Cincinnati*

The outstanding event of the calendar for the year was the testimonial dinner in honor of Professor Harris Hancock, who in June of 1937 retired from active teaching. As a permanent symbol of the esteem in which Professor Hancock is held, the Mathematics Club sponsored the painting of an oil portrait of him to be presented to the University.

During the year thirteen meetings were held, at which discussions centered around the following topics: Complex roots of equations, Hilbert's proof of the transcendence of $e$, A problem in dynamics, The cyclotomic equation, The harmonic oscillator in wave mechanics, Nomograms, Continuous geometry, A problem in physical chemistry, Geometric probability, A new type of sequence, Some parallelograms associated with a given parallelogram, Fermat's last theorem for the case $n=3$, Some properties of symmetric functions.

### *Mathematics Club, University of British Columbia*

In line with its purpose of being designed to acquaint its members with branches of mathematics not dealt with in lectures and regular courses and also to stimulate individual investigation, the following papers were presented and discussed: An introduction to algebra, Mathematics underlying *Alice in Wonderland*, Generalized coördinates, History of mathematics, Magic squares, Calculus of variations.

### *Pi Mu Epsilon, Pennsylvania State College*

Besides the various mathematics contests and social meetings sponsored by the chapter, the group centered their attention on the following problems presented for discussion: Mathematical solutions of an interesting problem, The history and evolution of $\pi$, The distribution of mass about an axis in a plane—about an axis in space, Elliptical harmonic motion, Newton versus Einstein in relativity, Mathematics in connection with vibrations, Tensor analysis.

### *Pi Mu Epsilon, University of Toledo*

The club entertained at their February and April meetings two guest speakers. Professor W. L. Ayres of the University of Michigan gave the talk at the initiation banquet in February on the subject, The four color problem. Mrs. Robert Ayers of the Toledo Museum of Art at the later meeting talked on the subject, Dynamic symmetry showing relationships between mathematics and art.

### *Junior Mathematical Club, University of Chicago*

Graduate students presented most of the talks at the regular meetings held throughout the year. The most interesting topics were: The geometric arches of E. H. Moore, Some identities of Cayley, Taylor and Laurent expansions, The Gamma function, Schwarz's lemma, Characteristic values for the third order linear homogeneous differential equation, Finite fields, Matrix approximation and the characteristic value problem, Historical trends in the Calculus of Variations, Convex and subharmonic functions, Moduls in quadratic fields and the composition of quadratic forms, Prime numbers. A joint meeting with the Physics Club was held, and informal discussions were encouraged at the customary afternoon tea preceding each meeting.

## *Pi Mu Epsilon, University of Georgia*

The book *Mathematics for the Million* served as the basis for the programs for most of the meetings throughout the year. Mathematics—the mirror of civilization, Mathematics in prehistory, The history of algebra, The history of arithmetic, Statistics, were discussed by students, and miscellaneous subjects were chosen for the other meetings. Some of the most interesting were: Relations between mathematics and philosophy, Newton's *Principia* in a modern age, Mathematics in medicine, Relations between mathematics and English grammar. Dr. Messick of Emory University was the guest speaker at the initiation banquet in January. He talked on the subject "Descartes."

## *Harvard Mathematical Club, Harvard University*

Many topics of interest to both undergraduate and graduate students of mathematics were discussed at the bi-weekly meetings of the club. Topics of lectures were: Harvard mathematicians and their works, Journals of mathematics, Postulational foundations of geometry, Famous mathematicians, Mathematical problems, Asymptotic expansion of functions defined by MacLaurin Series, Continued fractions, Binary notation in puzzles and games, Linear differential equations, Applications of group theory to differential equations, Dual numbers, Generalized Abelian groups, Critical points of harmonic functions.

## *Pi Mu Epsilon, University of Arkansas*

The chapter held ten meetings during the school year. At most of these meetings talks were given both by student members of the organization and by various members of the faculty. Among the topics discussed were: Mathematics and engineering, Teaching of mathematics in secondary schools, Electrical computing machines, History of the calendar, Lives of mathematicians at the time of Napoleon, The fourth dimension, Place of mathematics in education. Two banquets were held during the year at which time new members were initiated.

## *Mathematics Club, Case School of Applied Science*

Seven meetings were held throughout the school year, the first and last of which were combined business and social meetings held at the home of Dean T. M. Focke. At the special meetings, Dr. O. E. Brown and Dr. Max Morris of the department of mathematics spoke on the subjects, The application of determinants and projective transformations to nomography, and Complex elements in geometry, respectively. Other topics of meetings were: Geometric constructions, Probability, The theory of numbers, Continued fractions, Symmetric functions.

## *Pi Mu Epsilon, University of Kentucky*

A gift of one hundred dollars to the departmental library was made by the organization as its regular yearly contribution to show its active interest in mathematics. A banquet and a picnic in honor of the initiates were the social activities of the year. At the eight regular academic meetings the topics on the program were: A theorem on direction fields, The configuration of double points of cubics of a pencil, Integral sets with all ideals principal, Polyadics, A theorem on ternary forms, Summation of divergent series, Irrational numbers, Topology.

## *Pi Mu Epsilon, University of Illinois*

During the year there were twenty-five associate and forty-nine active members who met bi-monthly and heard the following papers: A brief history of Euclid's fifth postulate, A theorem concerning the greatest common divisor in Euclid's Algorithm, American mathematicians of today, An experimental formula concerning the expansion of binary forms, A criticism of Bell's *Men of Mathematics*, Mathematical economics, An historical survey of non-euclidean geometry.

### *Pi Mu Epsilon, Marquette University*

The annual Frumveller examination was held in May. There were fifteen high schools from Milwaukee County represented, with seventy-eight contestants. The highest score was made by James Doran of Pulaski High School who was awarded the one-year scholarship to Marquette. At the six meetings during the year guest speakers presented a variety of interesting subjects: The power of reason, New applications of polarized light, The total eclipse of 1937.

### *Mathematics Club, Massachusetts State College*

Of all the clubs listed with the department this one seems to be the most informal, having no officers, dues or outside speakers. Subjects for discussion are chosen most frequently by the students themselves, and they are allowed to work up the topic in their own way to develop their initiative and self-reliance. Six interesting meetings were held during the fall and winter terms, and reports were given upon a variety of subjects. Some of the most interesting follow: Certain higher plane curves, Solution of the cubic equation by Tartaglia, Hindu-Arabic notation, The mathematical exhibit at the Harvard Tercentenary, Dynamic symmetry, The Copernicus of antiquity, Alignment charts, The Richardson slide rule, Cartography, Theory of Numbers, Infinite series, Computation with large numbers.

### *Delta Nu Epsilon, Nebraska State Teachers College at Chadron*

The faculty and student members of the club took turns in presenting papers at the bimonthly meetings. Mathematics in mechanics, Einstein's basic formulas, Prime numbers, Mathematics of chance in card games, Linkages, Topology, and Newton's method of extracting roots, were presented and gave rise to interesting discussions. Both the president and vice-president of the club won scholarships permitting them to work on advanced degrees at other schools for 1938–39.

### *Mathematics Club, Trinity College, Washington, D.C.*

At most of the meetings held during the year, the subjects presented were discussed by outside speakers. Some of these were former members of the club. Their subjects were: Symmedian point of the triangle, Non-euclidean geometry, Teaching of algebra, Steiner's ellipse, Fermat's life and work, Mathematical puzzles, Inversion.

### *Kappa Mu Epsilon, Nebraska State Teachers College at Wayne*

An introduction on the relations between mathematics and the various other departments of the college curriculum made up the program at one of the most interesting meetings of the year. Six members gave short talks on Art, Language, History, Chemistry, Biology, and Physics, and the part they play in mathematics. Further topics presented at other meetings during the year included: Famous mathematicians, The linear and the circular slide rule, Nomography, Use and operation of electric calculator, Geometry as a type of reasoning, and a debate: Resolved that the mathematical contributions of Archimedes have been more powerful in the world than those of Euclid.

### *Mathematics Club, St. Xavier College, Chicago*

The lives and contributions of some of the great mathematicians and current views on the value and place of mathematics in the high school curriculum were the two general subjects for the papers presented during the year. The club helped in the mathematics exhibit sponsored by the Chicago Men's and Women's Mathematics Clubs at the Adler Planetarium.

### *Mathematics Club, Denison University*

The program for the year follows: Intuitive geometry, Origin of mathematical notations used in trigonometry and calculus, Counting in higher mathematics, Parallel coördinates, Harmonic relations.

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Connecticut State College, Storrs, Connecticut.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 383. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

The diameters from the vertices of the triangle $ABC$, in the circumscribed circle, cut the opposite sides in $E$, $F$ and $G$ respectively. $L$, $M$ and $N$ are the respective midpoints of $AE$, $BF$ and $CG$. Show that triangle $LMN$ is homologous to triangle $ABC$, and that the axis of homology is the orthic axis of the triangle.

E 384. *Proposed by V. Thébault, Le Mans, France.*

Determine the envelope generated by the polar line $w$ of the fixed point $W$, with respect to the circle $Q$, which is of constant size, but rolls around a fixed circle $C$. What happens if circle $C$ degenerates into a straight line, $L$?

E 385. *Proposed by C. W. Trigg, Los Angeles City College.*

What is the largest prime whose square contains no duplicate digits?

E 386. *Proposed by W. O. Pennell, Exeter, N. H.*

A certain irregular polygon has its $n$ sides skewed in space. From each of its $n$ vertices vectors are drawn to the mid-points of each of the $n-2$ non-adjacent sides. Show that the vector sum of these $n^2-2n$ vectors is zero.

### SOLUTIONS

E 346 [1938, 551]. *Proposed by Jack Lorell, Brooklyn, New York.*

Circles are constructed with the sides of a quadrilateral as diameters. Prove or disprove that at least one of the opposite pairs intersect.

*Solution by C. W. Trigg, Los Angeles City College.*

To disprove the proposition, it will suffice to cite one exception, an isosceles trapezoid with bases 2 and 26, equal legs of 13, and hence an altitude of 5. The distance between the midpoints of the legs is 13, whereas the sum of the radii of the circles on these legs as diameters is 13, so these opposite circles do not intersect. The distance between the midpoints of the bases, plus the radius of the circle on the smaller base, is 6, whereas the radius of the circle on the larger base is 13, so this circle wholly encloses the other. Therefore neither of the opposite pairs intersects, and the proposition is not true.

In their solutions, Paul Heinicke and E. P. Starke show that it is not possible for the two circles in each pair to be wholly external to each other, with no point in common.

E 347 [1938, 551]. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Que.*

The digits of a nine-place number are 1, 2, 3, 4, 5, 6, 7, 8 and 9, the order being determined by chance. Find the odds against the number being divisible by eleven.

*Solution by H. D. Larsen, University of New Mexico.*

In order that the number be divisible by 11, the sum of the 5 digits in the odd places must differ from the sum of the 4 digits in the even places by a multiple of 11. Since the nine digits add to 45, the two sets specified must add to 17 and 28 in one order or another.

A short calculation will show that there are 9 sets of 4 eligible digits adding to 17, and 2 sets adding to 28. Hence there are $11(4!)(5!)$ permutations of the 9 digits which yield a multiple of 11. The probability that a number is divisible by 11 is therefore $11(4!)(5!)/(9!)=11/126$, and the odds against it are 115 to 11.

Also received by W. E. Buker, Wm. Douglas, E. R. Heineman, Paul Heinicke, Herman Levy, E. P. Starke, C. W. Trigg and the proposer.

E 348 [1938, 551]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

In the triangle $ABC$, $M$ is placed on $AC$, $N$ on $AB$, and $P$ on $MN$, in such a manner that $MC/AM=AN/NB=MP/PN=r$, a variable parameter. Show that $P$ moves on a parabola, and determine its position with respect to the triangle.

*Solution by J. A. Bullard, University of Vermont.*

Let $A$ be the point $(-a, 0)$, $C$ the point $(a-h, -b)$, and $B$ $(a+h, b)$. Then $M$ falls at $[(a-ar-h)/(r+1), -b/(r+1)]$, and $N$ falls at $[(ar-a+rh)/(r+1), br/(r+1)]$. The coördinates of $P$ are consequently

$$x = a\left(\frac{r-1}{r+1}\right)^2 + h\left(\frac{r-1}{r+1}\right), \qquad y = b\left(\frac{r-1}{r+1}\right),$$

so that, eliminating $(r-1)/(r+1)$, we have $ay^2+hby-b^2x=0$.

This parabola is tangent to $AC$ at $C$ $(r=0)$, and is tangent to $AB$ at $B$ $(r=\infty)$. The point in which the parabola cuts the median through $B$ is given by $r=\frac{1}{2}$, that in which it cuts the median through $A$ by $r=1$, and that in which it cuts the median through $C$ by $r=2$. By substituting the coördinates of the vertex in the above equations, we find that the vertex tangent is determined by $r=(2a-h)/(2a+h)$, that is, the ratio of the projection of $AB$ on the median through $A$, to the projection of $AC$ on that same median.

For further discussion of this parabola, see this MONTHLY (vol. 42, 1935, p. 606; vol. 44, 1937, p. 368).

Also solved by C. W. Bruce, Paul Heinicke, D. L. MacKay, Edward S. Smith (with twelve pages of ramifications), C. W. Trigg and W. I. Thompson.

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3918. *Proposed by B. M. Stewart, University of Wisconsin.*

Given a block in which are fixed $k$ pegs and a set of $n$ washers, no two alike in size, and arranged on one peg so that no washer is above a smaller washer. What is the minimum number of moves in which the $n$ washers can be placed on another peg, if the washers must be moved one at a time, subject always to the condition that no washer be placed above a smaller washer?

For $k=3$ this problem is called "The tower of Hanoï" in Ball's *Mathematical Recreations*, and the solution is given as $2^n-1$.

3919. *Proposed by Richard Bellman, Brooklyn College.*

Prove that

$$\begin{vmatrix} \dfrac{x}{1-x} & 1 & 0 & 0 & 0 \cdots & 0 \\ \dfrac{x^2}{1-x^2} & \dfrac{x}{1-x} & 2 & 0 & 0 \cdots & 0 \\ \dfrac{x^3}{1-x^3} & \dfrac{x^2}{1-x^2} & \dfrac{x}{1-x} & 3 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \cdots & \vdots \\ \dfrac{x^r}{1-x^r} & \dfrac{x^{r-1}}{1-x^{r-1}} & \cdot & \cdot & \cdot \cdots & \dfrac{x}{1-x} \end{vmatrix} = \frac{r!x^{r(r+1)/2}}{(1-x)(1-x^2)\cdots(1-x^r)}.$$

3920. *Proposed by F. A. Lewis, University of Alabama.*

Set

$$A_i = \sum_{j=1}^{4} a_{ij}x_j, \quad B_i = \sum_{j=1}^{4} b_{ij}x_j, \quad C_i = \sum_{j=1}^{4} c_{ij}x_j, \quad i = 1, 2, 3, 4,$$

where the $x$'s represent independent variables and the determinant of the

coefficients of any four of the twelve linear forms does not vanish. Determine the number of distinct identities of the form

$$\alpha\prod_{i=1}^{4} A_i + \beta\prod_{i=1}^{4} B_i + \gamma\prod_{i=1}^{4} C_i \equiv 0$$

in the $x$'s, where $\alpha$, $\beta$, $\gamma$ are arbitrary constants not all are zero, and each coefficient in the various linear forms is zero or an $n$th root of unity, where $n$ is given in advance.

An example of such an identity is for $n=2$

$$(x_1^2 - x_2^2)(x_3^2 - x_4^2) - (x_1^2 - x_3^2)(x_2^2 - x_4^2) + (x_1^2 - x_4^2)(x_2^2 - x_3^2) \equiv 0.$$

An identity with the same coefficients in the various linear forms and $(\alpha, \beta, \gamma) = (k, -k, k)$ is not considered as distinct from the foregoing.

3921. *Proposed by V. Thébault, Le Mans, France.*

Let $BCA_1A_2$, $CAB_1B_2$, $ABC_1C_2$ be similar rectangles constructed upon the sides $BC=a$, $CA=b$, $AB=c$ of a triangle $ABC$ of area $S$, the three rectangles being all directed interiorly or all exteriorly, and $CA_1/a = AB_1/b = BC_1/c = k$. Let $A_h$, $B_h$, $C_h$, be points on $A_1A_2$, $B_1B_2$, $C_1C_2$ such that $A_1A_h/A_1A_2 = B_1B_h/B_1B_2 = C_1C_h/C_1C_2 = \lambda$. The straight lines $AB_h$, $BC_h$, $CA_h$ determine a triangle $\alpha\beta\gamma$ of area $S'$ similar to $ABC$, and

$$S' = (k \cot V - \lambda)^2 S/(k^2 + \lambda^2),$$

where $V$ is the angle of Brocard for $ABC$.

Problem 3850 [1937, 668] was incorrectly stated. Its last lines should be: Let $A_3$ be the symmetric of $A_1$ with respect to $A_2$, and similarly for $B_3$, $C_3$; then $AB_3$, $BC_3$, $CA_3$ meet in a point.

This follows from the present problem by taking $k=1$ for squares directed interiorly and $\lambda=2$.

## SOLUTIONS

3826 [1937, 251]. *Proposed by W. Macray, Clark Academy, N. Y.*

Solve the system of partial differential equations,

$$U\frac{\partial W}{\partial x} + 2W\frac{\partial U}{\partial x} = 0, \qquad U\frac{\partial W}{\partial x} + 2W\frac{\partial V}{\partial y} = 0, \qquad \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0.$$

*Solution by F. Underwood, University College, Nottingham, England.*

From equations (1) and (2) in the problem we have (4) $U_x = V_y$. From (3) and (4) result (5) $V_{xy} = U_{xx} = U_{yy}$, which may be written $(D^2 - D_1^2)U = 0$, where $D = \partial/\partial x$, $D_1 = \partial/\partial y$. The solution is (6) $U = f(x+y) + F(x-y)$. Then $V_x = f'(x+y) - F'(x-y)$, $V_y = f'(x+y) + F'(x-y)$. Hence (7) $V = f(x+y) - F(x-y) + C$. Writing (1) as $W_x/W = -2U_x/U$, we easily find (8) $W = \phi(y)/U^2$. The solution is given by (6), (7), (8) where $f$, $F$, $\phi$ are arbitrary functions and $C$ is an arbitrary constant.

Solved also by D. G. Bourgin and Barbara Ellis.

*Editorial Note.* Bourgin added a discussion of the case where $W$ depends on $x$ alone, showing that the functional forms of $f$, $F$, and $\phi$ are almost completely prescribed. Setting $U=M(y)W^{-1/2}$ we have $U_{xx}=M(y)(W^{-1/2})_{xx}$, $U_{yy}=M''(y)$ $(W^{-1/2})$. Hence $M''(y)/M(y)=(W^{-1/2})_{xx}/(W^{-1/2})$, and $M''(y)=k^2M(y)$.

Case A, where $k=0$. Then $M(y)=\alpha_1 y+\alpha_2$ and

$$W = (\beta_1 x + \beta_2)^{-2},\ U = (\alpha_1 y + \alpha_2)(\beta_1 x + \beta_2).$$

We may also write

$$\begin{aligned} U = {} & \tfrac{1}{4}\alpha_1\beta_1(x + y)^2 + \tfrac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1)(x + y) + \alpha_2\beta_2 \\ & - \tfrac{1}{4}\alpha_1\beta_1(x - y)^2 - \tfrac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)(x - y). \end{aligned}$$

For $V$ the signs of the terms in $(x-y)$ are changed and a constant $C$ is added.

Case B, where $k\neq 0$. Here $M(y)=A_1\cosh ky+A_2\sinh ky$, $W(x)=(B_1\cosh kx+B_2\sinh kx)^{-2}$, and

$$\begin{aligned} U &= [A_1\cosh ky + A_2\sinh ky][B_1\cosh kx + B_2\sinh kx] \\ &= \tfrac{1}{2}(A_1B_1 + A_2B_2)\cosh k(x + y) + \tfrac{1}{2}(A_1B_2 + A_2B_1)\sinh k(x + y) \\ &\quad + \tfrac{1}{2}(A_1B_1 - A_2B_2)\cosh k(x - y) + \tfrac{1}{2}(A_1B_2 - A_2B_1)\sinh k(x - y). \end{aligned}$$

For $V$ the signs of the terms with $(x-y)$ are changed and a constant $C$ is added.

If $W$ depends on $y$ alone, it may be taken arbitrarily; and $U=hy+\alpha$, $V=hx+\beta$, where $h$, $\alpha$, $\beta$ are arbitrary constants.

3827 [1937, 252]. *Proposed by V. Thébault, Le Mans, France.*

With three consecutive integers taken from 0, 1, 2, $\cdots$, 9 form a number of five figures such that its square is formed from the ten given integers. The solution is unique.

*Solution by C. W. Trigg, Los Angeles Junior College.*

$1023456789<N^2<9876543210$, so $31992\leqq N\leqq 99387$. As $N^2$ contains the ten digits it is congruent to zero, modulo 9, so $N\equiv 0 \pmod 3$. Any triad of consecutive digits is divisible by 3; so $N$ must be a permutation of three consecutive digits and their member which is a multiple of 3 repeated twice, or of the three digits and the non-multiples of 3 repeated once each. From a table of squares of the numbers from 1 to 10,000 can be read the first three (in general) and the last four digits of $N^2$ for the 300 eligible values of $N$. Duplicate digits are found in all but twenty of these, which when squared reveal the unique value $(55446)^2=3074258916$.

Solved also by W. E. Buker, E. P. Starke, and the proposer.

3828. [1937, 252]. *Proposed by V. Thébault, Le Mans, France.*

(a) The perpendiculars from a point $P$ to the straight lines $AQ$, $BQ$, $CQ$, which join the vertices of a triangle to an arbitrary point $Q$, cut the sides of the triangle $A_1B_1C_1$, determined by the perpendiculars to the lines $PA$, $PB$, $PC$,

drawn from the inverse points of $A$, $B$, $C$ in an inversion $(P, k)$, in three points of a straight line $\Delta$ perpendicular to $PQ$.

(b) If the point $P$ remains fixed and $Q$ describes a given straight line $d$, the straight line $\Delta$ passes through a fixed point.

*Solution by Otto J. Ramler, The Catholic University of America.*

(a) Draw the circles having $PA$, $PB$, $PC$ as diameters. Let $QA$, $QB$, $QC$ meet these circles in $A'$, $B'$, $C'$ respectively, Then since $PA'$, $PB'$, $PC'$, are perpendicular to $QA$, $QB$, $QC$ respectively it readily follows that $P$, $Q$, $A'$, $B'$, $C'$ are on a circle having $PQ$ for a diameter. Now subject the four circles $(AP)$, $(BP)$, $(CP)$, and $(QP)$ to an inversion $(P, k)$. The circles $(AP)$, $(BP)$, $(CP)$, invert into the straight lines $B_1C_1$, $C_1A_1$, $A_1B_1$, respectively determined by the perpendiculars to the lines $PA$, $PB$, $PC$, drawn from the inverse points of $A$, $B$, $C$ in the inversion $(P, k)$. The points $A'$, $B'$, $C'$ invert into the intersections of $PA'$, $PB'$, $PC'$, with $B_1C_1$, $C_1A_1$, $A_1B_1$ respectively, and since $A'$, $B'$, $C'$, $P$ lie on a circle through the center of inversion it follows that their inverses lie on a straight line $\Delta$. Since under the transformation of inversion angles are preserved, it follows that since $PQ$ is a diameter of the circle $(A'B'C'PQ)$ and inverts into itself, the line $\Delta$ must be perpendicular to $PQ$.

(b) If $Q$ describes a straight line $d$, the circles having $PQ$ as diameter all pass through $P$ and $F$, the foot of the perpendicular from $P$ upon $d$. Hence, the corresponding lines $\Delta$ all pass through a fixed point, the inverse of $F$ as to the circle of inversion.

Solved also by W. C. Janes, L. M. Kelly, and the proposer.

*Editorial Note.* Janes's solution used rectangular coördinates, while the proposer and Kelly used the geometry of inversion. It is simpler to use polar properties. Thus $ABC$ and $A_1B_1C_1$ are polar reciprocal triangles with respect to the circle $(P)$ with the center $P$. Let the perpendicular from $P$ to $QA$ cut $B_1C_1$ in $\overline{A}$. Then the polar of $\overline{A}$ goes through $A$ and it is perpendicular to $\overline{A}P$. Hence the polar of $\overline{A}$ is $QA$; and $\Delta$, the polar of $Q$, must be the perpendicular through $\overline{A}$ to $PQ$. If $\overline{B}$ and $\overline{C}$ are defined in the same way as $\overline{A}$, these three points lie on $\Delta$. If $Q$ moves on a straight line $d$, then $\Delta$, the polar of $Q$, must pass through $D$, the pole of $d$. This concludes the proof.

3830. [1937, 332]. *Proposed by Otto Dunkel, Washington University.*

In $n$ dimensional euclidean space, $n \geqq 2$, to the simplex $S$ there corresponds a simplex $S'$ such that the perpendiculars from each vertex $A_i$ of $S$ to the face of $S'$ opposite $A_i'$, $1 \leqq i \leqq n+1$, meet in a point $P$. Show that the perpendiculars from each vertex $A_i'$ of $S'$ to the face of $S$ opposite $A_i$ meet in a point $Q$; that is, $S$ and $S'$ are orthologic.

*Solution by the Proposer.*

It is assumed that neither $S$ nor $S'$ is degenerate, and this assumption will be used in the proof. Let the vectors from any chosen origin to the vertices of $S$ and $S'$ be $\mathbf{a}_i$ and $\mathbf{a}_i'$; and let the vector of $P$ be $\mathbf{y}$. Then

## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

A meeting of mathematicians of Pacific Northwest institutions was held at the University of Washington on April 1 and 2, 1939. The following mathematicians presented papers: Professor J. P. Ballantine, Dr. Harold Chatland, Harry Goheen, Professor F. L. Griffin, Dr. O. G. Harrold, Dr. P. G. Hoel, Professor Ralph Hull, Professor M. S. Knebelman, Dr. A. H. Taub, and R. L. Wertz. It was decided to have an annual meeting, to have the next meeting at Reed College in Portland, Oregon, and to consider at the next meeting the question of affiliation with one of the national mathematical groups.

The members of the Mathematical Association may recall that in 1932 a translation of the First Carus Monograph, *Calculus of Variations* by Professor G. A. Bliss, was published by B. G. Teubner of Berlin through special arrangement between the publisher and the Mathematical Association. The price of this translation by Dr. F. Schwank of Frankfurt is 7 marks, but through a special discount it is available to members of the Association for 3.70 marks. This edition incorporates a number of improvements made by Professor Bliss after the appearance of the Monograph in 1925.

A new publication outlet for manuscripts dealing with science and technology has been provided at Iowa State College, Ames, Iowa, by the recent organization of the Iowa State College Press, whose major purpose is "to serve learning, and particularly learning in fields of science and technology." The new press will consider for publication manuscripts from any source. The manufacture and sale of Iowa State College Press publications will be conducted by the Collegiate Press, Inc., Ames, Iowa.

Professor A. E. Landry of the Catholic University of America represented the Mathematical Association of America at the one hundred fiftieth anniversary of the founding of Georgetown University, May 28 to June 3, 1939.

Professor R. H. Reece of the New Mexico School of Mines represented the Mathematical Association at the fiftieth anniversary of the founding of the University of New Mexico, June 4–5, 1939.

Associate Professor L. M. Graves of the University of Chicago has been promoted to a full professorship.

Assistant Professor R. L. O'Quinn of Louisiana State University has been promoted to an associate professorship.

The following appointments to instructorships for the year 1939–40 are announced:

Bradley Polytechnic Institute: Ralph Johanson.
University of Chicago: Dr. O. F. G. Schilling.

# THE AMERICAN MATHEMATICAL MONTHLY

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.

## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-second Summer Meeting, Madison, Wis., September 4–7, 1939.
Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13.

ILLINOIS, Galesburg, May 12–13.

INDIANA, Muncie, April 28–29.

IOWA, Ames, April 21–22.

KANSAS, Topeka, April 1.

KENTUCKY, Murray, April 28–29.

LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13; WASHINGTON, D. C., December.

MICHIGAN, Ann Arbor, March 18.

MINNESOTA, Northfield, May 13.

MISSOURI, Springfield, April 28.

NEBRASKA, Lincoln, May 5.

NORTHERN CALIFORNIA, San Francisco, January 28.

OHIO, Columbus, April 8.

OKLAHOMA, Tulsa, February 10.

PHILADELPHIA, Bethlehem, Pa., December 2.

ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.

SOUTHEASTERN, Charleston, S.C., March 24–25.

SOUTHERN CALIFORNIA, Whittier, March 4.

SOUTHWESTERN, Alpine, Texas, May 2–3.

TEXAS, Abilene, March 31–April 1.

WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.


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## THE MARCH MEETING OF THE MICHIGAN SECTION

The sixteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 18, 1939. The chairman of the Section, Professor W. L. Ayres, presided at all sessions.

The attendance was about ninety, including the following thirty-nine members of the Association: N. H. Anning, L. A. Aroian, W. L. Ayres, J. W. Baldwin, W. D. Baten, J. B. Brandeberry, Angeline J. Brandt, R. V. Churchill, A. H. Copeland, C. C. Craig, P. S. Dwyer, Peter Field, K. W. Folley, R. E. Gaskell, T. H. Hildebrandt, J. D. Hill, L. A. Hopkins, L. S. Johnston, L. C. Karpinski, D. K. Kazarinoff, A. E. Lampen, C. E. Love, E. D. McCarthy, D. C. Morrow, A. L. Nelson, L. F. Ollmann, L. C. Plant, J. E. Powell, G. Y. Rainich, Gladdis E. Richards, C. C. Richtmeyer, L. J. Rouse, T. R. Running, E. R. Sleight, A. G. Swanson, T. O. Walton, Fern Welker, R. L. Wilder, Margarete C. Wolf.

Sixty persons attended the noon luncheon at the Michigan Union. At the business meeting which followed, A. E. Lampen, Hope College, was elected chairman, and P. S. Dwyer, University of Michigan, was elected secretary-treasurer. The main item of business dealt with plans for the meetings of the coming year. After some discussion it was left to the executive committee to determine the time and place of the fall meeting. With reference to the spring meeting, the executive committee was instructed to confer with the officers of the Michigan Schoolmasters' Club with respect to the possibility of scheduling a joint meeting next year. The afternoon session, which was held in the building of the Horace H. Rackham School of Graduate Studies, was followed by an inspection of the building and by an inspection of a book display arranged by Professor N. H. Anning.

The following papers were presented at the morning and afternoon sessions:

1. "A generalized Fibonacci series" by Professor L. S. Johnston, University of Detroit.

2. "Note on Sheppard's corrections" by Professor C. C. Craig, University of Michigan.

3. "On the space of convergent series" by Dr. J. D. Hill, Michigan State College.

4. Undergraduate papers.

(a) "Derivation of formulas for finding regression and correlation coefficients for the combination of sets of correlated data" by W. E. Swenson, Michigan State College, introduced by Professor Baten.

(b) "Introduction of infinite elements in geometry" by C. L. Dolph, University of Michigan, introduced by Professor Rainich.

(c) "Curves in one-to-one correspondence" by Mrs. Kathryn Crippen Warner, Michigan State College, introduced by Professor Grove.

5. "Napier and his logarithms" by Professor E. R. Sleight, Albion College.

6. "A series in matrices" by B. Pecherer and Professor G. Y. Rainich, University of Michigan. Mr. Pecherer was introduced by Professor Rainich, and the paper was presented by Professor Rainich.

7. "Some associated conics" by Professor Norman Anning, University of Michigan.

8. Three-minute talks.

Abstracts of these papers follow, numbered in accordance with the list of titles.

1. It is well known that the terms of the Fibonacci series are the coefficients of the power series expansion of $(1-x-x^2)^{-1}$. Power series expansion of $(1-x=x^2- \cdots -x^n)^{-1}$ gives an analogous series in which each term after the $n$th is the sum of the $n$ immediately preceding terms. Operating with this expansion, Professor Johnston exhibited two equal determinants of the $n$th order, each symmetric with respect to its secondary diagonal, the main diagonal of each consisting of identical elements, though not the same element for the two determinants. He exhibited also two functions of the roots of $1-x-x^2- \cdots -x^n=0$, each function being the product of $n$ polynomials with $n$ terms in each polynomial, the value of each product being equal to the value of each of the two determinants mentioned. This equality was proved without computing any root of the given equation, without expanding either product mentioned, and without expanding either determinant mentioned. The proof used theorems on the separation of a fraction into its partial fractions, theorems on the symmetric functions of the roots of an equation, and theorems on the sum of ingral powers of the roots of an equation.

2. Professor H. C. Carver seems to have been the first to point out ("The Fundamental Nature and Proof of Sheppard's Adjustments" *Annals of Mathematical Statistics*, vol. 7, 1936, pp. 154–163), that the usual formulas for Sheppard's corrections for central moments are strictly valid only if these moments are calculated about the mean of the ungrouped data. If, as must occur in practice, the moment to be adjusted is calculated from the mean of the grouped distribution, then an additional systematic error due to the variation of this mean with the different groupings possible with the same class-width is introduced. Dr. J. A. Pierce in his so-far unpublished dissertation at the University of Michigan, derived formulas of the type of Sheppard's corrections in which account is taken of this kind of systematic error as well. Professor Craig obtained results equivalent to Pierce's in a more elegant manner by the use of moment generating functions.

3. We denote by $(\gamma)$ the class of all sequences $x\equiv\{u_k\}$ for which $\sum u_k$ is convergent. With addition and multiplication by a constant defined as in the space (c) of convergent sequences, the class $(\gamma)$ satisfies the postulates for a linear space. Furthermore, if we set $\|x\|\equiv\sup_n|s_n|$, where $s_n\equiv u_0+u_1+\cdots+u_n$ it is seen that $(\gamma)$ becomes a Banach space. Doctor Hill considered transformations of $(\gamma)$ of the form $f_m(x)\equiv\sum a_{mk}u_k$, where $(a_{mk})$ is a given infinite matrix, and where $\{f_m(x)\}$, for every $x\epsilon(\gamma)$, is required to belong to (c) in the first case,

and to $(\gamma)$ in the second. The conditions on $(a_{mk})$ that characterize each case were found. As a corollary to the second case, he obtained the classical theorem of Mertens on the Cauchy multiplication of series. Finally, the category of certain interesting subsets of $(\gamma)$ was determined.

4. (a) Mr. Swenson attacked a rather practical problem. With $\sigma_x$, $\sigma_y$, $\bar{x}$, $\bar{y}$, $a$, $b$, and $r$ assumed to be known quantities for each set of variates, the author proceeded to derive general formulas for the regression coefficients $a$ and $b$ and the correlation coefficient $r$ for the combined sets. For purposes of illustration, the results of the analyses of hog production and corn prices for twenty-year periods from 1871 to 1930 inclusive, were considered. Interesting special cases of the general formulas for the regression coefficients were cited.

(b) Mr. Dolph defined projective geometry as the study of properties which remain invariant under projection. He started with the simplest concepts of euclidean geometry, plane, point, and line, and showed how to generalize by easy and natural steps until he reached the well-known theory of ideal elements in projective geometry.

(c) Mrs. Warner derived a set of formulas defining a one-to-one point correspondence between two curves. These formulas were then applied to the derivation of the equations of certain familiar formulas.

5. Professor Sleight gave an historical account of the development of logarithms according to Napier's plan, and of the important part which Briggs took in this invention. He also described and explained the use of Napier's rods.

6. Professor Rainich studied the problem in which a plane curve is given as a complex valued function of arc length used as a parameter. This function was developed into Maclaurin's series whose terms were then represented as products of two complex numbers, one of which is in all terms the unit tangent vector and the other an expression in terms of successive derivatives of the curvature. In the three-dimensional case matrices replace complex numbers; in particular, instead of the unit tangent vector we have the matrix whose elements are the components of the unit tangent, principal normal and binormal vectors, and the place of curvature is taken by a matrix whose elements are curvature, torsion, and zeros. The difficulty caused by absence of commutativity was overcome by using the transposed matrices, and a simple recursion formula for the terms was obtained.

7. Professor Anning showed that, if through a fixed point $O$ in the plane of a conic a line is drawn to cut the conic in $P$ and $Q$, and if $A$ and $G$ are points on the line such that $OA$ is the arithmetic and $OG$ the geometric mean of $OP$ and $OQ$, and if the line is rotated about $O$, the loci of $A$ and $G$ are conics homothetic to the original conic. Other common means were explored.

8. Members of the group discussed various topics which were, for the most part, of a geometrical nature.

P. S. DWYER, *Secretary*

## THE FEBRUARY MEETING OF THE OKLAHOMA SECTION

The regular meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association at Tulsa, Oklahoma, on Friday morning, February 10, 1939. Professor W. T. Short, chairman of the Section presided.

One hundred fifty representatives of high schools and colleges attended the meetings, including the following seventeen members of the Association: J. C. Brixey, J. H. Butchart, N. A. Court, A. H. Diamond, R. C. Dragoo, Dot Jeannette Gifford, H. L. Hall, J. O. Hassler, J. E. LaFon, J. S. Leech, W. C. Randels, Raleigh Schorling, W. T. Short, H. W. Smith, E. B. Wedel, B. S. Whitney, J. H. Zant.

At the annual business meeting the following officers were elected: Chairman, J. O. Hassler, University of Oklahoma; Vice-Chairman, A. H. Diamond, Oklahoma A. and M. College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following five papers:

1. "Remarks on the teaching of college geometry" by Professor J. H. Butchart, Phillips University.

2. "The historical background of the Weierstrass non-differentiable function" by Professor W. C. Randels, University of Oklahoma.

3. "The general equation of the second degree in plane analytical geometry" by Professor A. H. Diamond, Oklahoma A. and M. College.

4. "Special tetrahedrons" by Professor N. A. Court, University of Oklahoma.

5. "The place of mathematics in general education" by Professor Raleigh Schorling, University of Michigan.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Butchart pointed out possible simplification of several proofs in N. A. Court's text and advocated taking the student farther into a few topics. He also urged the adoption of symbolism to express some of the more frequently encountered relations.

2. Professor Randels gave an account of the background of the Weierstrass non-differentiable function as an example of the changes in the conceptions of mathematics which gave rise to modern analysis.

3. Professor Diamond gave an elementary means of identifying and graphing the general equation of the second degree without rotation of axes.

4. Professor Court considered the special tetragedron $(T)=ABCD$ in which the diametric opposite $U$ of the vertex $D$ on the circumsphere $(O)$ of $(T)$ lies on the circumcircle $(d)$ of the triangle $ABC$. The following special cases were considered. (1) The foot $D_h$ of the altitude $DD_h$ of $(T)$ from $D$ to the opposite face $ABC$ is the diametric opposite of $U$ on $(d)$. (2) The feet of the altitudes, issued from $D$, of the triangles $DAB$, $DBC$, $DCA$ are collinear. (3) The Monge point $M$ of $(T)$ lies in the face $ABC$, on the nine-point circle of the triangle

$ABC$, mid-way between $D_h$ and the orthocenter $H_a$ of the triangle $ABC$. (4) The centroid of the triangle $UD_hH_a$ coincides with the centroid of $ABC$. (5) The projection, upon $ABC$, of the vertex of the twin tetrahedron $(T')$ of $(T)$ which corresponds to $D$ is the diametric opposite of $M$ on the nine-point circle of the triangle $ABC$.

5. Professor Schorling considered the following topics: (1) The needs of secondary pupils as regards basic skills in mathematics. (2) The contribution of courses in the mathematics of the secondary school to the general reader. (3) The need for mathematics in the introductory courses in science of the secondary school. (4) The basic concepts and principles to be included in mathematics that are to contribute to general education.

J. C. BRIXEY, *Secretary*

## SIXTEENTH ANNUAL MEETING OF THE INDIANA SECTION

The sixteenth annual meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, April 28 and 29, 1939, at Ball State Teachers College, Muncie, Indiana.

Seventy-four registered at the meetings including the following twenty-five members of the Association: Emil Artin, Juna Lutz Beal, W. D. Cairns, J. E. Dotterer, Olive M. Draper, P. D. Edwards, W. R. Hardman, Cora B. Hennel, F. H. Hodge, L. P. Hutchison, M. W. Keller, D. A. Lehman, Florence Long, H. A. Meyer, D. H. Porter, H. R. Pyle, C. K. Robbins, L. S. Shively, D. R. Shreve, W. O. Shriner, Anna K. Suter, M. S. Webster, Agnes E. Wells, K. P. Williams, H. E. Wolfe.

At the business session on Saturday morning the following officers were elected for next year: Chairman, L. S. Shively, Ball State Teachers College; Vice-Chairman, Cora B. Hennel, Indiana University; Secretary, P. D. Edwards, Ball State Teachers College. The seventeenth annual meeting will be held at Earlham College, Richmond, Indiana.

Professor K. P. Williams made a report for the committee appointed to encourage and recognize superior preparation for the teaching of mathematics. On the basis of an examination conducted April 23, 1938, and April 22, 1939, a Certificate of Merit in Mathematical Preparation was awarded to Charles F. Brumfield of Ball State Teachers College, and to Richard E. Dietrich of Indiana University.

The annual dinner was held jointly with the Xi chapter of Sigma Zeta Honorary Science and Mathematics Society. Mr. James Findling, President of Xi chapter, served as toastmaster and introduced President L. A. Pittenger of Ball State Teachers College, who welcomed the visitors.

Following the dinner the first session of the Section was held, at which time Professor W. D. Cairns of Oberlin College gave an illustrated lecture on "The rôle of mathematics in seismology." Professor Cairns described the probable cause of earthquakes, the three main types of waves, the mathematical evidence

for their paths through or about the earth, the consequent deductions as to the nature of the earth's interior, including the conclusions as to the discontinuities in the earth's structure. He explained the theory of seismometers. Further topics which involved mathematical treatment were the correspondence between displacements on the instrumental records and displacements of the earth which these records are meant to give, the method of least squares in connection with the travel-time curves, and the consistency of various estimates of the earth's interior.

At the two sessions on Saturday the following program was given:

1. "Problem making" by Professor C. K. Robbins, Purdue University.
2. "Order relation in fields" by Professor Emil Artin, Indiana University.
3. "Some technical aspects of the mathematics of seismology" by Professor W. D. Cairns, Oberlin College.
4. "On the foundations of mathematics" by D. O. Schechter, Manchester College, introduced by Professor Dotterer.
5. "The generalization of the Eckhardt point" by Dr. D. R. Shreve, Purdue University.
6. "The general second degree equation without transformation of coördinates" by Professor K. P. Williams, Indiana University.
7. "A certain Lagrange interpolation formula" by Dr. M. S. Webster, Purdue University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Robbins discussed the matter of the construction of problems which would be suitable for textbook material. Several problems of different types were discussed. Among these was one consisting of two curves, each having rational coefficients so determined that the common tangents to the two curves have rational coefficients and are tangent at points whose coördinates are rational. Other interesting examples were given from the field of differential equations.

2. Hilbert's problem to characterize all geometric construction that can be carried out by means of ruler and compasses only, leads to the question, what elements of a field can be represented as a sum of squares? Professor Artin showed that if the element $-1$ is the sum of squares of elements of a field, then every element of the field is the sum of squares of elements of the field. So the interesting case is that of a field in which $-1$ is not a sum of squares. Such a field is called a real field. It can be shown that the real fields are identical with fields in which one can establish an order. An element of a real field is a sum of squares if, and only if, it is positive in whatever way one might order the field.

3. Professor Cairns gave a more technical consideration to those mathematical topics which could not be adequately treated in the general lecture of Friday evening.

4. Mr. Schechter discussed the three schools of thought that have been interested in the investigation of the foundation of mathematics, the formal, the

logical, and the postulational. The program as presented by each school was considered and the three schools were compared as to the actuality of the foundations, subject matter, method and structure, and consistency. From these comparisons, general conclusions were made concerning the nature of mathematics.

5. The generalized Eckhardt point was defined by Dr. Shreve, and two theorems of Eckhardt were given. The multiplicity of the Eckhardt point on the Hessian of the surface was determined; the configuration of Eckhardt points on the Segre Cubic Variety was discussed in detail.

6. The paper of Professor Williams considered the problems involved in the direct reduction of the equation

$$Ax^2 + By^2 + 2Hxy + 2Cx + 2Dy + E = 0$$

to the form

$$(x - \alpha)^2 + (y - \beta)^2 = e^2 \frac{(ax + by + c)^2}{a^2 + b^2}.$$

7. Dr. Webster discussed the Lagrange interpolation polynomial

$$l_k^{(n)}(x) \equiv \phi_n(x)/[(x - x_k)\phi_n'(x_k)],$$

where

$$\phi_n(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n), \qquad x_k = \cos [k\pi/n + 1].$$

He proved that in the interval $(-1, 1)$ the maximum of the absolute value of $l_k^{(n)}(x)$ is less than 2. This is the best possible inequality for all $n$ and $k$.

P. D. EDWARDS, *Secretary*

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## THE APRIL MEETING OF THE KANSAS SECTION

The twenty-fifth annual meeting of the Kansas Section of the Mathematical Association of America was held at the High School in Topeka, on Saturday, April 1, 1939. In the morning there was a joint session with the Kansas Association of Teachers of Mathematics, of which most members of the Section are also members. After the social hour and the luncheon, the two organizations met for separate programs. Professor C. B. Tucker, chairman of the Section, presided at the morning session as well as at the Section meeting.

The attendance was one hundred sixty-one, among whom were the following thirty-four members of the Association: Sister Ann Elizabeth, Sister Mary N. Arnoldy, Wealthy Babcock, E. A. Beito, Lois E. Bell, Florence L. Black, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, A. J. Hoare, Emma Hyde, H. E. Jordan, C. F. Lewis, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza Mossman, O. J. Peterson, P. S. Pretz, G. B. Price, C. B. Read, B. L. Remick, D. H. Richert, J. A. G. Shirk,

D. T. Sigley, G. W. Smith, W. T. Stratton, Sister M. Helen Sullivan, H. B. Thornton, C. B. Tucker, J. J. Wheeler, A. E. White.

The officers elected for the coming year are: Chairman, C. B. Read, University of Wichita; Vice-Chairman, G. B. Price, University of Kansas; Secretary, Lucy T. Dougherty, Junior College, Kansas City, Kansas. An invitation to meet at Wichita next year was presented by Professor Hoare on behalf of the high schools, the two universities and the mathematics club.

The morning program included a report by Professor U. G. Mitchell on the Joint Commission of the Association and the National Council and the progress of their work; also an address by Professor M. L. Hartung, University of Chicago, on "Mathematics in progressive education." At the meeting of the Section in the afternoon four papers were read, after which there was a round table discussion. The program follows:

1. "An historical discussion of the fundamental problem of abstract group theory" by Professor D. T. Sigley, Kansas State College.

2. "Projective vector algebra" by Dr. P. O. Bell, University of Kansas, introduced by Professor Mitchell.

3. "Summation of Fourier series" by Professor Paul Eberhart, Washburn College, introduced by Professor Harshbarger.

4. "Illustration of the use of lattice-points in the field of number theory" by Professor D. H. Richert, Bethel College.

5. Round Table discussion on freshman algebra, led by Professor Emma Hyde, Kansas State College.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. In 1854 Cayley began the publication of the number of abstract groups of a finite number $n$. Professor Sigley summarized the more important contributions during the last eighty-four years to the solution of this problem. The present status of the problem was given, with suggestions for future points of attack.

2. Notable algebras of segments have been developed by Hilbert, von Staudt, Schur, and Silberstein. Dr. Bell presented some main features of Silberstein's *Projective vector algebra*, published in 1919. Based only on the axioms of order and of connection, this algebra embraces only equality and vector addition and subtraction; the commutative and associative laws for addition apply, and the distributive law for scalar multiplication. The algebra has application to the generalization of ordinary metric theorems; its scope is the whole field of projective geometry. A linear vector operator for projective space is defined in much the same way as in ordinary space. Because of the distributive property of the linear operator and of its simple use in defining a general collineation in space, an effective means is obtained for studying the geometry of cyclic collineation groups.

3. Since experience shows that it is the exception rather than the rule for the Fourier series of an integrable function to converge, Professor Eberhart considered the effect of the application of certain methods of summation. He called

attention to the classical result of Féjer that the Fourier series of a Lebesgue integrable function is summable almost everywhere by the Césaro mean of order one. Various methods of summation have also been applied to the conjugate series, the derived series, and the conjugate derived series. Another type of result concerns the smoothing effect of certain methods of summation. The disappearance of Gibb's phenomena, and the convexity theorems of Féjer and Szász were mentioned.

4. Professor Richert stated that lattice-point systems have played an important part in the field of number theory and in other fields of mathematics. He first proved a fundamental theorem about the Farey series, in a simple way: If $p/q$ and $p'/q'$ are consecutive terms, then $p'q-pq'=1$. He showed next how lattice-point systems may be used to enliven the subjects of binary quadratic forms and continued fractions.

5. At last year's meeting of the Section there was a spirited discussion over two papers, "Vexing minor problems of the mathematics curriculum" by Professor Read, and "A program for the Association" by Professor Price. This Round Table was an outgrowth of that discussion. Professor Hyde gave the gist of the matter in her summary: The number of students seeking admission to Kansas colleges with less than one unit of high school algebra is increasing, and, since the state law requires that the state colleges admit all graduates from accredited high schools, the situation is serious, and the problem of what to do with those deficient in mathematics must be solved. A committee of which Professor Price is chairman, recommended that colleges offer a non-credit course to meet five times a week for one semester, to cover the first unit of high school algebra, all entering with less than one unit of algebra being required to take it. Those who prove to be unable to carry either of the regular college courses are also to be placed in this class. A committee of which Professor Peterson is chairman presented an outline course covering minimum essentials for the five-hour college course, and one for the three-hour course.

As a result of the facts presented and the discussion following, it was voted to give uniform tests in mathematics to entering freshman in all universities and colleges of the state next September, as a basis for meeting conditions as they seem to be. Professor Read appointed, as a committee to arrange for the tests, the heads of the mathematics departments of University of Kansas; Kansas State College, Manhattan; Kansas State Teachers College, Pittsburg; Baker University; and Southwestern College.

LUCY T. DOUGHERTY, *Secretary*

## ANNUAL MEETING OF THE WISCONSIN SECTION

The seventh annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the State Teachers College, Milwaukee, on Saturday, May 6, 1939. The meeting was presided over by the chairman of the

Section, Professor R. C. Huffer of Beloit College. Sessions were held in the morning and in the afternoon, with a luncheon at the Belleview Tea Shop.

There were forty-eight persons present, including the following twenty-one members of the Association: Leon Battig, Ethelywnn R. Beckwith, May M. Beenken, W. W. Bigelow, H. H. Conwell, Henry Ericson, H. P. Evans, W. L. Hart, R. C. Huffer, Elizabeth E. Knight, R. E. Langer, Morris Marden, Sister Mary Felice, R. E. Norris, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, P. L. Trump, M. J. Turner, J. I. Vass.

The business meeting was held at 2:00 P.M. at which the following officers were elected for the coming year: Chairman, Sister Mary Felice, Mount Mary College; Secretary, G. A. Parkinson, University of Wisconsin Extension, Milwaukee; Program Committee, Leon Battig, University of Wisconsin Extension, Sheboygan, and Irene Price, State Teachers College, Oshkosh. An invitation to hold the 1940 meeting at Mount Mary College, Milwaukee, was accepted by unanimous vote. Appreciation of the hospitality extended to the Section by the Milwaukee State Teachers College was expressed by rising vote. The following reports were made:

1. On the seminar for high school teachers held at the University of Wisconsin Extension Division, Milwaukee, during the past year, by Henry Ericson, Washington High School.

2. On the cooperative library project of the institutions of higher learning located in Milwaukee, by Professor Morris Marden, University of Wisconsin Extension Division.

3. On a project for making mathematical models and exhibits available to high schools, which is being undertaken in connection with a W.P.A. project, sponsored jointly by the University of Wisconsin and the Milwaukee Public Museum, by Professor G. A. Parkinson, University of Wisconsin Extension Division.

4. An informal report on the work of the Association committee which is considering the activities of the Association, by Professor R. E. Langer, University of Wisconsin.

The afternoon session was devoted to a panel discussion on "The effect of present day trends in high school mathematics upon college mathematics." The leaders were: Mary A. Potter, Racine High School; Dr. H. C. Trimble, University of Chicago; and Professor H. P. Evans, University of Wisconsin.

At the morning session the following four papers were presented:

1. "Diagonal functions" by Sister Mary Felice, Mount Mary College.

2. "Projective equivalents of the trigonometric functions" by Dr. May M. Beenken, State Teachers College, Oshkosh.

3. "Actuarial work—opportunities and preparation" by V. E. Henningsen, Assistant Actuary, Northwestern Mutual Life Insurance Company, introduced by the Secretary.

4. "The importance of a name—'College Preparatory Mathematics' or just 'Mathematics' " by Professor W. L. Hart, University of Minnesota.

Abstracts of the papers follow, numbered in accordance with their place in the list of titles:

1. Sister Mary Felice discussed the work of Charles L. Clarke, formerly chief engineer for the Edison Electric Company. Mr. Clarke found that certain types of experimental data could be fitted by means of a normal sine curve turned through an angle of 45°. In order to get a rectangular equation for such a curve, and for various modifications of it, Mr. Clarke defined the coördinates in terms of a series of functions which he called "Diagonal Functions." This paper discussed briefly the definitions of these functions and some of their properties. While these functions are somewhat similar in definition to the corresponding trigonometric functions, nevertheless they differ essentially from the latter in their properties.

2. Dr. Beenken considered the simple curves of the six trigonometric functions, the possibilities of projecting one into another, and the character of the projections of these curves. Illustrations of some typical projective equivalents were given.

3. Mr. Henningsen showed how the work of the actuary or "insurance mathematician" of a life insurance company includes the calculation of premiums, of dividends, and of reserve liabilities, the preparation of policy contracts, and mortality investigations. Outside of life insurance offices, actuaries are employed in state and federal service on social insurance problems or in administrative work. Admission into the Actuarial Society of America and the American Institute of Actuaries is gained through successful completion of a series of examinations covering pure and applied mathematics and other subjects related to insurance work. Comparatively, the actuarial field is a limited one, but opportunities are increasing through wider use of the services of persons with actuarial training within insurance companies themselves and also in government positions.

4. Professor Hart outlined a dual program in secondary mathematics, one part devoted to the needs of inferior students and the second part planned for superior students regardless of their collegiate intentions. His main attention was focused on the superior group; for it he advised a three-year or four-year program, including a substantial terminal course in applied mathematics with authentic cultural and practical objectives to be attained at the high school level. He recommended this rational approach, where secondary mathematics is caused to stand on its own merits without college props, in place of the approach which is symbolized by finally giving the name "college preparatory mathematics" to the secondary content. In support of this opinion he called attention to various unwelcome effects which may result from inferences based on the "college preparatory" label.

G. A. PARKINSON, *Secretary*

## THE MAY MEETING OF THE SOUTHWESTERN SECTION

The third annual meeting of the Southwestern Section of the Mathematical Association of America was held at Sul Ross State Teachers College, Alpine, Texas, on May 2 and 3, 1939, in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science. Professor R. S. Underwood, chairman of the Section, presided over the three sessions.

The attendance was twenty-five, including the following eleven members of the Association: C. A. Barnhart, J. W. Branson, C. A. Gilley, R. F. Graesser, H. D. Larsen, Roy MacKay, C. V. Newsom, E. J. Purcell, P. K. Rees, P. M. Swingle, R. S. Underwood.

At the business meeting the following officers were elected for the next year: Chairman, J. W. Branson, New Mexico State College; Vice-Chairman, E. J. Purcell, University of Arizona. It was voted to hold the next meeting at Tucson, Arizona, in conjunction with the meeting of the Southwestern Division of the American Association for the Advancement of Science.

The following resolutions were adopted by the Section:

1. The Southwestern Section of the Mathematical Association of America favors a modification of the organization of the national body of the Association to include a council of representatives, one from each Section, to govern the Association and determine its policies. 2. The Southwestern Section of the Mathematical Association of America favors a policy of devoting the *American Mathematical Monthly* essentially to expository material.

On Tuesday evening, May 2, a dinner was held for members of the Section and their guests. Dr. L. V. Robinson of Amarillo College was invited to speak on "Cepheid Variables." Professor R. S. Underwood of Texas Technological College followed with an address, "Are we alone in the universe?"

The Wednesday morning session was devoted to a symposium on topology. Professor Roy MacKay of Eastern New Mexico Junior College discussed "Homology Groups," and Dr. P. M. Swingle of New Mexico State College spoke on "Properties of the Straight Line in Topology."

The following ten papers were read, four on Tuesday afternoon, and six at the Wednesday afternoon session:

1. "The undergraduate curriculum in mathematics" by Professor C. V. Newsom, University of New Mexico.

2. "Reducing the number of failures in freshman mathematics" by Professor C. A. Murray, West Texas State Teachers College, read by Professor Branson. (This paper had been read at the Abilene meeting of the Texas Section and was presented to the Southwestern Section on invitation of the program committee.)

3. "Short-cuts in analytic geometry" by Professor R. S. Underwood, Texas Technological College.

4. "A cubic involution in $S_3$" by Professor E. J. Purcell, University of Arizona.

5. "Concerning transforms of Fuchsian groups" by Professor P. K. Rees, New Mexico State College.

6. "Remarks on a conjecture of Minkowski" by Dr. Douglas Derry, University of New Mexico, introduced by the Secretary.

7. "Concerning biconnected sets" by Dr. P. M. Swingle, New Mexico State College.

8. "On making a Peano continuum locally 1-connected by adding non-intersecting open-2-cells" by Professor Roy MacKay, Eastern New Mexico Junior College.

9. "The asymptotic representation of certain types of entire functions" by Professor C. V. Newsom and J. R. Ellis, the latter being introduced by Professor Newsom, University of New Mexico.

10. "Certain gamma functions as solutions of a difference equation" by Professor J. W. Branson, New Mexico State College.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Newsom discussed the undergraduate curriculum as it is related to those students interested in technical science, to those who desire some mathematics as a part of their general education, and to those who plan to become teachers of secondary mathematics. In particular, revisions in the mathematics program at the University of New Mexico were discussed and evaluated.

3. Professor Underwood discussed methods which he has used successfully to shorten the amount of time devoted to the straight line in analytic geometry. He showed how the slope-intercept, point-slope, two-point and intercept forms of the equations of the straight line can be replaced by one treatment which makes full use of the line's slope and its simplified equation. He also derived the formula for the distance from a point to a line, with a convenient rule of signs, without introducing the normal form.

4. Professor Purcell derived the equations of the following cubic involution and discussed its fundamental system: Consider a fixed conic $c_2$ intersecting two fixed lines $d$ and $d'$, skew to each other and to the conic. Through a general point $P$ passes one line $l$ intersecting $d$ once in $D$ and $c_2$ once in $A$, and another line $l'$ intersecting $d'$ once in $D'$ and $c_2$ once in $B$. Then $P'$, the intersection of $BD$ and $AD'$ is defined to be the correspondent of $P$.

5. Professor Rees considered the transformation $S=GTG^{-1}$ of a Fuchsian group in which $G$ is fixed and $T$ variable. The equation $r_s=r_t/k$ was considered, where $k$ is a non-negative real constant, and $r_s$ and $r_t$ are the radii of the isometric circles of $S$ and $T$ respectively. He derived algebraically a theorem which gives a necessary and sufficient condition for a maximum value of $r_s$, and pointed out its geometric significance.

6. Dr. Derry showed that Minkowski's conjecture on the boundary case of his linear form theorem, when the linear forms have unit determinant and rational coefficients with denominators a power of a fixed prime $p$, is equivalent to the following form: $\mathbf{G}$ is an Abelian $p$-group of order $p^{rn}$ and with rank less

than $n$. $\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n$ are $n$ cyclic subgroups of $\mathbf{G}$. For every sub-group $\mathbf{H}$ of $\mathbf{G}$ with cyclic factor group $\mathbf{G}/\mathbf{H}$ a series

$$\mathbf{G} = (\mathbf{C}_{r_1}, \cdots, \mathbf{C}_{r_k}, \mathbf{H}), \cdots, (\mathbf{C}_{r_1}, \mathbf{H}), \mathbf{H}$$

exists whose factors have order not greater than $p^r$. Then the groups $\mathbf{C}_1, \cdots, \mathbf{C}_n$, after a possible rearrangement of order, built a series

$$\mathbf{G} = (\mathbf{C}_1, \cdots, \mathbf{C}_n), \cdots, (\mathbf{C}), (\mathbf{E})$$

with factors of order $p^r$.

7. Dr. Swingle discussed several problems concerning bi-connected sets as illustrations of the types of problems in set theoretic topology.

8. Professor MacKay showed that, if any Peano continuum $P$ is imbedded in a subset of Hilbert Space and a denumerable set $Q$ of open 2-cells is spanned on a null basis of simple closed curves of $P$, the resulting $P+Q$ is locally 1-connected in the sense of homology.

9. Professor Newsom and Mr. Ellis presented a suggestion for unifying some of the recent work of Ford, Van Engen, Harp, and others concerning the asymptotic representation of entire functions. The basis of the unification was a recent theorem by Newsom (*American Journal of Mathematics*, July, 1938), pertaining to the character of certain entire functions. The paper was a preliminary report and considered, in particular, a generalization of series of the Bessel type.

10. Professor Branson discussed a product form of the gamma function, based on the definition of Weierstrass, and derived from a combination of the forms used by Weierstrass and Euler.

H. D. LARSEN, *Secretary*

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## THE SIXTEENTH ANNUAL MEETING OF THE NEBRASKA SECTION

The sixteenth annual meeting of the Nebraska Section of the Mathematical Association of America was held at Lincoln, Nebraska, on Friday afternoon May 6, 1939. Professor W. C. Brenke of the University of Nebraska was chairman. A combined meeting of the Section and the Nebraska branch of the National Council of Mathematics Teachers was held Saturday morning May 7, 1939.

The attendance was thirty-five including the following twenty-three members of the Association: M. A. Basoco, E. M. Berry, A. K. Bettinger, Jessie W. Boyce, W. C. Brenke, C. C. Camp, A. L. Candy, H. M. Cox, J. A. Daum, D. M. Dribin, W. A. Dwyer, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, Julia M. Hawkes, E. Marie Hove, J. M. Howie, W. C. Janes, F. E. Marrin, R. M. McDill, E. B. Ogden, T. A. Pierce, Lulu L. Runge.

Officers for the ensuing year were elected as follows: Chairman, A. K. Bettinger, Creighton University; Secretary-Treasurer, Lulu L. Runge, University of Nebraska; Member of Executive Committee, J. M. Earl, Municipal

University of Omaha. The next meeting will be held in Omaha at Creighton University.

The following program was presented:

1. "The $p$-adic method in modern algebra" by Dr. D. M. Dribin, University of Nebraska.

2. "Possible norms for a grading system" by Professor C. C. Camp, University of Nebraska.

3. "Errors in Steinhauser's twenty-place logarithm table" by Professor C. C. Camp, University of Nebraska.

4. "On certain psuedo-periodic functions" by Robert Hanna, Creighton University, introduced by Professor Dwyer.

5. "A mnemonic for settings of the slide rule" by Professor J. M. Earl, Municipal University of Omaha.

6. "The college course in calculus" by Professors J. M. Howie, Nebraska Wesleyan University, A. K. Bettinger, Creighton University, and H. P. Doole, University of Nebraska.

7. "Carl Friedrich Gauss, the mathematician" by Dr. D. M. Dribin, University of Nebraska.

8. "The personality of Carl Friedrich Gauss" by Helen Gauss, by invitation.

Abstracts of some of the papers follow, numbered in accordance with their place on the program.

1. If the ordinary notion of absolute value be replaced by a "$p$-adic valuation" in the sense of Hensel, a new type of number, the rational $p$-adic number, can be defined as the limit of a fundamental sequence of rational numbers. The transition to arbitrary fields is made similarly in terms of a valuation, and a distinction is drawn between Archimedean and non-Archimedean valued fields. Dr. Dribin indicated the importance of the $p$-adic method in modern algebra by a discussion of "local" properties of fields and algebras, analogous to the study of an algebraic function in the neighborhood of a point on its Riemann surface and of an algebraic curve in the neighborhood of one of its points. He also gave a short account of Hasse's application of the $p$-adic method to the arithmetic of quadratic forms.

2. Various schemes including the use of the normal probability curve have been devised for controlling college grades. Experience shows that a skewed distribution is more equitable than a symmetrical one. One scheme recommends the percentages 16.7, 33.3, 25, 18, 7, for grades A, B, C, D, E, respectively. A university committee fourteen years ago recommended various ranges of per cents for grades in the nineties, eighties, etc. and below 60. In the present paper Professor Camp recommends ranges based on past experience of the particular university, college, department or course with a particular norm for each of the following course levels: 1. those open to freshmen; 2. those open to sophomores but in the junior division; 3. those in the senior division but not carrying graduate credit; 4. graduate courses open to seniors.

3. If Steinhauser's table were free of error it would be superior to all other

such tables. The Secretary of the Royal Society of Edinburgh has recently returned to Professor Camp his copy after comparison with the unpublished manuscript of the late Dr. Sang. The results of this revealed some new errors both in Steinhauser's table and in Sang's manuscript. This paper is being written while Professor Camp is having table A compared with Thomson's new table, Logarithmetica Britannica. Several lists of errata in Steinhauser have been published, and it is planned that a more exhaustive supplementary list be published soon.

4. Certain pseudo-periodic functions exhibit rather interesting periodicity relations, from which Mr. Hanna obtained expressions for the theta-functions which are analogous to the exponential forms of the hyperbolic functions. Also, certain quotients of the form $\theta_1'/\theta_\alpha(x)$ give rise to expressions which might be of use in paraphrase.

5. Corresponding to any scale on a slide rule, a function may be defined such that the distance from the left index to any scale reading is the logarithm of the function of the scale reading. Computations are solutions of proportions involving the functions which correspond to various scales. By defining appropriately the left and right indices of these scales, the location of the decimal point in the result of a computation becomes automatic.

7. In this paper Dr. Dribin made a rapid survey of Gauss's contribution to mathematics and mathematical astronomy, discussing briefly his work in algebra and the theory of numbers, his computation of the orbit of Ceres, and his work in differential geometry and analysis.

8. In her very interesting paper Miss Gauss, who is a granddaughter of Carl Friedrich Gauss, related a great many interesting incidents in the life of the great mathematician, which were indicative of his character and personality.

T. A. PIERCE, *Secretary*

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## THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The seventeenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at The Citadel, Charleston, S. C., on Friday and Saturday, March 24–25, 1939. There were in attendance about one hundred and forty persons from forty-seven institutions, including the following fifty-nine members of the Association: D. H. Ballou, D. F. Barrow, Helen Barton, R. C. Blackwell, E. T. Browne, E. A. Cameron, T. C. Carson, W. B. Carver, Edna J. Cofield, E. C. Coker, W. A. Cordrey, R. W. Cowan, Forrest Cumming, U. P. Davis, D. C. Dearborn, W. G. Doyle, F. G. Dressel, L. A. Dye, R. L. Fritz, W. H. Gaver, A. M. Gignilliat, C. L. Hair, R. A. Hefner, Archibald Henderson, P. R. Hill, H. K. Holt, J. A. Hyden, J. B. Jackson, Rosa L. Jackson, Olive M. Jones, F. W. Kokomoor, J. W. Lasley, T. J. Leslie, T. G. Loudermilk, J. D. Mancill, S. W. McInnis, A. N. McPherson, C. F. Meyers, R. H. Moorman, W. P. Ott, D. D. Peele, Vertie D. Prince, Caroline M. Reaves, G. E. Reves, H. A. Robinson, J. A. L. Saunders, C. L. Seebeck, W. E. Sewell, T. M. Simpson,

H. E. Spencer, F. H. Steen, Ruth W. Stokes, Elizabeth C. Strayhorn, Cora Strong, J. M. Thomas, T. L. Wade, D. L. Webb, W. L. Williams, J. T. C. Wright.

Sessions were held Friday afternoon and evening and Saturday morning. Professor J. A. Hyden, chairman of the Section, presided, except Friday evening and part of Saturday morning when the Section was divided into subgroups according to the nature of the papers presented. Subgroups were presided over by Professors D. F. Barrow, E. T. Browne, W. P. Ott, and Ruth W. Stokes. On Friday evening a dinner was given in honor of the visiting speaker, Professor W. B. Carver of Cornell University, President of the Mathematical Association of America, whose two addresses were a highly valued contribution to the program. At this time Colonel C. L. Hair presided.

At the business session on Saturday the following officers were chosen for 1939–40: Chairman, C. L. Hair, The Citadel; Vice-Chairman, Forrest Cumming, University of Georgia; Secretary-Treasurer, H. A. Robinson, Agnes Scott College; Members of Executive Committee: F. W. Kokomoor, University of Florida; F. A. Lewis, University of Alabama; F. L. Wren, George Peabody College for Teachers. The next meeting was scheduled for March 1940, at the University of Georgia in Athens, Georgia.

The following twenty-nine papers were presented:

1. "On the defining relations of the simple group of order 360" by Professor F. A. Lewis, University of Alabama, by title.

2. "Logarithmic properties of a power of $x$" by Professor W. E. Sewell, Georgia School of Technology.

3. "A retrospect and prospect for mathematics in the Southeastern Section" by Chairman J. A. Hyden, Vanderbilt University.

4. "Diagonal combination and some applications" by Professor F. H. Steen, Georgia School of Technology.

5. "Descartes: mathematician and philosopher" by R. H. Moorman, George Peabody College for Teachers.

6. "The geometry of tensors" by Professor Archibald Henderson, University of North Carolina.

7. "The mathematical puzzle as a stimulus to mathematical work" by Professor W. B. Carver, Cornell University.

8. "A note on the plane sections of a cylinder and a cone" by Professor D. H. Ballou, Georgia School of Technology.

9. "An involutorial Cremona transformation of four dimensional space" by Professor L. A. Dye, The Citadel.

10. "A new type of measurement in mathematics" by Dr. F. S. Beers, Examiner, University System of Georgia, introduced by the Secretary.

11. "Closely packed spheres" by Professor W. B. Carver, Cornell University.

12. "Some properties of the crunodal cubic" by Professor R. V. Blair, Vanderbilt University, read by Professor A. N. McPherson.

13. "The Hessian of a cubic" by Professor R. T. Donnell, Cumberland University.

14. "Interpretation of several propositions of mathematical logic in terms of probability" by Dr. D. L. Webb, Georgia School of Technology.

15. "An analytical derivation of Lagrange's expansion and solution of Kepler's equation" by W. J. Mays, Vanderbilt University, introduced by the Secretary.

16. "Cubics having $r$-point contact with a plane curve" by R. C. Blackwell, University of North Carolina.

17. "Connected sets of limit points of sequences" by Professor H. G. Barone, The Citadel, introduced by Colonel Hair.

18. "The relation of the Lipschitz condition to the problem of extension of functions" by Dr. L. D. Rodabough, University of Alabama, introduced by the Secretary.

19. "On univalent Laplace transforms" by Professor N. N. Royall, Jr., The Citadel, introduced by Colonel Hair.

20. "The solution of a functional equation with trigonometric coefficients" by Dr. R. W. Cowan, University of Alabama.

21. "On the envelope of a one parameter family of extremals in the plane" by Professor J. D. Mancill, University of Alabama.

22. "Impulse functions" by Dr. F. G. Dressel, Duke University.

23. "On some uses of Plücker tensors which characterize the universality of tensor algebra" by Professor T. L. Wade, Jr., Mercer University.

24. "On measurable stochastic processes" by Warren Ambrose, University of Alabama, introduced by the Secretary.

25. "Vectors associated with a curve in a Riemann $n$-space" by C. L. Seebeck, Jr., University of North Carolina.

26. "Materials and devices as aids to the teaching of mathematics" by Professor Ruth W. Stokes, Winthrop College.

27. "A problem in a one quarter course in freshman mathematics" by Professor Eucebia Shuler, Georgia Southwestern College, introduced by the Secretary.

28. "Relating thinking in mathematics to thinking in life situations" by Professor B. P. Reinsch, Florida Southern College, read by title.

29. "Mathematics as a functional part of the secondary school curriculum" by Professor F. L. Wren, Peabody College, read by Professor J. T. C. Wright.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Professor Lewis was concerned with the defining relations of the simple group of order 360. The classes were catalogued and their relationships simplified and compared.

2. The logarithmic properties of a power of $x$ were discussed by Professor Sewell, and certain interesting comparisons were made.

3. Professor Hyden traced the mathematical work in the Southeastern Sec-

tion. He made some interesting comparisons between the work done in the colleges a century ago with the present.

4. Professor Steen showed how a simple "synthetic multiplication" process could be used to great advantage in division of polynomials, solution of polynomial equations, and in expansions into partial fractions.

5. Mr. Moorman depicted how the philosophy of Descartes sought to make the spirit of mathematics more fruitful. Mathematics influenced the *method* rather than the *content* of Descartes's philosophizing.

6. Professor Henderson illustrated his paper with lantern slides which showed the historical development from Möbius to the present for geometry of vectors and tensors. He applied the new technic of Appell and Thiry to certain salient geometric cases which embraced the transformations in euclidean space from mutually perpendicular axes to oblique curvilinear axes. The fundamental vectors and components of tensors of the first order were exhibited in a series of diagrams.

7. Professor Carver expressed the opinion that many boys and girls who are bored with the formal drill and the more useful applications of mathematics will show a keen interest in a clever, though entirely useless, puzzle. And in solving the useless puzzle he may do considerable work and become familiar with the very same mathematical processes which are needed to solve the more useful problems which arise in business and industry. He then gave examples of such puzzle material involving mathematics of the secondary school and college curriculum.

8. Professor Ballou examined the curves into which the plane sections of a cylinder and a cone are transformed when the cylinders or the cone are developed.

9. Professor Dye developed a Cremona transformation of order 25 in four-dimensional space by means of five pencils of primes and a pencil of quadric primals. The fundamental elements and their images were studied both algebraically and geometrically.

10. Dr. Beers illustrated how reading tests in the subject matter of mathematics have been validated against traditional problem-solving tests. His study suggested the possibility of great flexibility in the materials of instruction, and that differential degrees of understanding may become an objective of teaching.

11. Professor Carver showed that a sphere could be touched by twelve surrounding spheres of the same size in two distinctly different ways, the points of contact forming in one case a very regular pattern, and in the other case a rather unsymmetrical pattern. He exhibited models of the semi-regular polyhedron having vertices at the points of contact, and also of the polyhedron whose faces are the common tangent planes to the spheres at their points of contact. He showed that space could be filled up with closely packed spheres all of the same size with every sphere surrounded in the regular way, with every sphere surrounded in the irregular way, or with some spheres surrounded one

way and some the other; and that in each of these cases the fractional part of space that would be filled with the spheres would be $\pi/3\sqrt{2}$.

12. Professor Blair developed a particular form of the equation of the crunodal cubic which provided a ready means of proof of some interesting properties of curves of this type.

13. Professor Donnell was concerned with a comparison of roots of a cubic with the roots of its Hessian. Certain interesting applications were given.

14. Dr. Webb interpreted the operations of mathematical logic in terms of probability. Particular stress was placed upon the operation of implication and its use in the many-valued logics. Various propositions were interpreted in the light of probability.

15. A contour integral, due to Cauchy, leads to Lagrange's expansion in terms of complex variables. Mr. Mays gave conditions under which this expansion is valid. He showed that the Lagrange expansion form of the solution of Kepler's equation converges in general when the eccentric anomaly is less than $.662743 \cdots$.

16. Mr. Blackwell discussed some metric properties of a family of circular nodal cubics.

17. Professor Barone discussed certain theorems on connected sets of limit points of sequences. He pointed out restrictions that must be made in order that the transforms of the sequence have the same set of limit points.

18. Dr. Rodabough gave an example showing the existence of an open simply connected region in the plane bounded by a simple closed curve and a function of two real variables when the function satisfies certain conditions. When this example was generalized it was found to have a bearing upon the problem of the extension of the function.

19. Professor Royall showed that if a given function is triply monotonic in a given positive finite interval, and the Laplace transform converges in the positive half plane, the transform is univalent in the half plane. An example was given to show that the theorem was not true if the given function is merely doubly monotonic.

20. Dr. Cowan gave a solution of a functional equation with trigonometric coefficients by replacing the coefficients by exponential functions and the insertion of a parameter in the equation.

21. The purpose of Professor Mancill was to discuss the behavior of the envelope of a one parameter family of extremals in a neighborhood of a singular point. His results were used in a proof of the Jacobi condition for unilateral variations.

22. Dr. Dressel was concerned with sufficient conditions on a sequence of functions to insure that it lead to an impulse function.

23. Professor Wade showed that the numerical tensors, the $e$-systems and the Kronecker symbols were special cases of the more general Plücker tensors which are developed in the absolute differential calculus. The adequacy of tensor algebra to deal advantageously with problems of classical invariant theory was

brought out. A preliminary report on the use of tensor algebra in multivariate statistical analysis was made.

24. In this paper probability measures on the space of all real functions of a real variable were considered. Mr. Ambrose gave the product space of this space with the space of real numbers and further considered certain measurability conditions about functions of this product space.

25. Starting with the distance definition in a Riemann $n$-space Mr. Seebeck discussed the vectors associated with a curve. Certain theorems about ennuples were given.

26. In her talk Professor Stokes considered the subject matter of both high school and college mathematics from the standpoint of motivating and improving work by the use of certain materials and devices. She displayed models made of inexpensive materials, and charts showing the uses of mathematics. She called attention to certain free mathematical materials available from certain commercial companies and governmental agencies.

27. Assuming that mathematics has a unique contribution to make toward thought training and toward the development of better citizens, Professor Shuler gave some ideas concerning the type of work which should be given in the one quarter required course in a university system.

28. Professor Reinsch showed that by proper instruction mathematical thinking may have a transfer to thinking in real life situations. Some mathematical errors commonly made were classified and compared with corresponding common errors in everyday thinking.

29. Professor Wren stated that if mathematics is to make its full contribution in the attainment of the instructional objectives set up by the modern philosophy of education, teachers of mathematics must attempt to cultivate in the mind of every student an appreciation of mathematics as an effective mode of thinking as well as an understanding of its tool value in the practices of modern business and industry. The three cycles of mathematical instruction are preparation, foundation, and specialization. Each cycle is divided into three periods of apprehension, application, and abstraction.

H. A. ROBINSON, *Secretary*

# HAUSDORFF MATRICES

H. L. GARABEDIAN, Northwestern University

**Introduction.** The theory of summability of divergent sequences* is indebted to Felix Hausdorff for a highly significant contribution to this field. His elegant treatise on methods of summation and moment sequences [1] has been a source of satisfaction and an incentive to research to many mathematicians. Nevertheless, there remains a wide group, passively or actively interested in summability, who are not acquainted with this great work. This may in part be due to the difficulty of reading the paper. The object of this article is to present the principal ideas of Hausdorff's memoir in a form which should be readily intelligible to the student of analysis. The writer has undertaken this task with the aid of lecture notes from a course in Fourier's series given by J. D. Tamarkin of Brown University.

**1. Regular matrices.** We designate an infinite sequence of numbers by the symbol $\{s_n\}$. If, in particular, we are concerned with the infinite series $\sum_{\nu=0}^{\infty} u_\nu$, then $s_n = \sum_{\nu=0}^{n} u_\nu$. Let $A$ represent the triangular matrix:

$$A = (a_{mn}) = \begin{Vmatrix} a_{00} & 0 & 0 & \cdots \\ a_{10} & a_{11} & 0 & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix}.$$

Then, the sequence $\{t_m\}$, where

$$t_m = \sum_{n=0}^{m} a_{mn} s_n, \tag{1.1}$$

is called the *transform* of $\{s_n\}$ by the matrix $A$. We designate this transformation in symbolic form by

$$t_m = A\{s_m\}.$$

If $\lim_{m\to\infty} t_m$ exists and is equal to $s$, then the sequence $\{s_n\}$ is said to be *summable $A$ to the value $s$*. Moreover, in this case, the matrix $A$ is said to define a *method of summation* (*summability*) for the sequence $\{s_n\}$. If, for every convergent sequence, $\lim_{m\to\infty} s_m = s$ implies $\lim_{m\to\infty} t_m = s$, the matrix $A$ is said to be *regular*.

Necessary and sufficient conditions for the regularity of a matrix $A$ have been found by Silverman [2] and Toeplitz [3] to be

$$(i) \qquad \lim_{m\to\infty} \sum_{n=0}^{m} a_{mn} = 1, \tag{1.2}$$

* Among the introductory expositions on the theory of summability are those by Knopp [6, pp. 457–520], Bromwich [12, pp. 261–322], Hurwitz [13, pp. 17–36], and Carmichael [14, pp. 97–131]. These references may be found written out at length at the end of this paper.

$$(ii) \qquad \sum_{n=0}^{m} | a_{mn} | < M, \qquad (m = 0, 1, 2, \cdots),$$

$M$ independent of $m$,

$$(iii) \qquad \lim_{m\to\infty} a_{mn} = 0, \qquad (n = 0, 1, 2, \cdots).$$

As an example of a regular matrix we exhibit the $(C, \alpha)$ matrix, which defines Cesàro summability of order $\alpha$. Here, we have

$$(1.3) \qquad a_{mn} = \frac{\binom{\alpha + m - n - 1}{m - n}^*}{\binom{\alpha + m}{m}}.$$

The $(C, \alpha)$ matrix is regular for $\alpha > -1$.

**2. Diagonal matrices.** Henceforth we shall designate by $\{A\}$ the class of regular and permutable† matrices. Let $A_1$ and $A_2$ be any two members of this class. Then $A_1$ and $A_2$ define *consistent* methods of summation; that is, when both methods assign values to the same sequence these values are equal. For we assume that

$$A_1\{s_n\} = t_n' \to s',$$
$$A_2\{s_n\} = t_n'' \to s''.$$

Since $A_1$ and $A_2$ are regular, we have

$$A_2\{t_n'\} = A_2A_1\{s_n\} \to s',$$
$$A_1\{t_n''\} = A_1A_2\{s_n\} \to s'',$$

where the operations indicated are the conventional matrix multiplications. Since $A_1$ and $A_2$ are permutable, $A_1A_2 = A_2A_1$, and hence $s' = s''$.‡

We now define a matrix $A$ to be a *diagonal matrix* provided that each of its elements is zero except those on the principal diagonal; that is,

$$A = (\delta_{mn}\mu_m),$$

where

* This symbol is used to designate a binomial coefficient. Thus,

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

We define $\binom{m}{0} = 1$; $\binom{m}{n} = 0$ for $m < n$.

† This implies that if $A_1$ and $A_2$ are any two members of the class $\{A\}$, then $A_1A_2 = A_2A_1$.

‡ It is worthy of notice that regularity and permutability are sufficient, but not necessary, for the consistency of two summation methods. Examples of transformations which are consistent, but not permutable, are easy to find.

$$\delta_{mn} = 0, \qquad (m \neq n),$$
$$\delta_{nn} = 1.$$

We say, further, that a matrix $A$ is reduced to *diagonal form* by the triangular matrix $L$ provided that

$$LAL^{-1} = (\delta_{mn}\mu_m) = M.$$

We shall henceforth use the letter $M$ to represent a diagonal matrix.

Next we consider a fundamental theorem on diagonal matrices.

THEOREM 1. *The matrix $C_1$ of the $(C, 1)$ transformation is reduced to diagonal form by the matrix*

$$D = \left((-1)^n \binom{m}{n}\right).$$

In order to prove this theorem it is sufficient to verify that

$$DC_1D^{-1} = (\delta_{mn}\mu_m),$$

where $\mu_m = 1/(m+1)$. However, it is interesting and instructive to obtain $D$ by solving the matrix equation

$$LC_1L^{-1} = (\delta_{mn}\mu_m) \tag{2.1}$$

for $L$.

From (2.1) we have

$$LC_1L^{-1}L = (\delta_{mn}\mu_m)L,$$

or

$$LC_1 = (\delta_{mn}\mu_m)L. \tag{2.2}$$

This equation may be studied in the form

$$\sum_{k=n}^{m} l_{mk}a_{kn} = \sum_{k=n}^{m} \delta_{mk}\mu_m l_{kn}. \tag{2.3}$$

The left member of this equation is the element in the $m$th row and the $n$th column of the matrix $LC_1$, while the right member is the corresponding element of the matrix $(\delta_{mn}\mu_m)L$.

We recall that the elements $a_{mn}$ of the matrix $C_1$ are given by

$$a_{mn} = \frac{1}{m+1}.$$

Then, (2.3) becomes

$$\sum_{k=n}^{m} \frac{l_{mk}}{k+1} = \mu_m l_{mn}. \tag{2.4}$$

We have to find $l_{mn}$. Suppose that $m$ is fixed. Put $n=m$ in (2.4) and obtain

$$\frac{l_{mm}}{m+1} = \mu_m l_{mm}. \tag{2.5}$$

Since $L^{-1}$ must exist, $l_{mm}\neq 0$, $(m=0, 1, 2, \cdots)$. Then, from (2.5) we have

$$\mu_m = \frac{1}{m+1}.$$

Substituting this value in (2.4) we obtain

$$\sum_{k=n}^{m} l_{mk}\frac{m+1}{k+1} = l_{mn}, \qquad (n \leqq m). \tag{2.6}$$

Replace $n$ by $n+1$ in (2.6) to obtain

$$\sum_{k=n+1}^{m} l_{mk}\frac{m+1}{k+1} = l_{m,n+1}, \qquad (n < m). \tag{2.7}$$

Subtracting (2.7) from (2.6) we get the recursion formula

$$l_{mn}\frac{m+1}{n+1} = l_{mn} - l_{m,n+1}, \qquad (n < m),$$

or

$$l_{mn} = -\frac{n+1}{m-n} l_{m,n+1}, \qquad (n < m). \tag{2.8}$$

From (2.8) it is readily seen that

$$l_{mn} = -\frac{n+1}{m-n}\cdot -\frac{n+2}{m-n-1} l_{m,n+2}, \qquad (n < m-1).$$

By repeated application of the recursion formula (2.8) we obtain finally

$$l_{mn} = (-1)^{m-n}\frac{(n+1)(n+2)\cdots m}{1\cdot 2\cdot 3\cdots(m-n)} l_{mm}.$$

Any choice of $l_{mm}$, different from zero, will then yield a matrix of the desired type. We make the selection

$$l_{mm} = (-1)^m,$$

and thus obtain

$$l_{mn} = (-1)^m\binom{m}{n}$$

as the general element of the required matrix.

It is of some interest to display the matrix $D$ written out at length. We have

$$D = \left((-1)^n \binom{m}{n}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & -1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The theorem which follows exhibits an important property of the matrix $D$.

THEOREM 2. *The matrix $D$ is equal to its own inverse, that is,*

$$D = D^{-1}, \qquad \text{or} \qquad D^2 = I,$$

*where $I$ is the identity matrix.*

The verification of this statement is left to the reader.

**3. Matrices permutable with $C_1$.** The main theorem of this section determines the scope of the class of matrices permutable with $C_1$. For future reference, we define the matrix $M_1$ as follows:

$$M_1 = DC_1D = \left(\frac{\delta_{mn}}{m+1}\right).$$

THEOREM 3. *A necessary and sufficient condition that a triangular matrix $A$ be permutable with $C_1$ is that it can be reduced to diagonal form by the matrix $D$, that is,*

$$M = DAD, \qquad \text{or} \qquad A = DMD,$$

*where $M$ is a diagonal matrix*

Let us first assume that $A = DMD$, and prove that $A$ is permutable with $C_1$. We have

$$AC_1 = DMDDM_1D = DMM_1D,$$

and

$$C_1A = DM_1DDMD = DM_1MD.$$

Since diagonal matrices are always permutable, we have $MM_1 = M_1M$ and hence $AC_1 = C_1A$.

Next, we assume that $A$ is permutable with $C_1$, that is,

$$AC_1 = C_1A. \tag{3.1}$$

Let $A^* = DAD$. We wish to show that $A^* = M$, a diagonal matrix. From (3.1) we have

$$DADDC_1D = DC_1DDAD,$$

or

$$A^*M_1 = M_1A^*.$$

Thus $A^*$ and $M_1$ are permutable. We can now show that any triangular matrix which is permutable with a diagonal matrix whose elements in the principal diagonal are distinct and non-zero, is itself a diagonal matrix. To prove this, let $B=(b_{mn})$ be a triangular matrix and $N=(\delta_{mn}\mu_m)$ a diagonal matrix of the required type. We assume that $BN=NB$, or

$$B(\delta_{mn}\mu_m) = (\delta_{mn}\mu_m)B.$$

Then

$$\sum_{k=n}^{m} b_{mk}\delta_{kn}\mu_k = \sum_{k=n}^{m} \delta_{mk}\mu_m b_{kn},$$

or

$$b_{mn}\mu_n = \mu_m b_{mn},$$

whence

$$b_{mn} = 0, \qquad (m \neq n).$$

Hence $B$ is a diagonal matrix. We have thus proved that $A^*$ is a diagonal matrix.

THEOREM 4. *Any triangular matrix $A$ which is permutable with $C_1$ can be written in the form:*

$$A = \left(\binom{m}{n}\Delta^{m-n}\mu_n\right),\dagger$$

*where the $\mu_m$'s are the diagonal elements of the given matrix.*

We have

$$MD = (\delta_{mn}\mu_m)D = \left(\sum_{k=n}^{m}\delta_{mk}\mu_m(-1)^n\binom{k}{n}\right)$$
$$= \left((-1)^n\mu_m\binom{m}{n}\right).$$

Then, using Theorem 3, we have

$$A = DMD = \left(\sum_{k=n}^{m}(-1)^k\binom{m}{k}(-1)^n\mu_k\binom{k}{n}\right),$$

† The symbol $\Delta^k\mu_n$ is used to denote the $k$th difference of $\mu_n$:

$$\Delta^k\mu_n = \mu_n - \binom{k}{1}\mu_{n+1} + \binom{k}{2}\mu_{n+2} - \cdots + (-1)^k\binom{k}{k}\mu_{n+k}.$$

or, if we set $j=k-n$,

$$A = \left( \sum_{j=0}^{m-n} (-1)^j \binom{m}{j+n} \binom{j+n}{n} \mu_{j+n} \right). \tag{3.2}$$

Since

$$\binom{m}{j+n}\binom{j+n}{n} = \binom{m}{n}\binom{m-n}{j},$$

(3.2) may be written

$$A = \left( \binom{m}{n} \sum_{j=0}^{m-n} (-1)^j \binom{m-n}{j} \mu_{j+n} \right) = \left( \binom{m}{n} \Delta^{m-n}\mu_n \right).$$

It will be of interest to write down the Silverman-Toeplitz regularity conditions for matrices of the special form under consideration.

THEOREM 5. *Necessary and sufficient conditions that a matrix*

$$A = D(\delta_{mn}\mu_m)D = \left( \binom{m}{n} \Delta^{m-n}\mu_n \right)$$

*be regular are*

(*i*) $$\mu_0 = 1,$$

(*ii*) $$\sum_{n=0}^{m} \binom{m}{n} \left| \Delta^{m-n}\mu_n \right| \leqq M, \qquad (m = 0, 1, 2, \cdots),$$

*M independent of m*,

(*iii*) $$\lim_{m\to\infty} \binom{m}{n} \Delta^{m-n}\mu_n = 0, \qquad (n = 0, 1, 2, \cdots).$$

We form the transform of $\{s_n\}$ by $A$, where each element of $\{s_n\}$ is 1. We write

$$t_m = A\{s_m\} = DMD\{s_m\}.$$

Since $\{s_n\} = (1, 1, 1, \cdots)$, we have

$$D\{s_m\} = \{\Delta^m s_0\} = (1, 0, 0, \cdots),$$
$$MD\{s_m\} = (\mu_0, 0, 0, \cdots),$$
$$DMD\{s_m\} = (\mu_0, \mu_0, \mu_0, \cdots).$$

Thus, we have

$$t_m = \sum_{n=0}^{m} a_{mn} = \mu_0.$$

The Silverman-Toeplitz conditions for regularity of the matrix $DMD$ now reduce *a fortiori* to the conditions of our theorem.

A sequence $\{\mu_n\}$ such that

$$\Delta^m \mu_n > 0, \qquad (m, n = 0, 1, 2, \cdots),$$

has been designated by I. Schur [4] as a *totally monotone* sequence. We observe that for totally monotone sequences condition (*ii*) of the preceding theorem reduces to condition (*i*).

**4. Inclusion and equivalence relations.** A method of summation $A$ is said to *include* a method of summation $B$, symbolically $A \supset B$, provided that every sequence summable $B$ is also summable $A$ to the same value. Two methods of summation, $A$ and $B$, are said to be *equivalent*, symbolically $A \approx B$, provided that each method includes the other.

THEOREM 6. *Let $A = D(\delta_{mn}\mu_m^A)D$ and $B = D(\delta_{mn}\mu_m^B)D$ be triangular matrices permutable with $C_1$, and let $B^{-1}$ exist. Then, a necessary and sufficient condition that $A \supset B$ is that the matrix*

$$D\left(\delta_{mn}\frac{\mu_m^A}{\mu_m^B}\right)D$$

*be regular.*

Let $\{s_n\}$ be an arbitrary sequence summable $B$ to the limit $s$. Let $\{t_m^A\}$ and $\{t_m^B\}$ be the transforms of $\{s_n\}$ by $A$ and $B$ respectively. We have

$$t_m^B = B\{s_m\} \to s.$$

We must, then, find necessary and sufficient conditions in order that the transformation from $\{t_m^B\}$ to $\{t_m^A\}$ be regular.

Since $s_m = B^{-1}\{t_m^B\}$, we have

$$t_m^A = A\{s_m\} = AB^{-1}\{t_m^B\}.$$

Moreover, we have

$$AB^{-1} = DM_A DDM_B^{-1}D = DM_A M_B^{-1}D$$
$$= D\left(\delta_{mn}\frac{\mu_m^A}{\mu_m^B}\right)D.$$

Accordingly, a necessary and sufficient condition that $t_m^A \to s$ is that the matrix

$$D\left(\delta_{mn}\frac{\mu_m^A}{\mu_m^B}\right)D$$

be a regular matrix.

From this theorem we obtain *a fortiori* the following theorem on equivalence.

THEOREM 7. *Let $A$ and $B$ be triangular matrices permutable with $C_1$, and let $A^{-1}$ and $B^{-1}$ exist. Then, necessary and sufficient conditions that $A \approx B$ are that the matrices*

$$D\left(\delta_{mn}\frac{\mu_m^A}{\mu_m^B}\right)D, \qquad D\left(\delta_{mn}\frac{\mu_m^B}{\mu_m^A}\right)D$$

*be regular.*

**5. Moment sequences and Hausdorff matrices.** If the function $\phi(u)$ is integrable, say in the sense of Lebesgue,* on the interval (0, 1), symbolically $\phi(u) \subset L$, (0, 1), and if we define $\mu_m$ as follows:

$$\mu_m = \int_0^1 u^m \phi(u)du, \qquad (m = 0, 1, 2, \cdots), \tag{5.1}$$

we designate the sequence $\{\lambda_m\}$ a *moment sequence*. A sequence $\{\lambda_m\}$ such that $D(\delta_{mn}\lambda_m)D$ is a regular matrix is said to be a *regular sequence*. If $\{\lambda_m\}$ is a regular moment sequence the matrix $D(\delta_{mn}\lambda_m)D$ is called a *Hausdorff matrix*.

The theorem which follows is of fundamental significance.

THEOREM 8. *If $\phi(u) \subset L$, (0, 1), and if $\int_0^1\phi(u)du=1$, the moment sequence associated with $\phi(u)$ is regular.*

In order to establish this theorem we have merely to show that the conditions of Theorem 5 are satisfied.

Since $\mu_0=\int_0^1\phi(u)du=1$, condition (*i*) is fulfilled. Moreover, we have

$$\Delta^k\mu_m = \Delta^k\left[\int_0^1 u^m\phi(u)du\right] = \int_0^1 (1-u)^k u^m \phi(u)du.$$

Accordingly,

$$\begin{aligned}
\sum_{n=0}^{m}\binom{m}{n}\left|\Delta^{m-n}\mu_n\right| &= \sum_{n=0}^{m}\binom{m}{n}\left|\int_0^1 (1-u)^{m-n}u^n\phi(u)\right|du \\
&\leqq \sum_{n=0}^{m}\binom{m}{n}\int_0^1 (1-u)^{m-n}u^n\left|\phi(u)\right|du \\
&= \int_0^1\left[\sum_{n=0}^{m}\binom{m}{n}(1-u)^{m-n}u^n\right]\left|\phi(u)\right|du \\
&= \int_0^1 [(1-u)+u]^m\left|\phi(u)\right|du \\
&= \int_0^1 \left|\phi(u)\right|du\dagger \leqq M, \qquad (m = 0, 1, 2, \cdots),
\end{aligned}$$

* It would suffice for the purposes of this paper to require merely the Riemann integrability of the function $\phi(u)$.

† Notice that $\phi(u)\subset L$, (0, 1), implies $|\phi(u)| \subset L$, (0, 1).

$M$ independent of $m$, and condition (*ii*) is satisfied.

It remains to show that condition (*iii*) is satisfied. Let $n$ and $\epsilon>0$ be fixed. We must show that for some $m_0$

$$\left|\binom{m}{n}\int_0^1 u^n(1-u)^{m-n}\phi(u)du\right| < \epsilon, \qquad (m > m_0(n, \epsilon)).$$

We have

$$\left|\binom{m}{n}\int_0^1 u^n(1-u)^{m-n}\phi(u)du\right| \leqq I_1 + I_2, \tag{5.2}$$

where

$$I_1 = \binom{m}{n}\left|\int_0^\delta u^n(1-u)^{m-n}\phi(u)du\right|,$$

$$I_2 = \binom{m}{n}\left|\int_\delta^1 u^n(1-u)^{m-n}\phi(u)du\right|,$$

and $0<\delta<1$. Moreover,

$$\begin{aligned} I_1 &\leqq \int_0^\delta \binom{m}{n} u^n(1-u)^{m-n} \mid \phi(u) \mid du \\ &\leqq \int_0^\delta [(1-u)+u]^m \mid \phi(u) \mid du \\ &= \int_0^\delta \mid \phi(u) \mid du. \end{aligned}$$

Now, we may fix $\delta$ so that

$$I_1 < \frac{\epsilon}{2}\cdot \tag{5.3}$$

As for $I_2$ we have

$$\begin{aligned} I_2 &= O(m^n)\int_\delta^1 u^n(1-u)^{m-n}\phi(u)du^* \\ &= O(m^n(1-\delta)^{m-n}). \end{aligned}$$

Then, for $n$ fixed, there exists an $m_0$ such that

$$I_2 < \frac{\epsilon}{2}, \qquad (m > m_0). \tag{5.4}$$

From (5.2), (5.3), and (5.4) we have the desired result.

* If $g$ is a positive function of a variable which tends to a limit, we find it convenient sometimes to write $f=O(g)$ if $|f|/g<K$, where $K$ is a constant, and $f=o(g)$ if $f/g\to 0$.

We observe in passing that the most general regular moment sequence [5] is defined by the Stieltjes integral

$$\mu_m = \int_0^1 u^m dq(u), \qquad (m = 0, 1, 2, \cdots), \tag{5.5}$$

where

(*i*) $q(u)$ is of bounded variation in the closed interval $(0, 1)$;

(*ii*) $q(u)$ is continuous at $u=0$ and $q(0)=0$;

(*iii*) $q(1)=1$;

(*iv*) $q(u)=\frac{1}{2}[q(u-0)+q(u+0)]$, $0<u<1$.

We shall be concerned in this paper only with moment sequences of the type (5.1).

**6. Examples of Hausdorff matrices.** If $\{\mu_m\}$ is the moment sequence associated with

$$\phi(u) = \alpha(1-u)^{\alpha-1}, \qquad (\alpha > 0),$$

then $D(\delta_{mn}\mu_m)D$ is the matrix $C_\alpha$.

In order to prove this statement we observe first of all that

$$\mu_m = \alpha\int_0^1 u^m(1-u)^{\alpha-1}du,$$

and that the requirements of Theorem 8 are fulfilled. This is an Eulerian integral of the first kind which is easily evaluated. We get

$$\mu_m = \alpha B(m+1, \alpha) = \frac{\Gamma(m+1)\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} = \frac{1}{\binom{\alpha+m}{m}}.$$

In view of Theorem 4 it remains merely to verify that

$$\binom{m}{n}\Delta^{m-n}\frac{1}{\binom{\alpha+n}{n}} = \frac{\binom{\alpha+m-n-1}{m-n}}{\binom{\alpha+m}{m}}.$$

The familiar Hölder matrix $H_\alpha$ associated with summability $(H, \alpha)$ provides another example of a Hausdorff matrix. We define $H_\alpha$ as follows:

$$H_\alpha = D\left(\delta_{mn}\frac{1}{(m+1)^\alpha}\right)D, \qquad (\alpha > -1).$$

This matrix, which is regular for $\alpha > -1$, coincides with the Hölder matrix as originally defined for integral values of $\alpha$ [6].

The moment sequence $\{\mu_m\}$ associated with the function

$$\phi(u) = \frac{1}{\Gamma(\alpha)}\left(\log \frac{1}{u}\right)^{\alpha-1}, \qquad (\alpha > 0),$$

establishes the matrix $H_\alpha$ as a Hausdorff matrix. To prove this we have

$$\begin{aligned}\mu_m &= \frac{1}{\Gamma(\alpha)}\int_0^1 u^m\left(\log \frac{1}{u}\right)^{\alpha-1} du \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-(m+1)x}x^{\alpha-1}dx \\ &= \frac{1}{\Gamma(\alpha)(m+1)^\alpha}\int_0^\infty e^{-t}t^{\alpha-1}dt \\ &= \frac{1}{(m+1)^\alpha}\cdot\end{aligned}$$

Notice that the function $\phi(u)$ satisfies the conditions of Theorem 8.

If $\mu_m = \theta^m$, $0<\theta<1$, then the matrix $D(\delta_{mn}\mu_m)D$ defines the method of summation of Euler-Knopp [7]. When we set $\theta = 2^{-p}$ the symbol $(E, p)$ is used to designate Euler-Knopp summability of order $p$. The Euler-Knopp matrix is a Hausdorff matrix, since the sequence $\{\mu_m\}$ is generated by the integral (5.5) when $q(u)$ has a single discontinuity at the point $u=0$, $0<\theta<1$, and otherwise remains constant.

7. **Properties of the matrix** $C_\alpha$. The property of the matrix $C_\alpha$ embodied in the next theorem is widely known.

THEOREM 9. *If* $-1<\alpha_1<\alpha_2$, *then* $C_{\alpha_1} \subset C_{\alpha_2}$.

Let $\alpha$ and $\beta$ be arbitrarily chosen positive constants. In the integral

$$\mu_m = \int_0^1 u^m\phi(u)du,$$

set

$$\phi(u) = Au^{\beta-1}(1-u)^{\alpha-1}, \qquad (\alpha, \beta > 0),$$

where $A$ is a constant. The constant $A$ is a normalizing factor which we shall presently specify. Since $\alpha$, $\beta > 0$, we have $\phi(u) \subset L$. We obtain

$$\begin{aligned}\mu_m &= A\int_0^1 u^{m+\beta-1}(1-u)^{\alpha-1}du \\ &= A\,\frac{\Gamma(m+\beta)\Gamma(\alpha)}{\Gamma(m+\beta+\alpha)}\cdot\end{aligned}$$

Now, if we select

$$A = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)},$$

we get

$$\mu_m = \frac{\binom{m+\beta-1}{m}}{\binom{m+\alpha+\beta-1}{m}} = \frac{\mu_m''}{\mu_m'}, \tag{7.1}$$

where $\{\mu_m'\}$ and $\{\mu_m''\}$ are the diagonal elements of the matrices $C_{\beta-1}$ and $C_{\alpha+\beta-1}$ respectively. Now, by Theorem 8, the moment sequence (7.1) is a regular moment sequence. It follows by definition that the matrix

$$D\left(\delta_{mn}\frac{\mu_m''}{\mu_m'}\right)D$$

is a regular matrix. Thus it follows from Theorem 6 that for any $\alpha$, $\beta > 0$

$$C_{\beta-1} \subset C_{\alpha+\beta-1}.$$

This statement is equivalent to the statement of the theorem.

In what follows we shall use the symbol $C_A/C_B$ to designate the matrix

$$D\left(\delta_{mn}\frac{\mu_m^A}{\mu_m^B}\right)D,$$

where $\{\mu_m^A\}$ and $\{\mu_m^B\}$ are the diagonal elements of the matrices $C_A$ and $C_B$ respectively.

Moreover let $F_\alpha$ denote the matrix used in Theorem 9 when $\alpha = 1$:

$$F_\beta = \frac{C_\beta}{C_{\beta-1}}. \tag{7.2}$$

The diagonal elements of this regular matrix are given by

$$\mu_m^\beta = \frac{\beta}{\beta + m}, \qquad (\beta > 0).$$

We need the lemma which follows for use in a theorem on the matrix $F_\alpha$.

LEMMA 1. *Let the sequences* $\{\mu_m^i\}$, $(i = 1, 2, \cdots, k)$, *be regular sequences. Then, if* $\sum_{i=1}^k c_i = 1$, *the sequence* $\{\mu_m\}$, *where*

$$\mu_m = \sum_{i=1}^{k} c_i \mu_m^i, \qquad (m = 0, 1, 2, \cdots),$$

*is also regular.*

The condition imposed on the $c_i$'s insures that condition ($i$) of Theorem 5 be fulfilled. The other conditions of this theorem will be satisfied since the difference operation is linear.

THEOREM 10. *The matrix $F_\alpha$ is equivalent to the matrix $F_\beta$, that is,*

$$F_\alpha \approx F_\beta, \qquad (\alpha, \beta > 0).$$

The result

$$F_\alpha \subset F_\beta$$

will be established provided we can show that the matrix $F_\beta/F_\alpha$ is regular. The diagonal elements of $F_\beta/F_\alpha$ are given by

$$\frac{\beta(\alpha+m)}{\alpha(\beta+m)} = \frac{\beta}{\alpha} + \frac{\alpha-\beta}{\alpha}\cdot\frac{\beta}{\beta+m}.$$

Let $\{\mu_m'\} = (1, 1, 1, \cdots)$. Then

$$\frac{\beta(\alpha+m)}{\alpha(\beta+m)} = c_1\mu_m' + c_2\mu_m^\beta,$$

where $c_1=\beta/\alpha$, $c_2=(\alpha-\beta)/\alpha$, $c_1+c_2=1$. Then, by Lemma 1, the matrix $F_\beta/F_\alpha$ is regular.

By a repetition of this argument we can show that the matrix $F_\alpha/F_\beta$ is regular. This proves our theorem.

THEOREM 11. *If $k$, $p$, $r$ are positive integers such that $k=p+r$, then*

$$C_k \approx (C_1)^k \approx C_pC_r.$$

From Theorem 10 we have

$$F_\alpha \approx F_1, \qquad (\alpha > 0).$$

Then, since $F_1=C_1$, it follows that

$$F_\alpha \approx C_1, \qquad (\alpha > 0). \tag{7.3}$$

Since, from (7.2), we have

$$F_2 = \frac{C_2}{C_1}, \tag{7.4}$$

it follows from (7.3) and (7.4) that

$$\frac{C_2}{C_1} = F_2 \approx C_1,$$

whence

$$C_2 \approx (C_1)^2.\text{*} \tag{7.5}$$

To continue, from (7.2) we have

$$F_3 = \frac{C_3}{C_2}. \tag{7.6}$$

Then, from (7.3) and (7.6) we obtain

$$\frac{C_3}{C_2} = F_3 \approx C_1,$$

or

$$C_3 \approx C_1 C_2.$$

Using (7.5) we have

$$C_3 \approx C_1(C_1)^2 = (C_1)^3.$$

By induction it follows that

$$C_k \approx (C_1)^k.$$

Finally, we have

$$C_p C_r \approx (C_1)^p (C_1)^r = (C_1)^{p+r} = (C_1)^k.$$

This proves our theorem.

Since $C_1 = H_1$, then it follows from Theorem 11 that

$$C_k \approx (C_1)^k = (H_1)^k = H_k.$$

This constitutes a proof of the equivalence of the Cesàro and Hölder methods of summation for positive integral orders. Among the writers who obtained this result before Hausdorff, using longer and less elegant methods, are Knopp [**8**], Schnee [**9**], Ford [**10**], and Schur [**11**]. In the last section of this account Hausdorff's extension of this theorem to non-integral orders will be reproduced. The most recent and shortest proof of the equivalence theorem under discussion is due to Hille and Tamarkin [**5**].

**8. Essentially regular sequences.** A sequence $\{\mu_m\}$ is said to be *essentially regular* provided that

$$\text{(i)} \qquad \sum_{n=0}^{m} \binom{m}{n} \left| \Delta^{m-n}\mu_n \right| < M, \qquad (m = 0, 1, 2, \cdots),$$

$M$ independent of $m$,

* The reader is reminded that due to the structure of the matrices with which we are concerned matrix operations are made with almost the freedom permitted with elementary algebraic operations.

(ii) $$\binom{m}{n}\Delta^{m-n}\mu_n = o(1), \qquad (n = 0, 1, 2, \cdots).$$

The theorems developed in this section will be of aid in establishing the equivalence proof of §9.

THEOREM 12. *A linear combination of essentially regular sequences is an essentially regular sequence.*

This theorem follows immediately from the linearity of the difference operation.

THEOREM 13. *An essentially regular sequence for which $\mu_0 = 1$ is a regular sequence.*

This theorem is established with the observation that the conditions of our definition include all those of regularity except normalization.

THEOREM 14. *If $\Phi(x) \subset L$, $(0, \infty)$ and $\int_0^\infty \Phi(x)dx = 1$, then the sequence $\{\mu_m\}$, where*

$$\mu_m = \int_0^\infty e^{-mx}\Phi(x)dx,$$

*is a regular sequence.*

This is valid statement, since the integral merely provides an alternate expression for the moment sequence (5.1) effected by the substitution $u = e^{-x}$.

THEOREM 15. *If $\Phi(x) \subset L$, $(0, \infty)$, then the associated moment sequence is essentially regular.*

This follows from the proof of Theorem 8.

THEOREM 16. *The product of two essentially regular sequences is a regular sequence.*

Let $\{\mu_m'\}$, $\{\mu_m''\}$ be essentially regular sequences. We set $A = 1 - \mu_0'$, $B = 1 - \mu_0''$. Then, by Theorems 12 and 13, the sequences $\{\mu_m' + A\}$ and $\{\mu_m'' + B\}$ are regular sequences. We write

$$(8.1) \qquad \mu_m''' = (\mu_m' + A)(\mu_m'' + B) = \mu_m'\mu_m'' + B\mu_m' + A\mu_m'' + AB.$$

Now, the sequence $\{\mu_m'''\}$ is regular, since the product of two regular sequences is a regular sequence.* From (8.1) we have

$$\mu_m'\mu_m'' = \mu_m''' - B\mu_m' - A\mu_m'' - AB.$$

Then, by Theorem 12, our theorem is established.

THEOREM 17. *If $\Phi(x) \subset L$, $(0, \infty)$, and is absolutely continuous, and if $\{\mu_m\}$ is the moment sequence associated with $\Phi(x)$, then the sequence $\{v_m\}$, where*

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* The reader may easily verify this statement.

$$v_m = \begin{cases} \Phi(0) + \int_0^\infty \Phi'(x)dx, & (m = 0), \\ m\mu_m, & (m > 0), \end{cases}$$

*is essentially regular.*

By hypothesis we have

$$\Phi(x) = \Phi(0) + \int_0^x \Phi'(t)dt.$$

Moreover,

$$\Phi(\infty) = \lim_{x\to\infty} \Phi(x) = 0.$$

Accordingly, we write

$$\mu_m = \int_0^\infty e^{-mx}\left[\Phi(0) + \int_0^x \Phi'(t)dt\right]dx, \qquad (m > 0).$$

Then, we integrate by parts and obtain

$$\mu_m = \frac{\Phi(0)}{m} + \frac{1}{m}\int_0^\infty e^{-mx}\Phi'(x)dx,$$

or

$$m\mu_m = \Phi(0) + \int_0^\infty e^{-mx}\Phi'(x)dx.$$

Thus, by Theorems 12 and 15, our result is established.

**9. Equivalence of the Cesàro and Hölder methods of summation.**

THEOREM 18. *The $(C, \alpha)$ and $(H, \alpha)$ methods of summation are equivalent for all real values of $\alpha > -1$.*

Let $\{\mu_m'\}$ and $\{\mu_m''\}$ be the moment sequences associated with the functions

$$\Phi(x) = \frac{1}{\Gamma(\beta)}(e^{-x}x^{\beta-1}), \qquad (\beta > 0),$$

$$\Psi(x) = \frac{1}{\Gamma(\beta)}(1 - e^{-x})^{\beta-1}e^{-\alpha x}, \qquad (\alpha, \beta > 0),$$

respectively. It is clear that for $\alpha$, $\beta > 0$, $\Phi(x)$, $\Psi(x)$, $\Phi(x) - \Psi(x) \subset L$, $(0, \infty)$. Thus, the sequence defined by

$$\mu_m' - \mu_m'' = \int_0^\infty e^{-mx}[\Phi(x) - \Psi(x)]dx$$

is essentially regular.

If we can show that $\Phi'(x)-\Psi'(x)\subset L$, $(0, \infty)$, then Theorem 17 can be applied to show that $\{m(\mu_m'-\mu_m'')\}$ is essentially regular. We observe first of all that $\Phi(x)-\Psi(x)\to 0$ as $x\to\infty$. We must also show that $\Psi(x)-\Psi(x)\to 0$ as $x\to 0$. We have

$$\Gamma(\beta)[\Phi(x) - \Psi(x)] = e^{-x}x^{\beta-1} - (1 - e^{-x})^{\beta-1}e^{-\alpha x}$$
$$= x^{\beta-1}\left\{e^{-x} - e^{-\alpha x}\left(1 - \frac{x}{2!} + \frac{x^2}{3!} - \cdots\right)^{\beta-1}\right\}.$$

Setting

$$A(x) = -\frac{x}{2!} + \frac{x^2}{3!} - \cdots,$$

we have

$$\Gamma(\beta)[\Phi(x) - \Psi(x)] = x^{\beta-1}\{e^{-x} - e^{-\alpha x}(1 + A(x))^{\beta-1}\},$$

where $A(x)=O(x)$. Now

$$[1 + A(x)]^{\beta-1} = 1 + (\beta - 1)A(x) + \frac{(\beta - 1)(\beta - 2)}{2!}A^2(x) + \cdots$$
$$= 1 + B(x).$$

Observe that

$$\frac{B(x)}{A(x)} = \frac{[1 + A(x)]^{\beta-1} - 1}{A(x)} = O(1).$$

Then

$$\frac{B(x)}{x} = \frac{B(x)}{A(x)}\,\frac{A(x)}{x} = O(1),$$

and

$$B(x) = O(x).$$

Accordingly,

$$\Phi(x) - \Psi(x) = x^{\beta-1}O(x) = x^{\beta}O(1), \qquad (\beta > 0).$$

Hence, $\Phi(x)-\Psi(x)\to 0$ as $x\to 0$. Thus,

$$\int_0^\infty (\Phi'(x) - \Psi'(x))dx = 0,$$

and the sequence $\{m(\mu_m'-\mu_m'')\}$ is essentially regular.

Now, consider the sequence defined by

$$(m+\alpha)\mu_m'' - (m+1)\mu_m'.$$

This is equal to the expression

$$m(\mu_m'' - \mu_m') + \alpha\mu_m'' - \mu_m',$$

which, by Theorems 12 and 17, defines an essentially regular sequence. Let us express this sequence is explicit form. We have

$$\mu_m'' = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha+\beta+m)},$$

$$\mu_m' = (m+1)^{-\beta},$$

and hence

$$(m+\alpha)\mu_m'' - (m+1)\mu_m' = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+\beta+m)} - (m+1)^{1-\beta}.$$

As a special case take $\alpha+\beta=1$, $0<\alpha$, $\beta<1$. The general element of our sequence becomes

$$\text{(9.1)} \qquad \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+\beta+m)} - (m+1)^{\alpha} = \binom{m+\alpha}{m}\Gamma(\alpha+1) - (m+1)^{\alpha}.$$

This expression defines an essentially regular sequence. Moreover, we know that $(m+1)^{-\alpha}$ defines a regular sequence. By Theorem 16, the product of (9.1) by $(m+1)^{-\alpha}$ will define an essentially regular sequence. We get

$$\text{(9.2)} \qquad \frac{\binom{m+\alpha}{m}}{(m+1)^{\alpha}}\Gamma(\alpha+1) - 1.$$

To the sequence defined by (9.2) add the unit sequence and divide by $\Gamma(\alpha+1)$. By Theorem 12, we have left the essentially regular sequence

$$\left\{\frac{\binom{m+\alpha}{m}}{(m+1)^{\alpha}}\right\}.$$

Moreover, this sequence is regular, since the first element is 1. Hence, by Theorem 6, we obtain

$$C_\alpha \subset H_\alpha.$$

Had we used $\binom{m+\alpha}{m}^{-1}$ as a multiplier above instead of $(m+1)^{-\alpha}$ we should have obtained the result

$$C_\alpha \supset H_\alpha.$$

Hence, for $0<\alpha<1$ we have proved that

$$H_\alpha \approx C_\alpha. \tag{9.3}$$

Now, we show by induction that this relation holds for all $\alpha>0$. Let us assume that (9.3) is true for $\alpha\leqq m$, and prove that it remains true for $m<\alpha<m+1$. We have, from Theorem 10 and (7.2),

$$\frac{C_\alpha}{C_{\alpha-1}} = F_\alpha \approx C_1, \qquad (\alpha > 0).$$

Thus,

$$C_\alpha \approx C_1C_{\alpha-1}, \qquad (\alpha > 0).$$

Moreover, by definition, we have

$$H_\alpha = H_1H_{\alpha-1}, \qquad (\alpha > 0).$$

Then, for $\alpha-1<m$, we have

$$C_\alpha \approx C_1C_{\alpha-1} \approx H_1H_{\alpha-1} = H_\alpha,$$

or, for $m<\alpha<m+1$,

$$C_\alpha \approx H_\alpha.$$

This result is then established for $\alpha>0$.

We can also extend this result to hold for $\alpha>-1$. For $\alpha>0$ we have

$$H_1H_{\alpha-1} = H_\alpha \approx C_\alpha \approx C_1C_{\alpha-1} = H_1C_{\alpha-1},$$

and hence

$$H_{\alpha-1} \approx C_{\alpha-1}, \qquad (\alpha > 0).$$

Thus, we obtain the desired result.

Observe that for $\alpha$, $\beta$, $\alpha+\beta>-1$, we have

$$C_\alpha C_\beta \approx H_\alpha H_\beta = H_{\alpha+\beta} \approx C_{\alpha+\beta}.$$

Thus, we establish the statement

$$C_\alpha C_\beta \approx C_{\alpha+\beta}$$

as a corollary to our equivalence theorem.

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# A CLASS OF SURFACES APPLICABLE TO THE SPHERE

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**1. Introduction.** Most books on differential geometry give a very limited number of illustrations of surfaces of constant positive curvature (*i.e.*, applicable to the sphere), but give no general formula for such surfaces. While this paper does not give a general formula for all the surfaces applicable to the sphere of unit radius, a certain class of surfaces more general than one usually finds in the texts is given. This is accomplished by finding solutions for Gauss's and Codazzi's equations, regarding these as differential equations in $D$, $D'$, and $D''$—the fundamental coefficients of the second order. Because of difficulties of integration of these equations, certain restrictions will be made on the forms of $D$, $D'$, and $D''$.

**2. The forms of $D$, $D'$, and $D''$.** A sphere of unit radius is given by the equations

$$x = \cos u \cos v, \qquad y = \cos u \sin v, \qquad z = \sin v.$$

It is easily seen that the fundamental coefficients of the first order have the values

$$E = 1, \qquad F = 0, \qquad G = \cos^2 u.$$

From the well known expression for total curvature, we have the equation $(DD''-D'^2)/(EG-F^2)=1$, or

$$DD'' - D'^2 = \cos^2 u. \tag{1}$$

Henceforth in this paper we shall restrict our considerations to the cases where $D$, $D'$ and $D''$ are functions of $u$ only.

The two Codazzi equations are

$$\partial D/\partial v - \partial D'/\partial u = l'D + (m' - l)D' - mD'',$$
$$\partial D'/\partial v - \partial D''/\partial u = l''D + (m'' - l')D' - m'D''.$$

The $l$'s and $m$'s are Christoffel's triple index symbols of the second kind, and may easily be computed. Because of the above mentioned restriction any differential with respect to $v$ becomes zero. Putting in the values of the $l$'s and $m$'s, the Codazzi equations become

$$\partial D'/\partial u = D' \tan u, \tag{2}$$
$$- \partial D''/\partial u = \cos u \sin uD + D'' \tan u. \tag{3}$$

Solving the first of these by separating the variables, we have

$$D' = h \sec u, \tag{4}$$

where $h$ is the constant of integration. From the equations (1) and (4), we have

$$D = (\cos^2 u + D'^2)/D'' = (\cos^2 u + h^2 \sec^2 u)/D''. \tag{5}$$

Putting this value of $D$ in (3), we get

$$\partial D''/\partial u + \sin u \cos u(\cos^2 u + h^2 \sec^2 u)/D'' + D'' \tan u = 0. \tag{6}$$

In order to get a solution for this, let $D'' = fw$, $f$ and $w$ being functions of $u$. Then

$$\partial D''/\partial u = fdw/du + wdf/du,$$

and (6) becomes

$$fdw/du + w(df/du + f \tan u) + \sin u \cos u(\cos^2 u + h^2 \sec^2 u)/fw = 0. \tag{7}$$

By choosing $f = \cos u$, the coefficient of $w$ becomes zero, and (7) may be written

$$\cos udw/du + \sin u \cos u(\cos^2 u + h^2 \sec^2 u) \; w \cos u = 0.$$

Solving this by separating the variables, we obtain

$$w = \sqrt{k^2 - \sin^2 u - h^2 \sec^2 u}, \tag{8}$$

$k$ being another constant. We may therefore write

$$D = (\cos^2 u + h^2 \sec^2 u)/w \cos u,$$
$$D' = h \sec u, \qquad D'' = w \cos u.$$

**3. The tangents to the parameter curves, and the normal to the surface.** The notations and certain equations in the following are given by Eisenhart.*

* L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Chap.V.

Let $X_1$, $Y_1$, $Z_1$ denote the direction cosines of the tangent to the curves for $u$ constant; $X_2$, $Y_2$, $Z_2$ for $v$ constant; and $X$, $Y$, $Z$, the direction cosines of the normal to the surface. From pages 152–157 in Eisenhart, together with the values of the fundamental coefficients and the Christoffel symbols, the derivatives of these direction cosines may be computed in terms of the direction cosines. These results are listed below for the $X$'s. Similar expressions may be obtained involving the $Y$'s and the $Z$'s respectively.

$$\partial X_1/\partial u = (\cos^2 u + h^2 \sec^2 u)X/w \cos u. \tag{9}$$

$$\partial X_1/\partial v = -X_2 \sin u + Xh \sec u. \tag{10}$$

$$\partial X_2/\partial u = Xh \sec^2 u. \tag{11}$$

$$\partial X_2/\partial v = X_1 \sin u + wX. \tag{12}$$

$$\partial X/\partial u = -(\cos^2 u + h^2 \sec^2 u)X_1/w \cos u - hX_2/\cos^2 u. \tag{13}$$

$$\partial X/\partial v = -hX_1 \sec u - wX_2. \tag{14}$$

The author sees no way to get a *general* solution for this set of equations. Instead, then, we shall consider the problem of finding *any* solution that is more inclusive, and gives a larger family of surfaces than is generally given. At the same time, it will be seen that the surfaces generally given will be included in the family thus obtained.

Because of the mutual perpendicularity of the tangents and normal mentioned above, we have

$$Z_1^2 + Z_2^2 + Z^2 = 1.$$

By inspection, we may put

$$Z_1 = w/k, \qquad Z_2 = h \sec u/k, \qquad Z = -\sin u/k. \tag{15}$$

After experimenting with various forms for the $X$'s as given in equations (9) to (14), we have found that the following form for $X_2$ leads to a satisfactory solution of the problem. Let

$$X_2 = F(u)\phi'(v) + \Theta(u)\psi'(v). \tag{16}$$

Then, from (11), we obtain

$$X = (F'(u)\phi'(v) + \Theta'(u)\psi'(v))/h \sec^2 u. \tag{17}$$

From (10) we have

$$X_1 = (F'(u) \cos u - F(u) \sin u)\phi(v) + (\Theta'(u) \cos u - \Theta(u) \sin u)\psi(v). \tag{18}$$

Let

$$f'(u) = F'(u) \cos u - F(u) \sin u,$$

and

$$\theta'(u) = \Theta'(u) \cos u - \Theta(u) \sin u.$$

Then

$$f(u) = F(u) \cos u, \qquad \text{or} \qquad F(u) = f(u) \sec u,$$

or

$$F'(u) = \sec uf'(u) + \sec u \tan uf(u).$$

The values of $\Theta(u)$ and $\Theta'(u)$ are of the same form. Let us rewrite equations (16), (17) and (18), and at the same time put $\phi(v) = \sin rv$ and $\psi(v) = \cos rv$, $r$ being any constant.

$$(19)\quad \begin{aligned} X_1 &= f'(u) \sin rv + \theta'(u) \cos rv, \\ X_2 &= r \sec u(f(u) \cos rv - \theta(u) \sin rv), \\ X &= [(f'(u) \cos u + f(u) \sin u) \cos rv - (\theta'(u) \cos u + \theta(u) \sin u) \sin rv]r/h. \end{aligned}$$

Now $X_1Z_1+X_2Z_2+XZ=0$. Putting in this equation the relations (15) and (19), we obtain an equation in which every term contains sin $rv$ or cos $rv$. The equation will be satisfied if the coefficients of sin $rv$ and of cos $rv$ are each zero. Taking into account also the relation $h^2\sec^2 u+\sin^2 u = k^2-w^2$, we have, from the coefficient of cos $rv$

$$(20)\qquad \theta'(u) = [(k^2 - w^2)f(u) + \sin u \cos uf'(u)]r/wh.$$

If in equation (12) we put the value of $X_1$ and $X$ from the relations in (19), and then differentiate the second equation in (19) with respect to $v$, we shall have two expressions for $\partial X_2/\partial v$. Equating these two values we again may satisfy the resulting equation by putting the coefficients of sin $rv$ and cos $rv$ equal to zero. This gives two equations involving $f(u)$, $\theta(u)$, $f'(u)$ and $\theta'(u)$. Putting $\theta'(u)$ from (20) into these two equations, we get

$$- rwh \sec u\theta(u) - [\sin u(k^2 - w^2) + w^2 \sin u]f(u) - \cos u[\sin^2 u + w^2]f'(u) = 0,$$

$$rwh \sin u\theta(u) - r^2[h^2 \sec^2 u - \cos u(k^2 - w^2)]f(u) - \sin u[h^2 - r \cos^2 u]f'(u) = 0.$$

Multiplying the first of these equations by cos $u$, and the second by csc $u$, and adding, $\theta(u)$ will be eliminated. Also taking into account the value of $w$, this sum gives

$$- [f(u) \sin u \cos u + f'(u) \cos^2 u](k^2 - r^2) = 0$$

from which we find, $r = k$.

Hence, the relations (19) may be written,

$$(21)\quad \begin{aligned} X_1 &= f'(u) \sin kv + \theta'(u) \cos kv \\ X_2 &= k \sec u[f(u) \cos kv - \theta(u) \sin kv] \\ X &= [(f'(u) \cos u + f(u) \sin u) \cos kv - (\theta'(u) + \theta(u) \sin u) \sin kv]k/h. \end{aligned}$$

The symmetry of the functions $f(u)$ and $\theta(u)$ suggest the following forms for

the $Y$'s. It develops later that these values are the ones which will give the desired surfaces.

$$
\begin{aligned}
&Y_1 = \theta'(u) \sin kv - f'(u) \cos kv \\
(22)\quad &Y_2 = k \sec u[\theta(u) \cos kv + f(u) \sin kv] \\
&Y = [(\theta'(u) \cos u + \theta(u) \sin u) \cos kv + (f'(u) \cos u + f(u) \sin u) \sin kv]k/h.
\end{aligned}
$$

These, together with the values of the $Z$'s given in (15), shall be the final values of the direction cosines, except for the determination of the forms of $f(u)$ and $\theta(u)$.

4. **The forms of $f(u)$ and $\theta(u)$.** $X_2^2 + Y_2^2 + Z_2^2 = 1$. Putting in the values given in (15), (21) and (22), we see that the equation reduces to

$$k^4 f^2(u) + k^4 \theta^2(u) = k^2 \cos^2 u - h^2,$$

or

$$\left[\frac{k^2 f(u)}{\sqrt{k^2 \cos^2 u - h^2}}\right]^2 + \left[\frac{k^2 \theta(u)}{\sqrt{k^2 \cos^2 u - h^2}}\right]^2 = 1.$$

In order to solve this, let

$$(23)\quad \frac{k^2 f(u)}{\sqrt{k^2 \cos^2 u - h^2}} = \cos A, \quad \frac{k^2 \theta(u)}{\sqrt{k^2 \cos^2 u - h^2}} = \sin A, \quad \frac{k^2}{\sqrt{k^2 \cos^2 u - h^2}} = \frac{1}{t},$$

where $A$ is, of course, a function of $u$. Then

$$(24)\qquad f(u) = t \cos A, \qquad \theta(u) = t \sin A,$$

or

$$(25)\qquad f'(u) = -tA' \sin A + t' \cos A, \qquad \theta'(u) = tA' \cos A + t' \sin A.$$

Let us put the values of $f(u)$ and $f'(u)$ from (24) and (25) into (20) with $r = k$, and equate this value of $\theta'(u)$ to the one given in (25). We obtain

$$
\begin{aligned}
(26)\quad &[(k^2 - w^2)t \cos A + \sin u \cos u(-tA' \sin A + t' \cos A)]k/wh \\
&\qquad = tA' \cos A + t' \sin A.
\end{aligned}
$$

If we simplify this by putting in the value of $t$ from (23), by replacing $t'$ by $-\sin u \cos u/\sqrt{k^2 \cos^2 u - h^2}$, and by making other more or less obvious reductions, we get

$$(wh \cos A + k \sin A)[kwh - (k^2 \cos^2 u - h^2)A'] = 0;$$

whence

$$A' = dA/du = kwh/(k^2 \cos^2 u - h^2),$$

or

$$A = \int \frac{kwh\, du}{k^2 \cos^2 u - h^2}.$$

Hence, we have, from (24)

$$
(27)\qquad
\begin{aligned}
f(u) &= \frac{\sqrt{k^2\cos^2 u - h^2}}{k^2}\cos\int\frac{kh\sqrt{k^2 - \sin^2 u - h^2\sec^2 u}}{k^2\cos^2 u - h^2}\,du,\\
\theta(u) &= \frac{\sqrt{k^2\cos^2 u - h^2}}{k^2}\sin\int\frac{kh\sqrt{k^2 - \sin^2 u - h^2\sec^2 u}}{k^2\cos^2 u - h^2}\,du.
\end{aligned}
$$

**5. The values of** $x$, $y$, **and** $z$. It remains to express $x$, $y$, and $z$, the coördinates of a point on the surface, in terms of $u$ and $v$. From formulas on p. 157 in Eisenhart, we have

$$\frac{\partial x}{\partial u} = X_1, \qquad \text{and} \qquad \frac{\partial x}{\partial v} = X_2\cos u.$$

Putting in the values of $X_1$ and $X_2$ from (21) in these expressions, we obtain

$$\frac{\partial x}{\partial u} = f'(u)\sin kv + \theta'(u)\cos kv,$$

and

$$\frac{\partial x}{\partial v} = k[f(u)\cos kv - \theta(u)\sin kv].$$

In much the same way, we get

$$\frac{\partial y}{\partial u} = \theta'(u)\sin kv - f'(u)\cos kv,$$

$$\frac{\partial y}{\partial v} = k[\theta(u)\cos kv + f(u)\sin kv],$$

$$\frac{\partial z}{\partial u} = -\frac{w}{k}, \qquad \frac{\partial z}{\partial v} = \frac{h}{k}\cdot$$

Integrating these expressions, we finally have the desired class of surfaces:

$$
(28)\qquad
\begin{aligned}
x &= f(u)\sin kv + \theta(u)\cos kv,\\
y &= \theta(u)\sin kv - f(u)\cos kv,\\
z &= -\frac{1}{k}\int\sqrt{k^2 - \sin^2 u - h^2\sec^2 u}\,du + \frac{h}{k}v,
\end{aligned}
$$

where $f(u)$ and $\theta(u)$ are given by (27).

**6. Some special cases.** Let $d$ be the distance from a point on the surface to the $z$ axis. Then $d^2 = x^2 + y^2 = f^2(u) + \theta^2(u)$. Hence, in general, the curves $u =$ a constant are helices on a cylinder whose radius is $d$.

(a) Consider the case $h=0$. We have

$$x = \frac{1}{k}\cos u \sin kv, \qquad y = -\frac{1}{k}\cos u \cos kv, \qquad z = -\frac{1}{k}\int \sqrt{k^2 - \sin^2 u}\, du.$$

These are the well known surfaces of revolution which are applicable to the sphere, where $u=c$ are the parallels, and $v=c$ the meridians. Let us consider a few special values of $k$.

If $k=1$, we have

$$x = \cos u \cos v, \qquad y = -\cos u \cos v, \qquad z = -\sin u,$$

which is the sphere itself.

If $k<1$, $\sin u$ must be less than 1 if the surface is to be real. And hence $\cos u>0$, or $d$ must have a minimum value; $d=1/k\cdot\cos u$. A section of such a surface is shown in Fig. 1.

If $k>1$, $d$ has every value from 0 to $1/k$. The surface is a spindle-shaped surface shown in Fig. 2. If $\phi$ is the angle with which the surface cuts the $z$ axis, it can be shown that $\sin\phi = 1/k$.

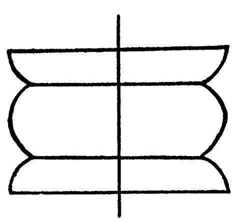

FIG. 1

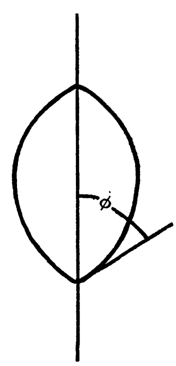


FIG. 2

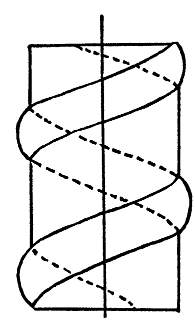

FIG. 3

(b) Consider the case $h\neq 0$.

If $h>k$, $f(u)$ and $\theta(u)$ become imaginary, and no real surface exists.

If $h=k$, the surface degenerates into the $z$ axis. Hence to have any real surface we must have $h<k$. Although it is interesting to study the surfaces that one gets for specific values of $h$ and $k$, the detail is rather lengthy. Let it suffice to say that the surfaces are helicoids somewhat like shown in Fig. 3, having for $d$ least value greater than zero, and a maximum value. The helicoids mentioned in Ex. 5, p. 291, of Eisenhart are special cases of these surfaces.

# A NEW PROOF OF STURM'S COMPARISON THEOREMS*

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**1. Introduction.** Sturm's comparison theorems are concerned with the zeros of the solutions of differential equations of the type

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] + g(x)y = 0 \qquad (0 < f). \tag{1}$$

For a long time the methods of proving these theorems were those of Sturm.† M. Bôcher‡ was the first to observe that it is useful to transform the differential equation of the second order into the corresponding Riccati differential equation

$$u' + hu^2 + gu = 0 \qquad \text{where} \qquad h = 1/f, \qquad yu(x) = fy'.$$

We shall go a step further and set $u(x) = \operatorname{ctg}\,\theta(x)$. We obtain the differential equation

$$\theta' = h\cos^2\theta + g\sin^2\theta, \tag{2}$$

*i.e.*, the equation which H. Prüfer§ obtained by replacing equation (1) by the system

$$y' = hz, \qquad z' = -gy, \tag{3}$$

and determining $y$ and $z$ by polar coördinates

$$y = \rho(x)\sin\theta(x), \qquad z = \rho(x)\cos\theta(x).$$

Then $\theta$ is determined by (2), and

$$\rho(x) = C\exp\frac{1}{2}\int_a^x (h-g)\sin 2\theta\, dx.$$

The aim of this paper is to show that the proof of the comparison theorems becomes especially simple after this transformation.‖ Instead of (1) or (3) I shall consider the more general system¶

$$y' = Py + Qz, \qquad z' = Ry + Sz.$$

2. THEOREM I. *If $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are continuous in the interval $a \leqq x \leqq b$, then the non-trivial solutions** $y(x)$, $z(x)$ of the system*

* Presented to the American Mathematical Society, September 1938.

† Journal de mathématiques pures et appliquées, vol. 1, 1836, pp. 106–186.

‡ Transactions of the American Mathematical Society, vol. 1, 1900, pp. 414–420.

§ Mathematische Annalen, vol. 95, 1926, pp. 499 ff., especially p. 503. Prüfer is not concerned with the comparison theorems. His aim is to obtain as simply as possible the oscillation and expansion theorems for Sturm's boundary problem. System (3) was also studied by Sturdivant. See Transactions of the American Mathematical Society, vol. 30, 1928, p. 560.

‖ For use of this method to prove separation theorems, see E. Kamke, Mathematica, vol. 15, 1939, pp. 201–203, and to prove oscillation theorems, E. Kamke, Mathematische Zeitschrift, vol. 44, 1939, pp. 619–658.

¶ Cf. Bôcher, Transactions of the American Mathematical Society, vol. 3, 1902, pp. 196–215.

** That is, the solutions with $|y(x)| + |z(x)| > 0$ in $a \leqq x \leqq b$.

$$y' = Py + Qz, \qquad z' = Ry + Sz, \tag{4}$$

*are the functions*

$$y = C\rho(x) \sin \theta(x), \qquad z = C\rho(x) \cos \theta(x), \tag{5}$$

*where $C$ is any constant different from zero, $\theta(x)$ is a solution of the differential equation*

$$\theta' = Q \cos^2 \theta + (P - S) \sin \theta \cos \theta - R \sin^2 \theta, \tag{6}$$

*with an initial value $0 \leqq \theta(a) < \pi$, and*

$$\rho(x) = \exp \int_a^x [P \sin^2 \theta + (Q + R) \sin \theta \cos \theta + S \cos^2 \theta] dx. \tag{7}$$

*Proof*: For every non-trivial solution $y$, $z$ of (4), there exists a constant $C \neq 0$ and functions $\rho(x)$, $\theta(x)$ which have continuous derivatives and for which $\rho(a) = 1$, $0 \leqq \theta(a) < \pi$, and (5) holds true. From (5) we have

$$\begin{aligned} y' &= C\rho' \sin \theta + C\theta'\rho \cos \theta, \\ z' &= C\rho' \cos \theta - C\theta'\rho \sin \theta, \end{aligned} \tag{8}$$

and so, because of (4),

$$\begin{aligned} \rho' \sin \theta + \theta'\rho \cos \theta &= (P \sin \theta + Q \cos \theta)\rho, \\ \rho' \cos \theta - \theta'\rho \sin \theta &= (R \sin \theta + S \cos \theta)\rho. \end{aligned} \tag{9}$$

Hence we get (6) and

$$\rho' = [P \sin^2 \theta + (Q + R) \sin \theta \cos \theta + S \cos^2 \theta]\rho, \tag{10}$$

and therefore (7), since $\rho(a) = 1$. Conversely (10) is a consequence of (7), and (9) a consequence of (6) and (10). Therefore the functions $y(x)$, $z(x)$, defined by (5), satisfy equations (4).

Thus system (4) reduces to the single equation (6). Since $\rho(x)$ never vanishes, the zeros of $y$ are given by $\theta = k\pi$ and the zeros of $z$ by $\theta = \pi/2 + k\pi$ where $k$ is an integer.

**3.** THEOREM II. *If $A(x)$, $B(x)$, $C(x)$ are continuous in the interval $a \leqq x \leqq b$ and if $\theta(x, \alpha)$ is a solution of the differential equation*

$$\theta' = A \cos^2 \theta + B \sin 2\theta + C \sin^2 \theta \tag{11}$$

*with the initial value $\theta(a, \alpha) = \alpha$, then we have*:

(a) *For every $\alpha$ there is exactly one solution $\theta(x, \alpha)$ and this exists throughout the interval $a \leqq x \leqq b$;*

(b) $\theta(x, \alpha + \pi) = \theta(x, \alpha) + \pi$;

(c) *If $\alpha$ increases from $\omega$ to $\omega + \pi$, then $\theta(x, \alpha)$ increases ($x$ being fixed) from $\theta(x, \omega)$ to $\theta(x, \omega) + \pi$.*

*Proof:* Since the function on the right of (11) is a bounded and continuous function of $\theta$, $x$ and has a continuous derivative with respect to $\theta$, the statement (a) is implied by the well known existence theorems of the theory of differential equations. If $\theta(x, \alpha)$ is a solution of (11), then obviously $\theta(x, \alpha)+\pi$ is also a solution, and it is a solution with the initial value $\alpha+\pi$; but since there is only one solution with a given initial value, the assertion (b) is true, and hence also (c), because $\theta(x, \alpha)$ is continuous and increases with $\alpha$.

**4.** THEOREM III. *Let $A_\nu(x)$, $B_\nu(x)$, $C_\nu(x)$ $(\nu=1, 2)$ be continuous in the interval $a \leqq x \leqq b$ and*

$$A_2 \geqq A_1, \qquad C_2 \geqq C_1, \qquad (A_2 - A_1)(C_2 - C_1) \geqq (B_2 - B_1)^2. \tag{12}$$

*If $\theta_\nu(x, \alpha)$ is the solution of the differential equation*

$$\theta' = A_\nu \cos^2 \theta + B_\nu \sin 2\theta + C_\nu \sin^2 \theta \tag{13}$$

*with the initial value $\theta_\nu(a, \alpha) = \alpha$, then*

$$\theta_2(x, \alpha_2) \geqq \theta_1(x, \alpha_1) \qquad \text{for} \qquad \alpha_2 \geqq \alpha_1. \tag{14}$$

*Proof*: Because of assumption (12) the quadratic form

$$(A_2 - A_1) \cos^2 \theta + 2(B_2 - B_1) \sin \theta \cos \theta + (C_2 - C_1) \sin^2 \theta$$

of the variables $\cos \theta$, $\sin \theta$ is positive semi-definite. Therefore we have for every $\theta$

$$A_1 \cos^2 \theta + B_1 \sin 2\theta + C_1 \sin^2 \theta \leqq A_2 \cos^2 \theta + B_2 \sin 2\theta + C_2 \sin^2 \theta$$

and so* for every $\alpha$

$$\theta_1(x, \alpha) \leqq \theta_2(x, \alpha),$$

and finally (14), since $\theta_2(x, \alpha)$ increases with $\alpha$. It is important to know when the equality in (14) is excluded for $x = b$.
Concerning this we can show the following:

**5.** THEOREM IV. *Let the assumptions of Theorem* III *be satisfied and moreover $B_2(x) \equiv B_1(x)$,† so that the last of the inequalities* (12) *is a consequence of the first two. Then we have*

$$\theta_2(b, \alpha_2) > \theta_1(b, \alpha_1) \qquad \text{for} \qquad \alpha_2 \geqq \alpha_1 \tag{15}$$

*if*

*either* $\alpha_2 > \alpha_1$
*or* $\alpha_2 = \alpha_1$ *and moreover* $A_2 > A_1$ *and* $|C_1| + |C_2| > 0$
*or* $C_2 > C_1$ *and* $|A_1| + |A_2| > 0$

*for at least one point of the interval $a \leqq x \leqq b$.*

---

* See for example E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig, 1930, p. 91. Of course one can prove this more directly.

† Without this assumption the text of the theorem would become more complicated.

*Proof*: Because of Theorem III we have

$$\theta_2(b, \alpha_1) \geqq \theta_1(b, \alpha_1)$$

and so (15) is true *a fortiori* for $\alpha_2 > \alpha_1$. Therefore we can now assume $\alpha_2 = \alpha_1 = \alpha$. In consequence of Theorem III and the above remark we have $\theta_2(b, \alpha) = \theta_1(b, \alpha)$ only if $\theta_2(x, \alpha) = \theta_1(x, \alpha)$ for every $x$ in the interval $a \leqq x \leqq b$. Now if $\theta(x)$ is a common solution of both of the differential equations (13), it follows by subtraction that

$$(A_2 - A_1)\cos^2\theta + (C_2 - C_1)\sin^2\theta = 0 \qquad \text{for} \qquad a \leqq x \leqq b. \tag{16}$$

If

$$A_2 > A_1 \qquad \text{or} \qquad C_2 > C_1$$

at any point $x_0$ and therefore also in some neighborhood of this point, then, because of (16),

$$\theta = \pi/2 + k\pi \qquad \text{or} \qquad \theta = k\pi \qquad (k \text{ an integer})$$

in this neighborhood. As $\theta$ is a solution of the differential equations (13), it follows that for this portion of the interval

$$C_1 = C_2 = 0 \qquad \text{or} \qquad A_1 = A_2 = 0.$$

This contradicts the assumptions of our theorem. Therefore the two equations have no common solution; that is, $\theta_2(x, \alpha)$ is never equal to $\theta_1(x, \alpha)$, and so (15) is true by virtue of (14).

Theorems III and IV are the comparison theorems. It only remains to show that the customary formulation is contained therein. This formulation* is the following:

**6.** THEOREM V. *Let $P_\nu(x), \cdots, S_\nu(x)$ be continuous* ($\nu = 1, 2$) *and*

$$Q_2 \geqq Q_1 > 0, \qquad R_2 \leqq R_1, \qquad P_2 - S_2 = P_1 - S_1. \tag{17}$$

*Let $y_\nu(x)$, $z_\nu(x)$ be a non-trivial solution of the system*

$$y' = P_\nu y + Q_\nu z, \qquad z' = R_\nu y + S_\nu z \tag{18}$$

*such that*

$$\begin{array}{llll} \textit{either} & & y_1(a) = 0, & \\ \textit{or} & y_1(a) \neq 0, & y_2(a) \neq 0, & \dfrac{z_1(a)}{y_1(a)} \geqq \dfrac{z_2(a)}{y_2(a)} \cdot \end{array} \tag{19}$$

*Then* (a) *$y_2(x)$ has at least as many zeros as $y_1(x)$ in the interval $a < x \leqq b$; if $x_n$, $\bar{x}_n$ are the nth zeros of $y_1$, $y_2$, then $\bar{x}_n \leqq x_n$; and moreover $\bar{x}_n < x_n$ if*

$$Q_2 > Q_1 \quad \textit{and} \quad |R_1| + |R_2| > 0, \quad \textit{or} \quad R_2 < R_1 \tag{20}$$

* See Bôcher, *loc. cit.*, footnote 3, p. 204.

*for at least one point of the interval* $a \leqq x \leqq x_n$.

(b) *if* $y_1$, $y_2$ *have the same number of zeros in the interval* $a < x < b$, *if further* $y_1(b) \neq 0$, $y_2(b) \neq 0$ *and assumption* (20) *is satisfied for at least one point of the interval* $a \leqq x \leqq b$, *then*

$$\frac{z_1(b)}{y_1(b)} > \frac{z_2(b)}{y_2(b)} \cdot \tag{21}$$

*Proof*: Both of the systems (18) can be transformed by Theorem I. We get the equations

$$\theta' = Q_\nu \cos^2 \theta + (P_\nu - S_\nu) \sin \theta \cos \theta - R_\nu \sin^2 \theta \tag{22}$$

with initial values $\theta_\nu(a)$ such that

$$0 \leqq \theta_\nu(a) < \pi. \tag{23}$$

The function $y_\nu$ vanishes when and only when $\theta_\nu(x)$ takes on a value $k\pi$ where $k$ is an integer. If $\theta_\nu$ has such a value at a point $x_0$, it follows from (22) and (17) that

$$\theta_\nu'(x_0) = Q_\nu(x_0) > 0.$$

That is, $\theta_\nu(x)$ is increasing at such a point and therefore passes through it only once. Hence to establish (a) it remains to show that $\theta_1(\xi) \leqq \theta_2(\xi)$ for every $\xi$ of the interval $a \leqq \xi \leqq b$.

Because of (5), assumption (19) says that

$$\sin \theta_1(a) = 0 \qquad \text{or} \qquad \operatorname{ctg} \theta_1(a) \geqq \operatorname{ctg} \theta_2(a),$$

and so because of (23)

$$0 \leqq \theta_1(a) \leqq \theta_2(a) < \pi.$$

Thus by Theorem III we have

$$\theta_2(\xi) \geqq \theta_1(\xi), \tag{24}$$

provided $a \leqq \xi \leqq b$. The equality sign can be removed in (24) when (20) is satisfied for $a \leqq x \leqq \xi$. Thus (a) is proved.

For the case (b) let $y_1$ and $y_2$ have exactly $N$ zeros. Then we have

$$N\pi < \theta_\nu(\mathrm{b}) < (N + 1)\pi.$$

By Theorem IV we have

$$\theta_2(b) > \theta_1(b),$$

and therefore

$$\operatorname{ctn} \theta_1(b) > \operatorname{ctn} \theta_2(b);$$

and this is equivalent to statement (21).

# SERIES FOR ALL THE ROOTS OF A TRINOMIAL EQUATION

ALBERT EAGLE, University of Manchester, England

The present paper gives three necessary series for *all* the roots of *any* trinomial polynomial equation. This work was first accomplished by Emory McClintock in 1895.* But though McClintock took many numerical examples (not all confined to trinomial equations) he did not explicitly set forth the three necessary series for such equations in terms of their algebraic coefficients in a form available for immediate use. It seems of interest, therefore, to present these here.

Let the trinomial equation be

$$z^{m+n} - pz^n + q = 0. \tag{1}$$

This may be written in the form

$$z^{m+n} - a^m z^n + a^m b^n = 0, \tag{2}$$

in which there is no restriction on $a$ and $b$ to be real. We will sometimes write $b \equiv \mu a$. When this is done it will be found that the condition that (2) should have equal roots is

$$m^m n^n = \mu^{mn}(m + n)^{m+n}. \tag{3}$$

If $|\mu|$ is less than the value given by (3) we will call $\mu$ "small"; if it exceeds this value we will call it "large."

If $|\mu|$ is very small, equation (2) clearly has $m$ roots approximately on a circle of radius $a$, and $n$ roots approximately on a circle of radius $b$, corresponding to the equations $z^m \cong a^m$ and $z^n \cong b^n$ respectively.

If $|\mu|$ is very large, equation (2) is approximately $z^{m+n} = -\mu^n a^{m+n}$, so that there are $m+n$ roots approximately on a concentric circle of radius intermediate between $a$ and $b$.

All the required series may easily be obtained from Lagrange's expansion. This may be succinctly expressed by saying that if

$$x = h + k\phi(x) \tag{4}$$

then the expansion of any function, $F(x)$, of $x$ is

$$F(x) = F(h) + k(\phi F') + \frac{k^2}{2!}(\phi^2 F')' + \frac{k^3}{3!}(\phi^3 F')'' + \cdots \tag{5}$$

where the dashes outside the brackets denote differentiation of their contents which are functions of $h$.

Writing equation (2) in the form

$$z^m = a^m - a^m b^n z^{-n},$$

* American Journal of Mathematics, vol. 17, 1895, pp. 89–110.

and putting $z^m \equiv x$, we have

$$x = a^m - a^m b^n x^{-n/m},$$

which is of the form of (4).

The expansion of $x^{1/m}$ will be found to give

$$x^{1/m} \equiv z = a - \frac{b^n a^{1-n}}{m1!} + \frac{(1-m-2n)b^{2n}a^{1-2n}}{m^2 2!} - \cdots \tag{6}$$

or

$$z = a\left\{1 - \frac{\mu^n}{m1!} - \frac{(m+2n-1)\mu^{2n}}{m^2 2!} - \frac{(m+3n-1)(2m+3n-1)\mu^{3n}}{m^3 3!} - \frac{(m+4n-1)(2m+4n-1)(3m+4n-1)\mu^{4n}}{m^4 4!} - \cdots\right\}. \tag{7}$$

As equation (2) is unaltered by replacing $a$ by $a\omega$, where $\omega$ is any of the $m$ $m$th roots of unity, if we change $a$ in (6) into $a\omega$ we shall obtain all the $m$ roots lying approximately on a circle of radius $a$. If the series in (7) is used $a$ must be changed into $a\omega$ and $\mu$ into $\mu/\omega$.

To obtain the roots which are approximately on a circle of radius $b$, we write (2) in the form $z^n = b^n + a^{-m}z^{m+n}$; putting $z^n \equiv x$ we have $x = b^n + a^{-m}x^{(m+n)/n}$. We now require the expansion of $x^{1/n}$. The series in (5) will be found to give

$$x^{1/n} \equiv z = b + \frac{b^{m+1}}{na^m 1!} + \frac{(2m+n+1)b^{2m+1}}{n^2 a^{2m} 2!} + \cdots \tag{8}$$

or

$$z = b\left\{1 + \frac{\mu^m}{n1!} + \frac{(2m+n+1)\mu^{2m}}{n^2 2!} + \frac{(3m+2n+1)(3m+n+1)\mu^{3m}}{n^3 3!} + \frac{(4m+3n+1)(4m+2n+1)(4m+n+1)\mu^{4m}}{n^4 4!} + \cdots\right\}. \tag{9}$$

Again, since (2) is unaltered by changing $b$ into $b\Omega$, where $\Omega$ is any of the $n$ $n$th roots of unity, replacing $b$ by $b\Omega$ in (8) will give all the present family of $n$ roots. In (9) we must obviously replace $b$ by $b\Omega$ and $\mu$ by $\mu\Omega$ to get the different roots.

It may be proved without much difficulty that both the series (7) and (9) are convergent so long as $|\mu|$ is less than or equal to the value of $|\mu|$ given by (3).

When this condition is not satisfied we will write $-a^m b^n \equiv c^{m+n}$ so that equation (2) becomes

$$z^{m+n} = c^{m+n} + a^m z^n, \tag{10}$$

giving, if we write $z^{m+n} \equiv x$,

$$x = c^{m+n} + a^m x^{n/(m+n)}.$$

Lagrange's expansion in (5) applied to this equation for the expansion of $x^{1/(m+n)}$ gives

$$x^{1/(m+n)} \equiv z = c + \frac{a^m c^{1-m}}{(m+n)1!} + \frac{(1+n-m)a^{2m}c^{1-2m}}{(m+n)^2 2!}, \tag{11}$$

or

$$z = c\left\{1 + \frac{\lambda^m}{(m+n)1!} + \frac{(1+n-m)\lambda^{2m}}{(m+n)^2 2!} + \frac{(1+2n-m)(1+n-2m)\lambda^{3m}}{(m+n)^3 3!} + \frac{(1+3n-m)(1+2n-2m)(1+n-3m)\lambda^{4m}}{(m+n)^4 4!} + \cdots\right\}, \tag{12}$$

where $\lambda \equiv a/c$.

If in (10) we replace $c$ by $c\epsilon$, where $\epsilon$ is any of the $(m+n)$ $(m+n)$th roots of unity, or, if in (12) we replace $c$ by $c\epsilon$ and $\lambda$ by $\lambda/\epsilon$, we shall obtain all the roots of the equation $z^{m+n} = c^{m+n} + \lambda^m c^m z^n$; and the series will be convergent whenever (7) and (9) are divergent, which, by (3), will be found to be when

$$|\lambda^m| \leqq (m+n)(m^m n^n)^{-1/m+n}.$$

Two of the above expansions for the roots of equation (1) were obtained by M. R. Birkeland* who seems not to have been familiar with McClintock's work. These results of Birkeland are quoted by Appell and Kampé de Fériet.†

Later, M. Birkeland returned to the problem and gave a complete solution of it.‡ But M. Birkeland expresses the roots of a trinomial equation of the $n$th degree, not in our simple series, but in terms of hypergeometric functions of the $(n-1)$th order. He requires $(n-1)$ of these functions as equivalent to the series in (12) and $(n-1)$ other functions as equivalent to the series in (7) and (9) together.

Thus for the equation

$$x^5 = x^2 + \beta,$$

which he takes by way of illustration, he gives for one root the expression

$$F\begin{pmatrix} -\frac{1}{15}, & \frac{2}{15}, & \frac{8}{15}, & \frac{11}{15} \\ \frac{6}{5}, & \frac{1}{3}, & \frac{2}{3}, & \zeta \end{pmatrix} + \frac{\beta}{3} F\begin{pmatrix} \frac{4}{15}, & \frac{7}{15}, & \frac{13}{15}, & \frac{16}{15} \\ \frac{4}{3}, & \frac{7}{6}, & \frac{2}{3}, & \zeta \end{pmatrix} - \frac{\beta^2}{3} F\begin{pmatrix} \frac{3}{5}, & \frac{4}{5}, & \frac{6}{5}, & \frac{7}{5} \\ \frac{5}{3}, & \frac{3}{2}, & \frac{4}{3}, & \zeta \end{pmatrix},$$

where $\zeta \equiv -5^5 2^{-2} 3^{-3} \beta^3$.

---

* Comptes Rendus, t. 171, 1920, p. 1370, and t. 172, 1921, p. 309.

† Fonctions hypergéométriques, p. 401.

‡ Mathematische Zeitschrift, vol. 26, 1927, p. 566.

These functions are cumbersome as first written out (and the complexity rapidly increases with the degree of the equation); but it will be found that the above expression reduces to the simple series

$$1+\frac{\beta}{3\cdot 1!}-\frac{6\beta^2}{3^2\cdot 2!}+\frac{8\cdot 11\beta^3}{3^3\cdot 3!}-\frac{10\cdot 13\cdot 16\beta^4}{3^4 4!}+\cdots,$$

which is immediately given by our series in (7) on putting $m=3$, $n=2$, $a=1$ and $\mu^2=-\beta$. It is to be noted that the series in (7) is equally simple whatever the degree of the equation.

M. Birkeland regards the fact that he has shown that the roots of a trinomial equation can be expressed in terms of higher order hypergeometric functions as the essence of his work on trinomial equations; and he especially emphasizes this point in a note in the *Comptes Rendus*, vol. 177, p. 25.

It may be noted that for the more general equation

$$af_1(z)+bf_2(z)+cf_3(z)=0$$

three different expansions for $z$ can theoretically be obtained by Lagrange's expansion. For if we divide the equation by $f_1(z)$, $f_2(z)$, or $f_3(z)$ in turn, to make one term constant, we may put either of the other two terms equal to a new variable $x$. It will be found that only *three* different expansions can be obtained in this manner, not *six*.

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## SERIES FOR ALL THE ROOTS OF THE EQUATION $(z-a)^m=k(z-b)^n$

ALBERT EAGLE, University of Manchester, England

This equation is closely related to the trinomial equation considered in the preceding paper. The former equation, when $m$ and $n$ are not restricted to be integers, can in fact be transformed into the present equation by a change of variable. In this equation we may evidently suppose that $m>n$; and, as by a simple change of variable we can make both $a=0$ and $b=1$, we will adopt

$$z^m=k(z-1)^n \tag{1}$$

as the canonical form. We will also write (1) in the equivalent forms

$$z^m=g^m(z-1)^n, \tag{2}$$

$$z^m=p^{m-n}(z-1)^n, \tag{3}$$

and

$$z^m=q^{-n}(z-1)^n. \tag{4}$$

When $g$ is small it is obvious from (2) that all $m$ roots are approximately on a circle of radius $g$ and center the origin. When $p$ is so large some that of the

roots have a modulus large compared with unity it is obvious from (3) that we shall have $m-n$ roots approximately on a circle of radius $p$. In this case $q$ will be small in (4); and it is evident from this equation that there will also be $n$ roots approximately on a circle of radius $q$ with center at $z=1$.

If we extract the $n$th root of (1) and differentiate and equate to zero, it is easily found that the condition that (1) should have equal roots (apart from the trivial case when $k=0$) is

$$k(m-n)^{m-n} = m^m n^{-n}, \tag{5}$$

with corresponding expressions in terms of $g$, $p$ and $q$.

To obtain the Lagrange's expansion in the case when $g$ is small we write (2) in the form $z=0+g(z-1)^{n/m}$. If we note that (2) is unaltered by changing $g$ into $g\omega$, where $\omega$ is any of the $m$ $m$th roots of unity, it will easily be found that the series

$$\begin{aligned} z = g\omega' - \frac{(2n)g^2\omega'^2}{m2!} + \frac{(3n)(3n-m)g^3\omega'^3}{m^2 3!} \\ - \frac{(4n)(4n-m)(4n-2m)g^4\omega'^4}{m^3 4!} + \cdots, \end{aligned} \tag{6}$$

where $\omega'$ represents any of the $m$ $m$th roots of *unity* when $n$ is *even* and of *minus unity* when $n$ is *odd*, gives the whole of the $m$ roots of (2) when $|g|$ is less than the critical value given by (5). [The change is effected by the powers of $(-1)^{n/m}$ which necessarily enter when $(z-1)^{n/m}$ is taken at the value $z=0$.]

When $|g|$ exceeds the critical value, to find the $(m-n)$ large roots we write equation (3) in the form $z^{m-n}=p^{m-n}(1-z^{-1})^n$. Extracting the $n$th root of this we have $z^{(m-n)/n}=p^{(m-n)/n}(1-z^{-1})$. Taking $z^{(m-n)/n}$ as a new variable, $x$, we have

$$x = p^{(m-n)/n} - p^{(m-n)/n}x^{n/(n-m)},$$

from which we can obtain the expansion of $z\equiv x^{n/(m-n)}$ by Lagrange's method.

We find*

$$\begin{aligned} z = p\epsilon - \frac{n}{m-n} - \frac{n\cdot m(p\epsilon)^{-1}}{(m-n)^2 2!} - \frac{n(m+n)(2m)(p\epsilon)^{-2}}{(m-n)^3 3!} \\ - \frac{n(m+2n)(2m+n)(3m)(p\epsilon)^{-3}}{(m-n)^4 4!} - \cdots, \end{aligned} \tag{7}$$

where $\epsilon$ is any of the $(m-n)$th roots of unity, gives the values of all the $(m-n)$ roots which lie approximately on a circle of radius $p$. It will be noticed that the center of this approximate circle tends to the value $z=-n/(m-n)$ as $p\to\infty$.

Lastly, to find the $n$ roots near $z=1$ when $z$ exceeds the critical value, we

* It will be noticed that the factors in the numerators, apart from the first factor $n$, are in A.P. with a common difference of $m-n$, and that the last factor is $Nm$ where $N$ is the number in factors in A.P.

return to equation (4), write it in the form $z=1+qz^{m/n}$, and readily find that the series

$$
\begin{aligned}
(8) \qquad z = 1 + q\Omega + \frac{2mq^2\Omega^2}{n2!} + \frac{(3m)(3m-n)q^3\Omega^3}{n^2 3!} \\
+ \frac{(4m)(4m-n)(4m-2n)q^4\Omega^4}{n^3 4!} + \cdots,
\end{aligned}
$$

where $\Omega$ is any of the $n$ $n$th roots of unity, gives all the roots in the neighborhood of $z=1$ when $|q|$ is less than the critical value given by (5).

As an interesting illustration of these three series we will take the equation

$$
(9) \qquad z^3 = \beta(z-1)
$$

to which any cubic equation can very readily be reduced. We will first suppose that $|\beta|>27/4$.

Equation (7) easily gives for two of the roots

$$
\begin{aligned}
(10) \qquad z = \pm\sqrt{\beta}\left\{1 - \frac{3}{2^3\beta} - \frac{3\cdot5\cdot7}{2^7\beta^2} - \frac{3\cdot7\cdot11\cdot13}{2^{11}\beta^3} - \frac{9\cdot11\cdot13\cdot17\cdot19}{2^{15}\beta^4} - \cdots\right\} \\
- \left\{\frac{1}{2} + \frac{1}{2\beta} + \frac{3}{2\beta^2} + \frac{6}{\beta^3} + \frac{55}{2\beta^4} + \cdots\right\}.
\end{aligned}
$$

The other root is immediately given by (8) as

$$
(11) \qquad z = 1 + \frac{1}{\beta} + \frac{3}{\beta^2} + \frac{12}{\beta^3} + \frac{55}{\beta^4} + \frac{273}{\beta^5} + \cdots.
$$

Obviously this root and the two given by (10) add up to zero as they should.

It should be noticed that (10) will give two complex roots, as it should do when $\beta$ is negative, as easily as it gives two real roots when $\beta$ is positive.

When $|\beta|<27/4$ we will write (9) in the form

$$
z^3 = \beta^3(z-1)
$$

to avoid fractional indices.

The series in (6) will then give (if we drop the $\omega'$)

$$
z = \beta - \frac{\beta^2}{3} + \frac{\beta^4}{3^4} + \frac{\beta^5}{3^5} - \frac{4\beta^7}{3^8} - \frac{5\beta^8}{3^9} + \cdots.
$$

To get the actual roots, $\beta$ must be replaced in this series by $-\beta$ and by $\frac{1}{2}\beta(1\pm i\sqrt{3})$ in turn. We then obtain

$$
(12) \qquad z = -\beta - \frac{\beta^2}{3} + \frac{\beta^4}{3^4} - \frac{\beta^5}{3^5} + \frac{4\beta^7}{3^8} - \frac{5\beta^8}{3^9} + \cdots
$$

for the real root, and

$$
(13)\qquad \begin{aligned} z &= \frac{1}{2}\left\{\beta + \frac{\beta^2}{3} - \frac{\beta^4}{3^4} + \frac{\beta^5}{3^5} - \frac{4\beta^7}{3^8} + \frac{5\beta^8}{3^9} - \cdots\right\} \\ &\pm \frac{i\sqrt{3}}{2}\left\{\beta - \frac{\beta^2}{3} - \frac{\beta^4}{3^4} - \frac{\beta^5}{3^5} - \frac{4\beta^7}{3^8} - \frac{5\beta^8}{3^9} - \cdots\right\} \end{aligned}
$$

for the two complex roots. It will be noticed that powers of $\beta$ which are multiples of three do not appear in these series and also that the three roots again add up to zero.

The series in (10) to (13) can also be obtained from the expansions given in the preceding paper.

It is interesting to compare the series given in (10) to (13) with the solution of $z^3=\beta(z-1)$ given by Cardan's method. This gives

$$
z = \left\{-\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \frac{\beta^3}{27}}\right\}^{1/3} + \left\{-\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - \frac{\beta^3}{27}}\right\}^{1/3}.
$$

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# MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This department of the* MONTHLY *affords a place for the discussions of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

## ON THE TRAINING OF TEACHERS FOR SECONDARY SCHOOLS

B. W. JONES, Cornell University

The Report on the Training of Teachers of Mathematics* has been of considerable interest in that it recognizes the responsibility of colleges toward teacher training and has set the pace for definite recommendations for such training. In this report, preparation for teachers in secondary schools is visualized as chiefly confined to the four undergraduate years and "any first year graduate curriculum . . . will be thought of principally as the first part of a complete [three year] program." For various reasons, some of which we shall mention, it is becoming increasingly necessary for prospective teachers in secondary schools to hold a master's degree. On the other hand, though a doctor's degree should help a student preparing to teach in secondary schools, such students at present seldom find it necessary to take as much graduate work as this degree requires. We, therefore, wish to present some principles which should underlie a program complete after one graduate year, a "complete" program being one

* This MONTHLY, vol. 42, 1935, pp. 263-277.

which completely prepares a student to continue his education on his own hook.

There will necessarily come into the discussion two terms: "research," by which we shall mean the process of discovering things new to the mathematical world; and "mathematical investigation,"* used to designate original work which is new to the individual concerned, but which *may* not be research; by the second term we do not mean the reading of mathematical books or articles unless the reader weaves around and through it the threads of his own investigation. In this connection we shall develop the idea of a third type of thesis between the research and exposition theses mentioned in the report. Also, by "teacher," unqualified, we mean a teacher of mathematics in secondary school (this is no oblique reflection on college teaching).

While many question the effectiveness of survey courses in mathematics, few quarrel with the reason for their existence, namely, the feeling that those who take only one year of this subject should have a different course from the first year course designed for those going on to further mathematics. It nevertheless seems very hard to forsake the notion that all graduate work—or at least the only worth while graduate work for the capable student—is that leading to the Ph.D. degree as at present constituted. The master's degree, to which we wish to devote our especial attention, is in most cases regarded only as a stepping stone towards the doctorate, even though for the majority it is the last step. It is not my aim to quarrel with the two parts, as such, of graduate training: that is, first, the laying of a broad foundation of knowledge and method in the various fields of mathematics; and, second, specialized studies leading to and giving training in mathematical investigation. But any program for the training of teachers must be tied to what they are to teach.† Breadth of knowledge and ability to do mathematical investigation are soon dissipated for the average teacher if the knowledge and the investigation are too far separated from what he teaches. One cannot effectively teach fractions in working hours and learn topology at other times; either fractions become dull or topology goes by the board. The path leading to research is, for most mathematicians, a *narrow* one. A teacher of mathematics in secondary schools needs *breadth* of outlook. To do research, one must be sophisticated—to teach, naïve. This is not to be taken to imply that teaching is incompatible with mathematical investigation. In fact, a teacher of any subject becomes dead as soon as he ceases to discover in his subject things new *to him*. But it does imply that research (as above defined) in mathematics is, for the most part, incompatible with good secondary school teaching. This is partly due to lack of time to engage in research, but the fundamental reason is that research is too far removed from the field of secondary school teaching. To one teaching long division, a knowledge of matrices is of

* This seems to be rather close to what R. L. Jeffery calls "productive scholarship," in this MONTHLY, vol. 42, 1935, pp. 364–369.

† In this connection, and, indeed, in comparison with our whole program, it is very enlightening to see the former German solution: see Richard Courant, Mathematical education in Germany before 1933, this MONTHLY, vol. 45, 1938, pp. 601–607.

little help, but some mathematical investigation of recurring decimals on the part of the teacher would result in a livening of the subject for him and his pupils. Similarly a knowledge of constructions with ruler and compasses and accompanying investigation would be of more use to a teacher of first year plane geometry than a course in algebraic geometry; and if he is interested in teaching his subject, he would likely enjoy the former more than the latter. Any candid person will admit that different objectives require different training. Let us, then, examine some of the differences between the needs of the prospective teacher and those of one who is preparing for research.

For instance, the report mentions an advanced course in synthetic euclidean geometry. This is not usually given, since research in this subject has largely gone out of style. But for a high school teacher such a course would be invaluable and would provide background much closer to the front than many another background course; it would also be a mine of unanswered questions for further investigation. This course would be just as difficult as any first year graduate course. (Let any skeptic try some stiff originals in synthetic geometry.) The history of mathematics has almost disappeared from our curriculum, but would be highly useful to a teacher.

Perhaps the most vital thing to any student of mathematics is insight.* We are apt to take it for granted that piling course on course (regardless of the course) will accomplish this. We must realize that there are two parts to insight: first, understanding a thing with respect to itself; and second, understanding a thing with respect to its ramifications; and the first part must largely precede the second. If a student knows high school algebra only formally, he cannot obtain insight into the theory of equations, determinants or the binomial theorem; nor, under these conditions, can the study of these more advanced subjects, however carefully taken, contribute greatly to an understanding of high school algebra. A knowledge of the reasons for the manipulations of algebra must precede all further understanding in the subject. On this basis, and on this basis *alone*, can be built work which, from its broader viewpoint, throws light on the simpler. For instance, *if* factoring is understood, it is illumined by a knowledge of the remainder theorem. If one cannot understand the simple, how can he understand the complex! A heavy slate roof does not lend stability to a shaky foundation. Of course, it is true that there are two types of prospective high school teachers: those whose grasp of mathematics comes too slowly for advanced graduate work, and those (increasing in number) who for other reasons prefer high school teaching. But most students who grasp the complex with difficulty can, with proper care, obtain a good grasp of what they will teach in high school. And even those whose strides are rapid need often to retrace their steps slowly lest they miss important things on the way. For a prospective teacher emphasis should be laid on firm understanding at the expense, if necessary, of coverage of ground. For instance, one could inculcate a good understanding of

* *Cf.* W. B. Carver, this MONTHLY, vol. 44, 1937, pp. 359–363; also Arnold Dresden, this MONTHLY, vol. 42, 1935, pp. 198–208.

the nature of calculus and its fundamental ideas by merely considering the differentiation and integration of polynomials. For the prospective teacher it would be better to omit other differentiation and integration if necessary to go deeply into the processes for polynomials. *Every* prospective teacher who has studied any integration should, for example, without any special preparation, be able to show easily that

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 < \int_1^n x^2 dx$$

for $n$ an integer greater than 1; this should take precedence over the technique of complicated integrations. We must face the fact that proficiency in techniques without an understanding of them is, for a teacher, *infinitely worse* than no techniques at all. The prospective teacher should be passed or failed not on the basis of the breadth of his smattering, but on the sureness of his understanding.

Also the secondary school teacher needs a broader training than one who is to teach in college. As was mentioned in the report, he must know the rôle of mathematics in economics, statistics and the sciences—things for which there is little place or time in the present graduate program. It is, in fact, this need, rather than the number of necessary courses in mathematics, which, with the required work in education and the training in mathematical investigation, make a fifth year of study increasingly necessary. While the student who is to go on to research can, in his undergraduate years, take elective mathematics courses, the prospective teacher must broaden the base of his knowledge.

As for the second part of the program, namely, training in mathematical investigation, it should first be remarked that we must not let our interest in publication, the corollary of research, hide the fact that *mathematical investigation* is the *sine qua non* of vitality in this subject. Though publication is probably the best way to "remain alive mathematically," it does not insure it, and there are many who, without publishing in mathematical journals have, by their pursuit of the unknown, continually advanced the frontiers of their own minds and have so kept an unfailing interest in the subject. Training for mathematical investigation is vital alike to him who seeks only a master's degree and to him who wishes eventually to do research. It is in this respect that our present program is weakest; evidence for this is the minority which continues to investigate mathematically. A *research* topic must usually be assigned the student, for he has little knowledge of what should be publishable. But a master's thesis need be merely a piece of mathematical investigation. In the first place, the topic *should be selected by the student.* This is an important part of the training, for if one is to conduct mathematical investigation on his own, he must select his problems on his own. As an aid to such selection the prospective teacher especially must be shown the beginnings of the many interesting by-paths in elementary mathematics. (A knowledge of integers in algebraic fields, however useful it is, leaves untouched many of the fascinating peculiarities of ordinary whole numbers.*)

* See, for example, Helen A. Merrill's fascinating little book: Mathematical Excursions, The Norwood Press, 1933.

Selection of a thesis topic by the prospective teacher would be made easier if we were not so careful to answer all the interesting questions we raise in our courses. At precisely this point in the second part of the program is the chief difference between the needs of the prospective teacher, or indeed anyone who does not go beyond the master's degree, and of the future Ph.D. For the latter, many, though not all, of the questions raised in courses must be answered, so that he may later build on these results; his unanswered questions come later. For the former, a broader, less complete course is necessary—a course in which a *few* basic methods are *learned* and many questions left unanswered. For a man to go on in any subject after he has left college, it is necessary that the topic be not complete to him. For instance, if a college English teacher wants his students to read Shakespeare after they have left college, he will not attempt to make them think their knowledge of Shakespeare is complete (or as complete as necessary) at the close of his course, but will attempt, on the basis of some knowledge on their part, to show them what of interest remains for them to explore on their own.

Secondly, while there should, of course, be guidance in writing the thesis, it should be from the student's point of view. The results of a problem selected by the student may be evident to the instructor from his advanced viewpoint, but he must curb his desire to see it done the elegant way. The student must be shown how to explore these new trails himself; and merely reading travelogues of other people or being led by a guide does not take the place of, nor do him as much good as, his own explorations. Incidentally, I have often seen ardent mountain climbers reach with great labor a spot much more easily attained by the highroad; a spot, which, moreover, was not worth attaining the easy way. Many a college teacher before his class does all the exciting discovering himself and leaves them the drudgery. The measure of the merit of a master's thesis should not be the importance of the result or its "interest" to the professional mathematician (it seldom is of much importance to the mathematical world in any case), but rather the progress the student has made over *his* previous knowledge and the initiative and originality *he* has shown in making it. In other words, we must permit the master's thesis to be elementary, but we must demand that it be the student's own. A master's degree must imply that the holder is capable of improving his mathematical knowledge with his own paper and pencil. Especially for him who is not to go on, preparation for a master's degree should give training on a more elementary level for mathematical investigation, just as preparation for a Ph.D. degree gives training (or is supposed to) for mathematical investigation on a more advanced level. Let us not be accused of preferring unfinished and poorly built skyscrapers to closely knit and complete bungalows.

It is the habit of those teaching in colleges and universities to say "we are the people," and disparage all "lower" instruction. But we merely perpetuate a vicious circle when we, on the one hand, complain of the quality of instruction in high schools, and, on the other hand, tell our better students that they are too good to go into high school teaching. If we think conditions are bad for a good

mathematician in a high school, we should feel their challenge to good men who can go in to set them aright. We must realize that there are many ways to serve mathematics: advance its frontiers, recruit its army, spread its propaganda. They require different aptitudes, but who shall say that one is more important than the other! Contrast, for example, the general practitioner in medicine with one doing research in it. The profession is not well served if the medical scientist looks down on the practitioner or disparages his service any more than it is served if the practitioner proceeds without regard to the scientist. The latter depends upon the former for the application of his discoveries and, in many cases, for the testing of his results. The former knows many things the latter does not. To try to establish which is on a "higher level" is only to encourage division. Each must be self-respecting in his own sphere and have an appreciation of the vital contributions of those in other spheres. So it is, in a slightly different way, in mathematics. While a high school teacher seldom can contribute to the advancement of scientific knowledge, he has a delicate and vital service to mathematics to perform. To say that he teaches in high school because he isn't good enough to teach in college is similar to saying that he teaches because he isn't good enough to be a principal or superintendent. Administration and teaching require different aptitudes, just as do college and high school teaching. It is a very difficult thing to give high school students a feeling for sound mathematics and develop ability in it. Later instruction is easy compared with it. (It is my private opinion that an undergraduate would learn much more college mathematics under a high school teacher who was competent in high school mathematics than would a high school student under a college teacher who knew well his college mathematics.) The choice is not between "higher" and "lower," but should be based on individual capabilities and inclinations. Only by cooperation (this implies meeting on a basis of equality) between college teachers and secondary school teachers can the best interests of mathematics be served and a completely worth while program for teacher training be worked out.

I have heard it said in connection with a proposed course of study for teacher-training that while the program advocated will be the best for the run-of-the-mine teacher, anyone really good in the subject matter should take the usual master's degree. It seems to me that such an attitude defeats the program from the start. Allowing for individual differences, it should be definitely the best for its purpose. The good student as well as the poor needs breadth of background and training in the kind of mathematical investigation which fits in with his teaching. In comparison with the usual master's degree the standards for the master's degree for the prospective teacher should be stricter in some respects (*e.g.*, in thorough understanding or in initiative in the thesis problem) and not as strict in others (*e.g.*, in degree of advancement of courses).

In brief, the course of training for a prospective teacher of mathematics in secondary schools should differ from that for one going on to research in the following fundamental ways: first, it should be broader; second, it should lay

chief emphasis on thorough understanding; third, it should give by the close of the first graduate year material for mathematical investigation as well as training in it. The standards, though different, must be just as strict.

We are justly proud of the advancements in mathematical knowledge made in this country, but we know that a very large part of this present supremacy is due to the influx of gifted men from other lands. We must increasingly rely on ourselves. Even research men begin their mathematics in secondary schools, and most men end it there. If we are to grow mathematicians, and if we are to make mathematics a living subject to all educated people, we must definitely concern ourselves with the improvement of teaching in secondary schools and realize that our present shoe does not fit all feet.

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## QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, New York

*The Department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

### THE RATIO TEST FOR CONVERGENCE OF SERIES

N. J. LENNES, Montana State University

The "ratio test" for the convergence of an infinite series is often stated:

*A series* $\Sigma_0^\infty a_i$ *is convergent if* $\lim_{i\to\infty}(a_{i+1}/a_i)=p$, *where* $|p|<1$. *If this limit is* $+1$ *or* $-1$ *the series may or may not be convergent. Obviously the limit may not exist and the series still be convergent.*

A reader of one of my own publications, identity of reader unknown to me, sends me through a publisher the following example,

$$1+\frac{1}{2}-\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}-\frac{1}{2^6}-\frac{1}{2^7}+\frac{1}{2^8}+\frac{1}{2^9}-\cdots,$$

for which $\lim_{i\to\infty}(a_{i+1}/a_i)$ fails to exist, although the series converges. This same reader then says that the condition should be stated $\lim_{i\to\infty}|a_{i+1}/a_i|=p$, *etc.*

It is obvious that this emendation does not take us very far, though it does make the test cover the series that he gives.

The purpose of this note is to give a convergent series in which the ratios $a_{i+1}/a_i$ have every real number, and also $+\infty$ and $-\infty$, as limiting values.

The series is constructed as follows:

*Step 1.* Arrange the set of all rational numbers between 0 and 1 in countable order as follows. Consider the set of rational fractions $m/n$, $0<m/n<1$, the fractions being in their lowest terms. For $m+n=0$, 1, or 2, there is no such fraction. For higher values of $s$ in $m+n=s$ we make a table as shown at the right.

| $s$ | $m/n$ |
|---|---|
| 3 | $\frac{1}{2}$ |
| 4 | $\frac{1}{3}$ |
| 5 | $\frac{1}{4}$, $\frac{2}{3}$ |
| 6 | $\frac{1}{5}$ |
| 7 | $\frac{1}{6}$, $\frac{2}{5}$, $\frac{3}{4}$ |
| ... | ....... |

*Step 2.* For each of these fractions construct four series of the following type.

For $m/n=\frac{1}{2}$:

(a) $(1/2)+(1/2)^2+(1/2)^3+(1/2)^4+\cdots$; for $i$ odd, $a_{i+1}\div a_i=1/2$;

(b) $(1/2)^2+(1/2)+(1/2)^4+(1/2)^3+\cdots$; for $i$ odd, $a_{i+1}\div a_i=2$;

(c) $(-1/2)+(-1/2)^2+(-1/2)^3+(-1/2)^4+\cdots$; for $i$ odd, $a_{i+1}\div a_i=-1/2$;

(d) $(-1/2)^2+(-1/2)+(-1/2)^4+(-1/2)^3+\cdots$; for $i$ odd, $a_{i+1}\div a_i=-2$.

For $m/n=\frac{1}{3}$:

(e) $(1/3)+(1/3)^2+(1/3)^3+(1/3)^4+\cdots$; for $i$ odd, $a_{i+1}\div a_i=1/3$;

(f) $(1/3)^2+(1/3)+(1/3)^4+(1/3)^3+\cdots$; for $i$ odd, $a_{i+1}\div a_i=3$;

(g) $(-1/3)+(-1/3)^2+(-1/3)^3+(-1/3)^4+\cdots$; for $i$ odd, $a_{i+1}\div a_i=-1/3$;

(h) $(-1/3)^2+(-1/3)+(-1/3)^4+(-1/3)^3+\cdots$; for $i$ odd, $a_{i+1}\div a_i=-3$.

Continue in an obvious manner for $m/n=1/4$, $m/n=2/3$, and so on.
Each of these series is absolutely convergent and the sum of the absolute values in every case is $m/(n-m)$. Multiply each series by $(n-m)/m$, making the sum unity. Finally multiply the first series by 1, the second by 1/2, the third by $(1/2)^2$, and so on. Then the sums are 1, 1/2, $(1/2)^2$, $(1/2)^3, \cdots$. Let the resulting sequence of series be denoted by $S_1, S_2, S_3, \cdots$.
*Step 3.* Using the terms of this sequence of series build a series $U_1+U_2+U_3t+\cdots$ as follows.

For the first four terms in the $U$ series use the first four terms of $S_1$; for the next two terms of the $U$ series use the first two terms of $S_2$. Then use in order the 5th and 6th terms of $S_1$, the 3rd and 4th terms of $S_2$, and the 1st and 2nd terms of $S_3$. Then the 7th and 8th terms of $S_1$, the 5th and 6th terms of $S_2$, and so on.

The point is that every pair of odd-even terms of any $S_i$ appears as a pair of odd-even terms in the $U$ series.

The $U$ series so constructed has the following properties.
(a). It is absolutely convergent with sum 2.
(b). For any rational number $k(k\neq 0, 1, -1)$, there is an infinite number of even numbered terms such that when any one of them is divided by the term immediately preceding it the quotient is $k$.

Using the general theorem,

*If $f(x)$ has as limiting values a set of numbers everywhere dense on an interval (possibly on the whole set of real numbers) as $x\to a$ (possibly infinity), then $f(x)$ has as limiting values every number on the continuum on that interval,*

we have that $U_{i+1}/U_i$ has every real number, and also $\infty$ and $-\infty$, as limiting values, as $i\to\infty$.

*Note by the Editor.* Dr. J. B. Rosser has suggested the following scheme for constructing a series of the desired type. Let $a_1, a_2, \cdots$ be a sequence of real

numbers having for limiting values all real numbers and $\pm\infty$. For instance, we may use the set of all rational numbers, ordered in a manner similar to that explained in Step 1. Now define a sequence $b_1, b_2, \cdots$ as follows:

$$b_{2i-1} = 1, \qquad b_{2i} = a_i, \quad \text{if} \quad |a_i| \leqq 1; \qquad i = 1, 2, \cdots;$$
$$b_{2i-1} = 1/a_i, \quad b_{2i} = 1, \quad \text{if} \quad |a_i| > 1; \qquad i = 1, 2, \cdots.$$

Evidently $|b_i| \leqq 1$, and $b_{2i}/b_{2i-1} = a_i$. Let

$$U_1 = b_1, \; U_2 = b_2, \; U_3 = b_3/2, \; U_4 = b_4/2, \; U_5 = b_5/2^2, \; U_6 = b_6/2^2, \; \cdots.$$

Then $\Sigma U_i$ converges absolutely, and $U_{i+1}/U_i$ has as limiting values every real number and $\pm\infty$. The same scheme can evidently be used to construct an absolutely convergent series of complex numbers for which $U_{i+1}/U_i$ has as limiting values every complex number and $\infty$.—R.J.W.

## NOTE ON RELATIVISTIC ORBITS

W. G. LEAVITT, Princeton University

There are a few possible orbits which were not mentioned in my paper in the January MONTHLY.* There we considered only those orbits projected from an apse, and since nearly all orbits have at least one apse, this was in general sufficient. Of those which do not, one arises when there is a single zero of the right-hand side of the differential equation [(1), p. 26], say $a<0$; then if $m>0$, we know that $u \geqq a$. It is thus possible for $u$ to vary from $+\infty$ to 0. Since in our case we have divided $m$ and $r$ by $1/a$, this corresponds to $m<0$, $u \leqq 1$ and $u$ varying from $-\infty$ to 0. There being no reason to rule out negative values of $u$, this creates no difficulty, and the orbital equation is that of region VII [Fig. 1, p. 28].

Another case arises when two or more of the roots coincide. Suppose all three roots are equal [point $B$, Fig. 1]; then $\alpha=\beta=1$, and [from (6), p. 27] the differential equation is

$$\left(\frac{du}{d\phi}\right)^2 = 2m(u-1)^3. \tag{1}$$

Thus if $u=1$ initially, $u\equiv 1$ and the orbit is circular. However, we may have initially $u>1$ (since $m>0$), in which case the orbit may range from $+\infty$ to approach the circle $u=1$ asymptotically. In our case ($u=1$ initially) this portion of the orbit will never be reached. The orbital equation on curve $C_1$ below $B$ may be obtained from the equation for region IV [(13), p. 29] by setting $\alpha=\beta$. Thus we have

$$u = 1 + (1-\alpha)\tan^2 \tfrac{1}{2}p\phi, \qquad p^2 = 2m(1-\alpha). \tag{2}$$

* This MONTHLY, January 1939, pp. 26–32. References in what follows to equations and figures in that paper will be enclosed in brackets.

From the transformation

$$p\phi = p\phi' + \pi$$

we have that at $\phi'=0$, $\lim_{\alpha\to 1}\phi=\infty$. Applying this transformation to equation (2) we obtain

$$u = 1 + (1-\alpha)\cot^2 \tfrac{1}{2}p\phi'$$

which in the neighborhood of $\frac{1}{2}p\phi'=0$, may be expanded as follows:

$$u = 1 + (1-\alpha)\left(\frac{1}{\frac{1}{2}p\phi'}\right)^2 = -\frac{2}{3} + O(\tfrac{1}{2}p\phi')^2.$$

Inserting the value of $p$ from (2) and passing to the limit, we have

$$u = 1 + \frac{2}{m\phi'^2}\cdot$$

Thus at $\phi'=0$, $u=\infty$ and at $\phi'=\pm\infty$, $u=1$.

For two roots equal we have the two cases:

1. The two upper roots, say $u=a$, equal. If $m<0$ only one root may be greater than zero, hence we need consider only $m>0$ (if all three are less than zero we divide by the inverse of one of them and obtain again $m>0$). If initially $u=a$ the orbit is, as above, circular [$C_2$ below $B$, Fig. 1]. Let $u=b$ be the least root; we then have $u\geqq b$. If $u=b>0$ initially, the orbit will be asymptotic circular [$C_1$ above $B$] and approaches $u=a$ for $\phi=\pm\infty$. If $b<0$, since for $m>0$ only one root may be negative, we may have either $b\leqq u\leqq 0$, which gives a quasi-hyperbolic orbit [$C_1$ left branch], or $0\leqq u<a$ and $u$ varying from $u=0$ asymptotically to $u=a$ [$C_1$ left branch, negative values of $u$]. If initially $u>a$, we proceed from the orbit for $u<a$ [$C_1$ above $B$]. This is a specialization from region I, so [in (9), p. 29] we set $1=b$ (the least root) and $\alpha=\beta=a$, from which we have

$$u = b + (a-b)\tanh^2 \tfrac{1}{2}p\phi, \qquad p^2 = 2m(a-b);$$

and by the transformation

$$p\phi = p\phi' + \pi i,$$

we obtain the orbit for $u>a$,

$$u = b + (a-b)\coth^2 \tfrac{1}{2}p\phi'.$$

Thus $u=\infty$ for $\phi'=0$, and $u=a$ for $\phi'=\pm\infty$.

2. The two least roots ($u=b$) equal. As above, if $u=b$ initially, the orbit is circular [$C_2$ above $B$]. If $m>0$, then only $u\geqq a$ is possible and we have an ordinary captured orbit [$C_1$ below $B$]. If $m<0$ and upon dividing by $1/b$ we obtain again case 1. The orbits for both of these cases are more easily obtained by direct solution of the differential equation.

One other point should be mentioned. In our paper [p. 31] we obtained, for the real case, the relation $m \leqq 1/2$. This does not limit the possible value of the mass but, since we have divided by the apsidal distance, implies that there are no apses inside $r=2m$. This may be shown in somewhat greater generality. Assume the proper time, $ds$, and the Schwarzschild time, $dt$, to be real, whence [from (2) and (3), p. 26] $h$ and $c$ are real. Then we have [(1) p. 26]

$$\left(\frac{du}{d\phi}\right)^2 = \left(2mu^2 + \frac{2m}{h^2}\right)\left(u - \frac{1}{2m}\right) + \frac{c^2}{h^2}\cdot$$

Thus if $m>0$, for $u>1/2m$ we have $(du/d\phi)^2>0$; if $m<0$, $u<1/2m$ implies $(du/d\phi)^2>0$. Thus there are no zeros of $du/d\phi$ and hence no apses for $0 \leqq r < 2m$ $(m>0)$ or $0 \leqq -r < -2m$ $(m<0)$.

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## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 513 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Compendio di Meccanica Razionale, Parte Prima*: Cinematica Principi e Statica. Seconda Edizione Riveduta. By T. Levi-Civita and U. Amaldi. Bologna, Zanichelli, 1938. 12+423 pages.

*Compendio di Meccanica Razionale, Parte Seconda*: Dinamica, Cenni di Meccanica dei Sistemi Continui. Seconda Edizione Riveduta. By T. Levi-Civita and U. Amaldi. Bologna, Zanichelli, 1938. 8+310 pages.

*College Algebra.* By H. L. Rietz and A. R. Crathorne. Fourth edition. New York, Henry Holt and Co., 1939. 18+298 pages. $1.65.

*Mêchanique Statistique Quantique.* By F. Perrin. (Traité du Calcul des Propabilités et de ses Applications, vol. 2, no. 5.) Paris, Gauthier-Villars, 1939. 224 pages. 100 fr.

*Calculus.* By F. H. Miller. New York, John Wiley & Sons; London, Chapman & Hall, 1939. 14+419 pages. $3.00.

*Essentials of Analytic Geometry.* By R. W. Brink. New York and London, D. Appleton-Century Company, 1939. 11+233 pages. $2.25.

*Fünfstellige Logarithmen und Zahlentafeln.* Zürich, Orell Füssli, 1939. 184 pages. 4.50 fr.

*Mathematics in Action.* By W. W. Hart and Lora D. Jahn. New York, D. C. Heath and Co., 1939. 9+344 pages. $0.88.

*Les Fonctions de Matrices, I. Les Fonctions Univalentes.* By H. Schwerdtfeger. (Actualités Scientifiques et Industrielles, 649, Exposés de Géométrie, publiés sous la direction de E. Cartan, X.) Paris, Hermann et Cie, 1938. 59 pages. 20 fr.

*Les Fonctions Multivalentes.* By M. Biernacki. (Actualités Scientifiques et Industrielles, 657; Exposés sur la Théorie des Fonctions, publiés sous la direction de Paul Montel, XI) Paris, Hermann et Cie, 1938. 66 pages. 20 fr.

*La Notion de Point Irrégulier dans le Problème de Dirichlet.* By Florian Vasilesco. (Actualités Scientifiques et Industrielles, 660; Exposés sur la Théorie des Fonctions, publiés sous la direction de Paul Montel, XII.) Paris, Hermann et Cie, 1938. 61 pages. 20 fr.

*La Régularisation des Fonctions.* By S. Mandelbrojt. (Acualités Scientifiques et Industrielles, 733; Exposés sur la Théorie des Fonctions, publiés sous la direction de Paul Montel, XII.) Paris, Hermann et Cie, 1938. 61 pages. 20 fr.

*Plane Trigonometry.* By W. T. Stratton and R. D. Daugherty. New York, Prentice-Hall, 1939. 7+88 pages. Trigonometric Tables, 118 pages. $2.25.

*Fabre and Mathematics and Other Essays.* By Lao G. Simons. (The Scripta Mathematica Library, Number 4.) New York, Scripta Mathematica, 1939. 5+101 pages. $1.00.

*Wahrscheinlichkeitsrechnung für Nichtmathematiker.* By Karl Dörge and Hans Klein. Berlin, Walter de Gruyter, 1939. 113 pages. RM 6.00.

*Elements of the Topology of Plane Sets of Points.* By M. H. A. Newman. Cambridge, at the University Press, 1939. 8+221 pages. $3.50.

*Modern-School Geometry.* By R. R. Smith and J. R. Clark. (Schorling-Clark-Smith Modern-School Mathematics Series). Yonkers-on-Hudson, New York, World Book Company, 1939. 8+248 pages. $1.28.

*Protons, Neutrons, Neutrinos.* By Jacques Solomon. (Collection de Physique Mathématique, Fascicule VI.) Paris, Gauthier-Villars, 1939. 13+228 pages. 100 fr.

*Les Nouvelles Méthodes du Calcul des Probabilités.* By Louis Bachelier. Paris, Gauthier-Villars, 1939. 8+71 pages. 25 fr.

*Mathematics of Statistics.* By John F. Kenney. New York, D. Van Nostrand Co., 1939. Part One, 10+248 pages. $2.50. Part Two, 9+202 pages. $2.25. Both parts in one volume, $4.00.

*Probability, Statistics and Truth.* By Richard v. Mises. Translated by J. Neyman, D. Sholl, and E. Rabinowitch. New York, The Macmillan Co., 1939. 16+323 pages.

*Contributions to the Mechanics of Solids*, dedicated to Stephen Timoshenko by his friends on the occasion of his sixtieth birthday anniversary. New York, The Macmillan Co., 1939. 10+277 pages. $5.00.

*Complex Variable and Operational Calculus with Technical Application.* By N. W. McLachlan. Cambridge, the University Press, 1939. 11+355 pages. $6.50.

*Einführung in die algebraische Geometrie.* By B. L. van der Waerden. (Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen, Band LI.) Berlin, J. Springer, 1939. 7+247 pages. RM 19.50.

*An Introduction to Modern Geometry.* By Levi S. Shively, New York, Wiley & Sons; London, Chapman and Hall, 1939. 11+167 pages. $2.00.

*Analytic Geometry*. By Roscoe Woods. New York, The Macmillan Company, 1939. 14+294 pages. \$2.25.

*Outline of the History of Mathematics*. By R. C. Archibald. Fourth edition. Oberlin, Ohio, Mathematical Association of America. 1939. 66 pages. \$0.50.

*Algebraische Kurven, II. Allgemeine Eigenschaften*. By H. Wieleitner. Neue Bearbeitung. (Sammlung Göschen Band 436.) Berlin, Walter de Gruyter & Co., 1939. 122 pages. RM 1.62.

*Synthetic Projective Geometry*. By R. G. Sanger. New York and London, McGraw-Hill Book Co., 1939. 9+175 pages. \$2.00.

*Geschichte der Mathematik I Bis 17. Jahrhunderts*. By H. Wieleitner. (Sammlung Göschen Band 226.) Berlin, Walter de Gruyter & Co., 1939. 136 pages. RM 1.62.

*Geschichte der Mathematik II. Von 1700 bis zur Mitte des 19. Jahrhunderts*. by H. Wieleitner. (Sammlung Göschen Band 875.) Berlin, Walter de Gruyter & Co., 1939. 154 pages. RM 1.62.

## REVIEWS

*Panmagische Quadrate und Magische Sternvielecke*. By F. Fitting. Leipzig, K. F. Koehlers Antiquarium, 1939. 70 pages. RM. 2.

This is the fourth number in the series *Scientia Delectans*.

The first fifty pages of the book deal with the construction and enumeration of panmagic, or pandiagonal, magic squares of even order. Since pandiagonal magic squares whose order is of the form $2(2n+1)$ do not exist, the squares treated here are of order $4n$.

A number of writers on magic squares have given the impression that the construction of magic squares of even order is much more difficult than for squares of odd order. One of the principal purposes of the author is to show that this view is without foundation.

This he does by building up his magic squares by means of several component squares of very simple form. He then shows that, in general, the greater the number of components used, the easier the method is to apply; and that the greater the number of prime factors in $(4n)^2$, the greater the number of possible components. Furthermore, the author's method is powerful in that it enables one, in a purely mechanical way, to construct a surprisingly large number of pandiagonal magic squares of any given order, of the form $4n$.

The last twenty pages of the book deal with magic star-polygons. Many examples of magic stars have appeared from time to time, but the treatment here is of particular interest because it presents the beginning of a theory of the construction and enumeration of such figures. The method used is that of components, analogous to that employed in the first part of the book in studying magic squares.

G. E. RAYNOR

*Methods of Statistical Analysis*. By C. H. Goulden. New York, Wiley; London, Chapman and Hall, 1939. 7+277 pages.

"Of making many books there is no end" seems to apply with particular patness to books dealing with statistical methods which have recently been developed. The present book, which belongs to this class, is a textbook for students of statistics who have gone beyond the elementary stage, those who understand the fundamental principles and are ready to prepare themselves for practical statistical work.

With the exception of two chapters, one on tests of significance with small samples, and another on the design of simple experiments, the first part of the book treats such topics as the calculation of mean and standard deviation, frequency tables, theoretical frequency distributions (especially the binomial and normal distributions), regression, and correlation. While this is review material for the type of student for which the book is designed, it is presented in a way which gives an insight into the structure of actual experiments and a taste of the sort of calculation which the student may expect to meet later in the book and in practice.

There follow two short chapters on chi square and its application to testing goodness of fit and independence. Yates's correction for continuity is explained.

In the first chapter devoted to the analysis of variance occurs a good discussion of the question of selecting a valid "error" variance for making tests of significance. The ability to select such an error probably comes only with experience, but the author points out, by means of examples, the appropriate error to use in various circumstances, and gives some idea of what the requisites of a valid error are.

This chapter is followed by a long chapter entitled "The Field Plot Test," which, in the opinion of the reviewer, is the most important and most valuable portion of the book. It is divided into several parts, the first of which discusses soil heterogeneity, replication, randomization, and error control, and such standard designs as randomized blocks, the Latin square, factorial experiments, and split-plot arrangements. In the discussions of the last-named type of experiment there is a complicated, but carefully explained, example of breaking up degrees of freedom. Further consideration is given here to the question of selecting an appropriate error term. The next part takes up orthogonality and confounding, partial confounding, and the recovery of information. The third part deals with incomplete block methods for testing a large number of varieties.

Next occur chapters on the application of the analysis of variance to regression, both linear and non-linear, Fisher's method of fitting least-squares polynomials by successive steps is well illustrated. By this method a straight line is fitted first, then a parabola, then a cubic, and so on. At each stage one degree of freedom is consumed in fitting, and the variance represented by this degree is tested for significance against the regression error. If there is no gain in precision, the curve of next lower order previously fitted is taken as giving the best fit.

After a chapter on the analysis of covariance, the book concludes with a

chapter on miscellaneous applications, which includes a discussion of the estimation of missing values and methods of randomization.

Tables of $t$, $\chi^2$, and Snedecor's $F$ are included.

Adverse criticism is entirely on minor points. For example, the statement near the bottom of page 10 that the mean deviation is zero might be objected to on the ground that by "mean deviation" is usually meant the mean absolute deviation, and this is not zero. On page 12, equation (5) can hardly be termed an exact relation; it would be exact only if the numerator were the population standard deviation. On page 22 (line 16) it would be better to say that $y$ is the frequency density at the point $x$.

The author might well have explained on page 42 that the $s$ of (6) is the estimated standard deviation of the difference $x_1 - x_2$, and that the $s$ of (8) is the estimated standard deviation of $x_1$ and of $x_2$, it being assumed that they have a common variance. Formula (7) gives the estimated standard deviation of this mean difference when (6) is used, while (9) gives the estimated standard deviation of this mean difference when (8) is used. The fact that the $T$'s are totals should be explicitly stated. The subscript 1 is omitted from $\bar{x}$ in the equation following (7) and from the $T$ in (8). The denominator $N-1$ in (6) and (8) should be enclosed in parentheses and the same remark applies to the $N-1$ in the last line of fine print on page 36.

On page 72 (line 1) the word "approximately" is unnecessary. The same word however is necessary in describing the normal distribution of $z'$ on the following page.

The book is as much a treatise on experimental design in agronomy as on statistical methods. It is replete with worked-out examples of the size, regarding the amount of computation involved, actually met in practice, it being the contention of the author that miniature examples, although perhaps desirable in explaining fundamental concepts, are likely to give the student a false impression of what is required of the practicing statistican. A careful study of the book, including the solution of a representative set of the exercises, with which it is plentifully supplied, should prepare the student for applying modern statistical methods intelligently, not merely mechanically, to agronomic and similar problems.

P. R. RIDER

*Der Deutsche Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts.* E.V. 1891–1938. By Wilhelm Lorey. Frankfurt am Main, Verlag Otto Salle, 1938. 165 pages. RM. 3. Kart.

As set forth by the author, this book presents an account of the German Association for the Advancement of Instruction in Mathematics and Natural Science from its inception to the present time, together with an historical review of the role education has played in these fields during the last fifty years. In seven short chapters we follow the development of the Association, its pre-

history and organization, the first Congress at Jena in 1890, its expansion to include all branches of natural science, the World War and inflational period, the influence of the Third Reich, and the last congress in Munich in 1938. The author has summarized the work of each congress throughout this period of time and has accumulated some very worth while biographical material. We see the impetus given the Association by J. C. V. Hoffmann, the zeal of Felix Klein for its reorganization, and the contribution of many contemporary German mathematicians and scientists, as Günther, Pietzker, Withing, Lietzmann, and Wieleitner.

Thoroughly indexed, this book is made still more valuable to the reader by carefully prepared annotations for each chapter and the inclusion of twenty-eight illustrations of outstanding members of the Association. We have here, written in beautifully clear style, a book which presents information obtainable at no other source, of interest to the historian, the educator and the teacher. It is a worth while reference book to present day trends in education and science in Germany.

R. A. HARRISON

*General Mathematics*. By C. H. Currier, E. E. Watson, and J. S. Frame. The Macmillan Company, 1939, revised edition. 382 pages. $3.00.

This book includes material from algebra, trigonometry, analytic geometry, and calculus, and is intended for use in first-year college taken by students who will not pursue the subject further. The present edition, which has been revised by the original two authors and J. S. Frame, differs in many respects from the former. The changes occur mostly in the chapters on trigonometry, exponents and logarithms, and conic sections. Throughout the new edition, the problem material has been considerably enriched with a greater number of concrete applications. The omission of the chapter on determinants and the enlarging of the tables at the end of the book are certainly improvements.

The course is treated systematically, clearly, and in a manner not too detached from the experience of the students. Historical excursions and comments make the book interesting. However, it leaves the impression that, while the material is presented from the purely mathematical point of view, the rigorousness could be improved in several places without making the book more difficult. For example, the introduction of $a^0$ and $a^{-n}$ on page 46, leaves the reader uncertain whether the authors are supposed to have proved that $a^0=1$ and $a^{-n}=1/a^n$, or have defined these values to avoid exceptions in the laws of exponents. Like most other textbooks which include practical problems, *General Mathematics* is inconsistent in its own use of the rules of accuracy in computation which it develops. The explanation of the rules of accuracy might well be made more convincing to the student.

HELEN A. INFELD

*Synthetic Projective Geometry*. By R. G. Sanger. New York and London, McGraw-Hill Book Co., 1939. 9+175 pages. $2.00.

In reply to the oft-repeated question: how should this subject be presented in a first course, the author of this volume has selected the classic treatise of Cremona as his guide, but has made the important improvement of removing all metric applications until the defining elements have all been established. Not so much detail in the discussion of homology is necessary, in consequence, but otherwise the development closely follows the scheme of Cremona until the earlier author introduces cross-ratio, which in the present case is not used until the purely synthetic treatment of the first and second order is completed, including the generation of a conic by Steiner's method, the theorems of Pascal, and Brianchon and Desargues on a pencil of conics, and polarity. The fundamental theorem of Pappus thus appears as a special case of that of Pascal when the conic is composite. A chapter of nearly thirty pages near the end of the volume, devoted to metrical properties, develops many properties of lines and of conics. from projective ones. A short glance is thrown upon the generation of quadric surfaces and of cones by means of projective pencils of planes, and the concept of tangent planes established. Spatial intuition is used freely, and no attempt at rigor is claimed. From his premises, the author has established a serviceable introduction to his subject.

VIRGIL SNYDER

*Essentials of Analytic Geometry*. By Raymond W. Brink, University of Minnesota. New York, D. Appleton-Century Company, 1939.

This book is an abridged edition of his earlier text entitled "Analytic Geometry, Revised Edition." The shorter book is better suited to the needs of a usual course in analytic geometry of four or five hours a week for one quarter, although the reviewer feels that it is well for even freshman college students to have books containing more material than is actually needed for the course. Such would serve as a source for reference for topics which can be used for extra work by the brilliant students in the class, or as sources for information on analytic geometry needed later in following courses in mathematics. For the ordinary student, however, a compact text covering only the material needed in the course is a delight, and to this end this new book will serve a definite purpose.

The usual topics are treated in very orthodox fashion. It is good to see a text which covers preliminary material and finishes the work through the straight line in forty-six pages. Perhaps if the author had followed this work on the straight line with his twelve page chapter on circles the students would attack the chapters on "The Locus of an Equation" and "Locus Problems" with a little more experience in the type of analysis needed in the discussion of graphs. The chapter on tangents is an isolated topic which initiates the student into some ideas used in calculus later, and for this reason has some merit. On the other hand, it could easily be omitted if a class were pressed for time. The

chapter on conic sections is vitalized by interesting paragraphs on tangent conics and systems of conics, and particularly by a paragraph on the sections of a cone. This is a topic which should be included in every analytic geometry so that the instructor can give some argument for the use of the term "conic sections."

The text closes with a brief chapter on transcendental curves and a chapter on solid analytic geometry, followed by a short appendix and four pages of tables.

The text is compactly written with frequent appearance of heavy type to stress important concepts and formulas. The analysis in discussion of the various topics is direct and easily understood by the student reader. There seems to be a generous selection of problems and exercises. The author in his introduction says that he has stressed methods, and he has accomplished this.

CORNELIUS GOUWENS

*Algebraische Kurven, II. Allgemeine Eigenschaften.* By H. Wieleitner. Second edition. Sammlung Göschen 436. Berlin, Walter de Gruyter, 1939. 122 pages. RM 1.62.

This little volume aims to give the essential elementary properties of algebraic plane curves. It includes the intersection theorems, including Cramer's paradox, and the multiplicity of common singular points in the simplest cases; polar properties with application to diameters; the detailed development of Plücker's formulas, and an extensive treatment of the Hessian and Steinerian curves at singular points; a chapter on rational curves, with their parametric representation. A very brief outline of Cremona transformations is followed by a more detailed treatment of quadratic transformation, including circular inversion. A chapter on cubic curves is limited to the configuration of points of inflexion and the constant cross-ratio theorem. A similar discussion of the general quartic and its systems of bitangents closes the volume. The book is provided with a brief bibliography, and an index. The mechanical make-up is excellent.

VIRGIL SNYDER

*College Algebra.* By H. L. Rietz and A. R. Crathorne. New York, Henry Holt and Company. Fourth Edition, 1939. xv+298 pages. $1.85.

The first edition of this excellent text was published in 1909. The second and third editions appeared in 1919 and 1929 respectively. The present reviewer has compared this fourth edition in reasonable detail with the third edition. The essential character of the text is unchanged. It contains less of elementary review and more of advanced material, such as an introduction to the theory of limits and infinite series, than appears in many current texts on college algebra. The changes noted in the present edition consist, to a large extent, in rearrangement of material within the same chapter or in transfer of material to another chapter. By the latter means, the treatment of exponents, radicals and complex

numbers is condensed into one chapter and this change may be regarded as an improvement. The chapters on systems of quadratic equations, inequalities, ratio proportion and variation, the theory of equations, and partial fractions have been enlarged from twenty-five to forty per cent. Some of the present lists of problems are almost identical with those of the third edition. Others have been materially extended and some are entirely new. The changes noted seem fully to justify the new edition.

J. V. McKelvey

*Geschichte der Mathematik*. By Dr. Heinrich Wieleitner, Berlin, Walter de Gruyter & Co., 1939 (Two volumes. Sammlung Göschen Band 226 and Band 875, respectively, 136 pages and 154 pages). RM 1.62.

Someone has called the Sammlung Göschen "The Five Foot Shelf of Mathematics." All fifty volumes of this collection should be in our mathematical libraries and should be read. They are so well-written, so brief, and so handy to carry around. These books have, to quote the motto of a well-known magazine, "infinite riches in a little room." These two volumes by Dr. Wieleitner live up to the reputation made by the other volumes in the collection.

Each book of this history of mathematics has an excellent table of contents, page of references to the literature, index of names, and index of subjects. Volume I treats of the progress of mathematics from the ancient times to the beginning of the 18th century. Volume II carries the story down to the middle of the 20th century.

The different branches of mathematics (such as geometry, algebra, trigonometry, etc.) are systematically taken up, so the books are not arranged entirely according to time, country, or mathematicians. General discussions of trends, of historical and philosophical backgrounds, are interspersed. These add greatly to the value of the books. Due credit is given where credit is due. Especially fair-minded is the discussion of such an episode as the Newton-Leibnitz quarrel. The author succeeds all along, while using few words, in giving a vivid picture of the times and the settings, as well as of the results achieved mathematically.

Let us compare these two books with their predecessors (of which these are actually a revision), namely the Geschichte Vol. I by Gunther (Sammlung Schubert, Vol. XVIII, 1907, Göschen, Leipzig, 407 pages) covering the period from ancient times to Descartes, Vol. II (1) by Wieleitner (Vol. LXIII, 1911, 215 pages) discussing the time from Descartes to the beginning of the 18th century, and Vol. II (2) by Wieleitner (Vol. LXIV, 1921, 184 pages) treating geometry and trigonometry. We should probably prefer reading the new history and referring to the older volumes for more details.

Dr. Wieleitner has done well in his choice of what facts to include, in his control of such a sweeping subject, in his analysis of what influenced mathematics and how mathematics influenced other subjects. It took a combination of excellent mathematician and good writer to produce two such interesting and

valuable little books. It will inspire any mathematician, and even perhaps give him hints on how to approach his own problems of study or research, to read these books. He will realize anew how ancient is our field of study, how slowly and painfully it developed until recent times, how and why discoveries (or inventions, if you wish to call them so) were made independently and then perhaps lost for a while, and also how rapid is the tempo of the present day in mathematics. The author has succeeded in giving the reader the feeling that we are caught up in a great and growing flood of discoveries and of new ideas that shows no sign of ever ebbing.

A. D. CAMPBELL

*Fünfstellige Logarithmen und Zahlentafeln.* By Erwin Voellmy. Zürich, Orell Füssli, 1939. 184 pages.

These tables, with explanations in German and in French, are printed on a large open page and in unusually clear type. Logarithms to five places are followed by five-place logarithmic tables of sines, cosines, tangents, and cotangents, printed on paper of a different color, with supplementary tables for small angles; these are followed by four-place tables of natural trigonometric functions (on white paper) with arc and chord lengths and radius minus apothegm in intervals of one degree for arcs up to 180°.

A table of squares, cubes, square-roots, cube-roots, reciprocals and other functions is provided, followed by various functions of $\pi$, of arcs on an ellipse, powers of $e$, factorials, interest tables, future values, present worth and tables connected with the Swiss insurance system, chemical tables, lists of physical constants, astronomical constants, and tables derived from the recently corrected tables of the geodesic survey of Switzerland.

Finally, some thirty pages of formulas from algebra, plane and solid geometry, and trigonometry are provided. The frontispiece is a copper plate of Just Bürgi, the Swiss co-founder of logarithms.

The reviewer was unable to find any typographical errors.

VIRGIL SNYDER

*Mathematik und Verkehr.* By H. Kellerer. Leipzig and Berlin, B. G. Teubner, 1938. 48 pages.

In this little book Mr. Kellerer describes a number of elementary statistical methods for which illustrations are taken from many different forms of transportation and communication throughout the world, but the bulk of the material describes the development of German transportation in recent years.

Among the subjects he includes in this study are the flights of the dirigible Hindenburg, passenger and freight traffic on German railroads, traffic accidents on the main German automobile highways and streets, the distribution of radios in the different branches of the national economy, bicycles per capita in different European countries, criteria for determining the rank of different countries in the use of automobiles, German daily newspapers, improvements in German

naval construction, distribution of traffic accidents in Berlin according to the hours of the day, variation in the consumption of electricity throughout the day, and the fluctuations in the telegraph business of the German post office. It is difficult to know whether the author is more interested in presenting the statistical means used in analyzing this great variety of subjects or in the discussion of the subjects themselves.

The statistical concepts described in Part I are average values, dispersion, and percentages. In the author's search for an index to be used in the comparison of the degree of motorization in the United States; Great Britain and Ireland, France, Germany, Italy, and Australia, he introduces the only concept that may be considered different from that ordinarily presented under the subjects treated in Part I. Mr. Kellerer maintains that an index of motorization which considers both the population and land area of a country is better than one which uses merely the land area or population. He proposes the index $100K/\sqrt{FE}$, in which $K$ is the number of automobiles, $E$ the population, and $F$ the area of land. The denominator, $\sqrt{FE}$, is the product of the population times the distance between nearest neighbors if the population were distributed uniformly over the country with one person at the center of each square of side equal to $\sqrt{F/E}$. The author states that his suggestion does not present an ideal index, and although he does not specifically say so, he would agree, no doubt, that in such matters the choice of an index depends primarily on the use that is to be made of it.

The discussion in Part II includes frequency and cumulative tables, frequency and cumulative curves, concentration curves, and correlation diagrams or dot charts. A concentration curve is used in this book to indicate the concentration of the sales of copies of daily German newspapers in a small number of separate newspapers. This is shown by plotting the per cent of daily separate newspapers from zero to one hundred on one axis and the copies of these same newspapers sold, as percentages of all newspapers sold, on the other axis.

Part III is devoted to a discussion of time series and contains illustrations of such customary concepts as index numbers, seasonal variation, fitting of trends by least squares, moving average, and the exponential or growth curve.

The statistical technique presented in each part is of an elementary nature, but is nevertheless very useful in analyzing the data on transportation and communication.

This little volume will be particularly interesting to those concerned with the study of different methods of travel, and also to those interested in reading a development of elementary statistical or mathematical tools in an informal manner as they are needed and used. The author has assumed no knowledge of traffic or statistics on the part of his readers, and he has written the book in such a way as to hold the interest of many individuals not particularly inclined toward statistical studies. Viewed in this light, the book is a worthwhile presentation of a few very elementary but useful statistical tools.

J. M. THOMPSON

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.*

## BEGINNING AN ACADEMIC YEAR

At the beginning of another year, program committees will be engaged in making an outline of the year's work and activities. From yearly reports received, we find that the most generally accepted year's calender includes several general meetings at which papers on pertinent subjects are presented by students and faculty members. Certain clubs have found that a contest indulged in on campus or with clubs of nearby schools has created a friendly mathematical rivalry. Or the meeting with a nearby group may take any one of several other forms. For when a group of people with such similar interests get together, the field is unlimited. And, of course, interspersed here and there throughout the year are each club's own traditional social gatherings. With this general club program in mind we make the following suggestions and hope that they may prove useful to the program committees.

PAPERS. In the consideration of club topics we have noted with a great deal of pleasure, from hastily scanning some of the new reports which have just come to our desk, that several clubs have turned to the various departments of the MONTHLY for material on which they based their year's program. We should like heartily to endorse this policy, and call the attention of each club to the splendid articles, Questions, Discussions and Notes, Problems and Solutions, Mathematical Education, and Recent Publications. You will find the MONTHLY and other mathematical periodicals fine source books as well as reference material.

BIBLIOGRAPHY. Reminder: Please keep on file bibliographies on all subjects presented this winter and send any you think of general interest to this department of the MONTHLY to add to our growing list of club topics and their bibliographies. By the way, remember there is a fine list of club topics which has been published from time to time over a period of several years under this department. You might find there some new topic or question for discussion with your group this year.

JOINT MEETING. A great many groups have found that a joint meeting with the club from some other department on their campus has proved very worth-while and stimulating. Have you a club with whom you could share a speaker and a discussion which would be of mutual interest and benefit? Or is there some mathematics group near enough to encourage one or more joint meetings combining a speaker or a contest with social "get-to-gether" such as are enjoyed by groups around Boston, Milwaukee, and New York? Or have you tried getting in touch with the better high school students in mathematics through a competitive examination or mathematics program of popular appeal? Delta Rho of Southern Illinois State Normal University at Carbondale found a Mathematics Field Day to hold immeasurable opportunities. To quote from the report received of the meeting the chairman has this to say: "Too, the experience gained by these college students and the stimulation of their own interest in mathematics and mathematical activities make up no small part of the benefits of the occasion to the college."

CONTESTS. A contest may prove instructive as well as entertaining, such as the one sponsored annually by Brooklyn College among members of its own department, and

5. Pi Delta Theta, honorary mathematics fraternity at the University of Denver, sent us the following variation of the "How old is Ann?" type of problem:

"The combined ages of the University of Denver and Pi Delta Theta are 86 years. The University of Denver is fifteen-sixteenths as old as Pi Delta Theta will be when the University of Denver is nine-sixteenths as old as Pi Delta Theta will be when Pi Delta Theta is twice as old as the University will be when the University of Denver is twice as old as Pi Delta Theta. When was Pi Delta Theta founded, and how old is the University of Denver?" Your solution may be sent to Pi Delta Theta as well as to this department. Pi Delta Theta would welcome such contacts with other groups. May we add the suggestion that similar problems be sent this club or exchanged with other groups. This department will gladly publish lists of clubs willing to send as well as receive such challenges.

NEW CLUBS. Members of mathematics departments in universities, colleges, normal schools and junior colleges in which no mathematics club exists, may find good opportunity to form such groups during the coming year. There are many schools where active interest can be encouraged. This department can think of no better Christmas greeting than a note containing the name of a newly formed club, its officers, and plans.

HONORARY FRATERNITIES. Many schools may be interested in forming honorary groups to affiliate with one of the national mathematics fraternities. Officers of these two organizations to whom inquiries may be directed are:

*Pi Mu Epsilon*: President General, Professor W. E. Milne, Oregon State College, Corvallis, Oregon; Secretary-Treasurer General, Professor John S. Gold, Bucknell University, Lewisburg, Pennsylvania.

*Kappa Mu Epsilon*: President Pythagoras, Professor C. V. Newsom, University of New Mexico, Albuquerque, New Mexico; Secretary Diophantus, Miss E. Marie Hove, State Teachers College, Wayne, Nebraska.

## SLIDES AND FILMS

We are happy to announce that arrangements have been made through Dr. T. C. Fry, research mathematician, and Mr. John Mills, Director of Publication, with the Bell Telephone Laboratories in New York City for a loan of the film on the *Isograph* to mathematics departments or clubs. The Isograph is described in two articles by Dietzold and Mercner;* briefly it might be stated that it is an instrument designed by Dr. Fry and constructed in the laboratories to find quickly and easily the roots of polynomials irrespective of whether they are real or complex. The program consists of a 16 mm. (silent) film, slides, and a prepared lecture, and is recommended for showing before junior, senior, and graduate students. This program has had a preview before several groups, including the Mathematics Club of Stanford University, and a joint meeting of the New Jersey College for Women and Upsala College groups with the Montclair State Teachers Mathematics Clubs. At the latter meeting, Mr. R. L. Dietzold himself gave the lecture that will accompany the slides and film. Those in attendance found that it gave them a particularly good insight into the use of mathematics in the work of the Bell Telephone Laboratories and into the operation of this complicated instrument.

* R. L. Dietzold, The Isograph—A Mechanical Root Finder, Bell Laboratories Record, XVI, 1937, pp. 130–134.

R. O. Mercner, The Mechanism of the Isograph, Bell Laboratories Record, XVI, 1937, pp. 135–140.

Clubs may borrow this material without expense other than the shipping cost of returning the film or forwarding it to the next designated club. In order that an itinerary may be made for shipping the film from one school to another, it is requested that applications be filed before October 30th with Miss L. E. Smith, Librarian, Bell Telephone Laboratories, 463 West Street, New York City. Requests should include both a preferred and an alternate date.

## PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND H. S. M. COXETER

### ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems. *Note.* The editors wish to express to W. F. Cheney, Jr. their gratitude for his splendid work in editing this department for many years, and their regret that he has found it necessary to give up the work at this time.—E. J. M.

### SOLUTIONS

E 344 [1938, 478]. *Proposed by V. W. Graham, High School, Dublin, Ireland.*

$ABCD$ is a parallelogram with vertices named in order around the perimeter. $DA$ and $CB$ are produced to $P$ and $Q$ respectively, so that $AP=BQ$. Any point $X$ is taken on $AB$. $PX$ meets the diagonal $BD$ at $Z$. $QZ$ meets $DC$ at $Y$. Prove that $AX=DY$. Conversely, if $AX=DY$, prove that $Q$, $Y$, and $Z$ are collinear.

*Solution by E. R. Heineman, Texas Technological College.*

Extend $PZ$ until it meets $BC$ at $S$. Extend $QY$ until it meets $AD$ produced at $T$. Then from similar triangles, $AX/XB=PA/BS$, and $DY/YC=DT/QC$. In order to prove that $AX=DY$, we merely need to show that $PA/BS=DT/QC$. Replacing $PA$ by $QB$, and $QC$ by $PD$, then interchanging the means, we find that we must prove that $QB/DT=BS/PD$. These ratios are the same because each of them equals $BZ/ZD$, a consequence of two pairs of similar triangles.

To prove the converse, let $QZ$ meet $DC$ at $U$. Then by the preceding proof $AX=DU$. But by hypothesis, $AX=DY$. Hence $U$ and $Y$ must coincide, and $QZY$ is a straight line.

Also solved by W. E. Buker, W. B. Clarke, Wm. Douglas, Herman Levy, D. L. MacKay, C. W. Trigg and the proposer.

E 345 [1938, 551]. *Proposed by F. E. Wood, Northwestern University.*

Let $S=a+b+c$, and $T=ab+ac+bc$, where $a$, $b$ and $c$ are the sides of a triangle. Show that $3T\leqq S^2<4T$. What are the analogous inequalities for a tetrahedron?

*Solution by Paul Heinicke, St. Louis, Missouri.*

$2ab\leqq a^2+b^2$, and similarly for $2ac$ and $2bc$. Adding, simplifying, and then adding $2T$, there results immediately $3T\leqq S^2$.

Since $a$, $b$ and $c$ are the sides of a triangle, $|a-b|<c$, and similarly for $|a-c|$ and $|b-c|$. Squaring, adding, simplifying and adding $2T$, gives $S^2<4T$.

Therefore $3T \leqq S^2 < 4T$.

In the tetrahedron, let $S$ equal the sum of the six edges, and $T$ equal the sum of their fifteen products in pairs. We write the fifteen possible equations like $2ab \leqq a^2+b^2$, add them, divide by 5, add $2T$, and get $12T/5 \leqq S^2$.

Writing the three inequalities like $|a-b|<c$ for each of the four faces, squaring, adding, transposing all square terms to the left and all product terms to the right, halving and adding $2T$, we get $S^2<3T-$(the sum of the three products of pairs of opposite edges). But since we know only that the quantity in parenthesis is positive, we have $S^2<3T$.

Consequently, $12T/5 \leqq S^2 < 3T$.

H. D. Larsen submitted a generalized solution in which $S$ represented the sum of $n$ numbers, and $T$ represented the sum of their $n(n-1)/2$ products in pairs. By methods analogous to those above he deduced that $2nT/(n-1) \leqq S^2$, and if the $n$ numbers represent the edges of a polyhedron whose faces are all triangular, then $S^2<3T$.

In his solution, C. W. Trigg let $A$, $B$, $C$ and $D$ represent the areas of the faces of a tetrahedron, with $S$ their sum and $T$ the sum of their six products in pairs. By analogous methods he showed that $8T/3 \leqq S^2 < 4T$.

Also solved by E. R. Heineman, D. L. MacKay, E. P. Starke and the proposer.

## ADVANCED PROBLEMS

*Send all communcations about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3922. *Proposed by V. Thébault, Le Mans, France.*

The triangle $BAC$ is right angled at $A$; the squares $CAA_1C_1$ and $ABB_1A_2$ are constructed exteriorly on the sides $CA$ and $AB$; and $M$ is the foot on $BC$ of the exterior bisector of angle $A$. (1) Show that the polygon $P'$, the antipedal of $M$ with respect to polygon $P \equiv BB_1A_2CC_1A_1$, is inscribed in the circle $\Sigma$, passing through $M$ and concentric with the square constructed interiorly on the hypotenuse $BC$. (2) Express the radius $r$ of $\Sigma$ as a function of elements of triangle $BAC$, and obtain the condition that $r=BC$. (3) Show that the areas of $P$ and $P'$ are equal.

*Note.* The antipedal triangle of a point $M$ with respect to a triangle $ABC$ is determined by the intersection of the perpendiculars to $MB$ and $MC$, to $MC$ and $MA$, and to $MA$ and $MB$ at the respective points $A$, $B$, $C$.

3923. *Proposed by R. E. Gaines, University of Richmond.*

It is known that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus. Show that the diameter $d$ of the circle is given by $d \sin\alpha \sin\beta \sin\gamma = a$, where $\alpha$, $\beta$, $\gamma$ are the angles which the tangents make with the axis of the parabola, $y^2 = 4ax$.

## SOLUTIONS

3831 [1937, 332]. *Proposed by V. Thébault, Le Mans, France.*

Given a triangle $ABC$ and a variable point $M$ on the circumcircle $(O)$, show that (a) The parallels to $MA$, $MB$, $MC$ drawn from the orthocenter $H$ meet the corresponding sides in three points on the polar $\Delta$ of the point $M'$, diametrically opposite to the point $M$ on $(O)$, with respect to the conjugate circle; and that this line passes through the mid-point of the straight line segment $MH$: (b) The line $\Delta$ envelops the conic with foci $O$ and $H$ inscribed in the triangle $ABC$: (c) The orthogonal projection of the point $M'$ on the line $\Delta$ is the orthopole of the reciprocal transversal of that straight line with respect to triangle $ABC$.

*Editorial Note.* A straight line $d$ cuts the sides $BC$, $CA$, $AB$ of a triangle in $L$, $M$, $N$. Let $L'$ be the symmetric of $L$ with respect to the mid-point of $BC$; $M'$ and $N'$ are defined in a similar manner. Then $L'$, $M'$, $N'$ lie on a straight line $d'$ which is called the reciprocal transversal of $d$ with respect to triangle $ABC$.

Let $A'$, $B'$, $C'$ be the feet of the perpendiculars from the vertices of triangle $ABC$ upon the straight line $d$ in its plane. Then the perpendiculars from $A'$ to $BC$, from $B'$ to $CA$, from $C'$ to $AB$ meet in a point $D$ which is called the orthopole of $d$ with respect to $ABC$.

*Solution by J. R. Musselman, Western Reserve University.*

Let us choose for our circle, with center $O$, the unit circle, whence we can denote the coördinates of the points $A$, $B$, $C$ and $M$ respectively by $t_i$ $(i=1, 2, 3, 4)$ where the $t_i$ are turns, *i.e.*, $|t_i| = 1$. We shall denote the elementary symmetric functions of $t_1$, $t_2$ and $t_3$ by $\sigma_i$; those of $t_1$, $t_2$, $t_3$ and $t_4$ by $S_i$.

Any four concyclic points have associated with them a point $P$ and a line; the point, with coördinate $S_1/2$, is the midpoint of the segment joining the orthocenter of any three of the points to the fourth, and the line which is perpendicular to $OP$ at $P$, with the equation

$$S_3 x + S_1 S_4 \bar{x} = S_1 S_3 \tag{1}$$

shall be shown to be the line $\Delta$ of the problem.

The equation of the parallel to $MA$ through $H$ is

$$x + t_1 t_4 \bar{x} = \sigma_1 + t_1 t_4 \sigma_2 \sigma_3^{-1}.$$

The intersection of this with the side $BC$

$$x + t_2t_3\bar{x} = t_2 + t_3$$

is the point $\alpha$, whose coördinate is given by

$$(1 - t_1^2t_4\sigma_3^{-1})\alpha = \sigma_1 + t_4.$$

The intersections of the parallels to $MB$ and $MC$ through $H$ with their corresponding sides are points $\beta$ and $\gamma$ whose coördinates differ from those of $\alpha$ by having $t_2^2$ and $t_3^2$ respectively in place of $t_1^2$. The points $\alpha$, $\beta$ and $\gamma$ lie on a line $\Delta$ whose equation is

$$(2) \qquad (t_4\sigma_2 + \sigma_3)x + t_4\sigma_3(\sigma_1 + t_4)\bar{x} = (\sigma_1 + t_4)(t_4\sigma_2 + \sigma_3).$$

But (2) in terms of symmetric functions of the four $t_i$ is

$$S_3x + S_1S_4\bar{x} = S_1S_3$$

or precisely (1). Hence the line $\Delta$ really belongs to all four points. *Given four points $T_iT_jT_kT_l$ on a circle, if through the orthocenter $H_l$ of triangle $T_iT_jT_k$ we draw lines parallel to $T_lT_i$, $T_lT_j$, $T_lT_k$ cutting $T_jT_k$, $T_kT_i$, $T_iT_j$ in points $\alpha_l$, $\beta_l$, $\gamma_l$ respectively, the twelve points $\alpha$, $\beta$, $\gamma$ thus obtained are collinear.* This line passes through the midpoint of the segment joining the orthocenter of any three of the points to the fourth.

The equation of the conjugate circle of $ABC$ is

$$(3) \qquad (x - \sigma_1)(\bar{x} - \sigma_2/\sigma_3) = (\sigma_1\sigma_2 - \sigma_3)/2\sigma_3.$$

The coördinate of $M'$, the diametrically opposite point of $M$ on the circumcircle, is $-t_4$, and the polar line of $M'$ as to the circle (3) turns out to be precisely (2) or (1). Hence *given any four concyclic points $T_iT_jT_kT_l$, the polar line of $T_i'$ the diametrically opposite point of $T_i$, $(i=1, 2, 3, 4)$, as to the conjugate circle of triangle $T_jT_kT_l$ is the line $\Delta$.*

If in equation (2) we let $t_4$ vary, the envelope of $\Delta$ is the locus of $\Delta_2$ in problem 3758 [1937, 668], a conic inscribed in $ABC$ having $O$ and $H$ as foci. Hence the line $\Delta$ is a common tangent of the conics inscribed in triangles $ABC$, $ABM$, $BCM$ and $CAB$ having $O$ as a common focus and their orthocenter as the other focus.

The reciprocal transversal of $\Delta$ as to $ABC$ is the line $S_1x+S_3\bar{x}=0$, which passes through $O$. The orthopole of this line as to $ABC$ is the point $(\sigma_1+\sigma_3S_1S_3^{-1})/2$, a point on the ninepoint circle of $ABC$ and also on the line $\Delta$. In fact it is the orthogonal projection of $M'$ on $\Delta$. Similarly, if we determine the orthopoles of the reciprocal transversal of $\Delta$ as to the triangles $ABM$, $BCM$ and $CAM$ we obtain three other points lying on $\Delta$, the orthogonal projections on $\Delta$ of the points diametrically opposite to $C$, $A$ and $B$ respectively.

Solved also by the proposer.

*Editorial Note.* The conjugate circle $(H)$ of triangle $ABC$ has its center at the orthocenter $H$, and the triangle is self polar with respect to it. Its radius is real if $H$ is outside of $ABC$ and a pure imaginary if it is inside. It follows that the nine point circle $(N)$ with center $N$ and the circumcircle $(O)$ are inverses with respect to $(H)$, a vertex $A$ and the foot of its altitude $H_a$ being inverse points. The polar of $(O)$ with respect to $(H)$ is a conic $(OH)$ tangent to the sides of $ABC$ and with its foci at $O$ and $H$, and $(N)$ is the auxiliary circle corresponding to the focal diameter. The projections of $O$ and $H$ on any tangent to $(OH)$ lie on $(N)$. Also $(N)$ and $(O)$ have $H$ as center of similitude with the ratio $1:2$. A line $l$ through the circumcenter $O$ cuts $(O)$ in $M$ and $M'$. The inverse of $M'$ with respect to $(H)$ is a suitable intersection $M_1'$ of $M'H$ with $(N)$. The polar $m'$ of $M'$ is then the perpendicular to $M'H$ at $M_1'$. Let $m'$ cut the sides of $ABC$ in $\alpha$, $\beta$, $\gamma$; then the polar with respect to $(H)$ of any one of the last points, say $\gamma$, passes through $M'$ and $C$. Hence $M'C$ is perpendicular to $H\gamma$ and to $CM$, and thus $CM$ and $H\gamma$ are parallel. Thus $\alpha, \beta, \gamma$ are the points in the first part of the problem, and they lie on $\Delta = m'$. If $HM_1'$ is produced to meet $(O)$ at $M''$, then $M_1'$ is the midpoint of $HM''$, and $M''M$ is parallel to $m'$. Hence $m'$ passes through the midpoint of $HM$. Let $m$, the polar of $M$, cut $m'$ in $D$; then the pole of $l$ is $D$. As $l$ turns about $O$, $D$ describes the directrix of $(OH)$ corresponding to the focus $H$, since this directrix is the polar of $O$ with respect to $(H)$. All these results are special cases of the generalization to $n$ dimensions in the solution of problem 3821 [1939, 241].

Let the projections of $A$, $B$, $C$ on any line $l$ through $O$ be $A_1$, $B_1$, $C_1$; then the perpendiculars from these last points to the corresponding sides of $ABC$ meet in the orthopole $L$ of $l$. We consider now nine positions of $l$ through $O$, and it will be shown that $L$ for each $l$ lies on $l'$, the reciprocal transversal for $l$, and also on $(N)$; and that $HL$ is perpendicular to $l'$. (a) Let $l$ be parallel to a side, say $AB$. Since the distance between $l$ and $AB$ is one half of $HC$, the reciprocal transversal $l'$ is the perpendicular bisector of $HC$, and it is also tangent to $(OH)$. The figure $HAA_1H_c'$ is a parallelogram, where $H_c'$ is the midpoint of $HC$. Similarly, $HBB_1H_c'$ is a parallelogram; and hence $H_c'$ is the orthopole of $l$, and it is on $l'$ and $(N)$. (b) Let $l$ pass through a vertex, say $A$, and let $AH$ cut $(O)$ in $K$. The midpoint of $HK$ is $H_a$; $B_1$ is the midpoint of the chord $BB_1B_2$ of $(O)$; and $AB_2 = AB$. Hence angle $AKB_2$ is $C$ or $\pi - C$, and in either case $KB_2$ is parallel to $HB$. Therefore $B_1H_a$ is parallel to $BH$ and perpendicular to $CA$. Similarly, we show that $C_1H_a$ is perpendicular to $AB$. Hence $H_a$ is the orthopole of $l$ and it lies on $(N)$. Also $HH_a$ is perpendicular to $BC$, the reciprocal transversal for $l$. (c) Let $l$ pass through a midpoint of a side, say $C'$ on $AB$, cutting $CA$ in $L_2$. Then $A_1 = B_1 = C'$, and hence $C'$ is the orthopole of $l$ and it lies on $(N)$. It also lies on the transversal $l' = C'L_2'$ reciprocal to $l$, where $CL_2 = L_2'A$. In what follows it will be shown that $l'$ is tangent to $(OH)$, and it then follows that the projections of $O$ and $H$ on $l'$ lie on $(N)$. If $BC$ and $CA$ are unequal, $C'$ cannot be the projection of $O$ on $l'$, and so $C'$ must be the projection of $H$ on $l'$. If $BC = CA$,

then obviously $C'$ is the projection of $H$ on $l'$. The perpendicularity may also be proved independently by elementary geometry.

As $l$ turns about $O$, $l'$ cuts projective ranges of points on the sides of $ABC$, and therefore envelops a conic tangent to its sides. It has been shown above that the perpendicular bisectors of $AH$, $BH$, $CH$ are positions of $l'$; and hence this envelope is $(OH)$. As $l$ turns about $O$, $l'$ passes through all positions of tangents to $(OH)$. It now follows that the reciprocal transversal to any tangent to $(OH)$ must pass through $O$.

With $O$ as origin of rectangular coördinates the equation of $l$ is $y-\lambda x=0$. The coördinates of $L$ are easily found to have the form of fractions with numerators and denominators of the second degree in $\lambda$, and the two denominators have a common quadratic factor in $\lambda$. The equation of $l'$, with zero for the right member, has coefficients and constant term of the second degree in $\lambda$. The insertion of the coördinates of $L$ in the left member of the equation for $l'$ reduces the latter to a polynomial of the fourth degree in $\lambda$. Since this polynomial is zero by (a), (b), (c) for more than four distinct values of $\lambda$, it must be zero for all values. Hence $L$ lies always on $l'$. The condition for the perpendicularity of $HL$ and $l'$ reduces to the vanishing of a polynomial of the fourth degree in $\lambda$. It follows in the same way as above that $HL$ is always perpendicular to $l'$. Moreover $L$ lies on $(N)$. We have now proved that the orthogonal projection of $M'$ on $m'=\Delta$ is the orthopole of the transversal reciprocal to $m'$, and that this reciprocal transversal passes through $O$.

3832 [1937, 393]. *Proposed by Don Wallace, Charlottesville, Va.*

Given the circle whose center is the circumcenter and which passes through the incenter: prove that (a) it cuts the altitudes at a distance from the vertices equal to the circumradius, and (b) it cuts the perpendicular bisectors of the sides at a distance twice the inradius from the reflections of the circumcenter in the sides.

*Solution by J. H. Butchart, Phillips University.*

In the first two lines of the problem the words circumcenter and incenter should be interchanged. Let $I$ be the incenter and $O$ the circumcenter, and let $P$ be the farther point from $A$ determined by the given circle on the altitude from $A$. Since $AI$ bisects the angle $PAO$, $AP$ equals $AO$, which is the first statement for the altitude from $A$. If $A'$ is the midpoint of the side opposite $A$, let $Q$ be the second point where $A'O$ meets the given circle. It is clear that the inradius is the arithmetic mean of $A'O$ and $A'Q$, which amounts to statement (b).

*Editorial Note.* C. W. Trigg showed that the problem as stated is not true for any triangle.

## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

The Statistical Laboratory of Iowa State College, of which Professor G. W. Snedecor is director, has recently made a project agreement with the Bureau of Agricultural Economics of the United States Department of Agriculture. This provides for joint research in the statistics of agriculture and associated statistical theory. The project aims to satisfy the need for agricultural information of greater scope, higher degree of precision and increased detail.

Dr. W. G. Cochran, mathematician in the statistical department of Rothamsted Experimental Station, has been appointed professor of mathematical statistics at Iowa State College, to belong to the staffs of the agricultural experiment station, mathematics department, and the statistics laboratory. In the laboratory he will serve the cooperative project above mentioned, as well as the longer-established functions of the laboratory—the coordination of all statistical teaching, counsel and research of the college. Dr. Cochran was visiting professor of mathematical statistics at Iowa State College in October and November, 1938.

The Southern Intercollegiate Mathematics Association held its sixth annual meeting in Jackson, Mississippi, at Millsaps College, May 13, 1939. Southern Methodist University was awarded the S. I. M. A. Cup for having the highest average in the final examination. The following students won the right to wear the S. I. M. A. Key because they made the highest grades in the Association in their respective subjects: Southern Methodist University: Merle Mitchell, Julia Smith, Wilbur Teubner; Mississippi Women's College: Cleo White, Mary Emma Fancher, Nina Pearl Byrd, Louise Tate; North Texas State Teachers College: William Tittle; Centenary College: William E. Steger.

The Division of Mathematics of Harvard University has awarded the William Lowell Putnam Memorial Scholarship for 1939 to E. L. Kaplan of the Carnegie Institute of Technology. As previously announced, this scholarship was to be awarded to one of the five ranking highest in the Putnam Competition for the year.

At the recommendation of Professor R. C. Archibald, the American Mathematical Society has generously given to the Mathematical Association volumes 1 to 5, 7 and 10 of the *Bulletin*, thus enabling the Association to complete its file.

On May 31 Professor W. L. Ayres of the University of Michigan gave two addresses before the department of mathematics of Wayne University. The titles were, "The coloring of maps" and "Peano spaces and their decomposition."

Dr. E. R. Hedrick, provost of the University of California at Los Angeles,

gave the commencement address at the University of Missouri. At this time the degree of doctor of laws was conferred upon Dr. Hedrick.

Professor J. H. Van Vleck of Harvard University gave a series of lectures at the Institut Henri Poincaré at the University of Paris during the month of May. He also took part in a symposium on Magnetism at Strasbourg.

At the University of Oregon, Professor E. E. DeCou, after thirty-seven years of service, retired as head of the department on July 1, 1939, with the title of professor emeritus. He will teach part time the coming year. Associate Professor A. F. Moursund is now head of the department, Dr. K. S. Ghent and Dr. T. S. Peterson are promoted to assistant professorships, and Dr. C. F. Cossack is appointed an instructor.

Dr. P. O. Bell of the University of Kansas has been promoted to an assistant professorship.

Assistant Professor H. A. Bender of the University of Akron has been promoted to an associate professorship.

Professor H. F. Blichfeldt of Stanford University retired in June 1938 with the title of professor emeritus.

Dr. J. W. Blincoe of the University of Tennesee has been promoted to an assistant professorship.

Dr. Joseph Bowden, after forty-one years of service at Adelphi College, has retired with the title of professor emeritus.

Assistant Professor I. W. Burr of Antioch College has been promoted to an associate professorship.

Professor C. E. Comstock of Bradley Polytechnic Institute has retired with the title of professor emeritus.

Assistant Professor T. F. Cope of Queens College has been appointed an associate professor.

H. M. Cox, formerly assistant examiner in the University System of Georgia, has been appointed assistant professor and assistant to the Bureau of Instructional Research at the University of Nebraska.

Assistant Professor H. A. DoBell of the New York State College for Teachers has been promoted to a professorship.

Dr. E. L. Dodd of the University of Texas has been selected as research professor for the year 1939–40, one professor at the University being selected in this capacity each year with a reduced teaching schedule.

Dr. Otto Dunkel, professor of mathematics and astronomy at Washington University, retired in June 1939 after twenty-three years of service there.

Professor J. A. Eiesland of West Virginia University has been given the title of professor emeritus.

Assistant Professor C. H. Forsyth of Dartmouth College has been promoted to a professorship.

Dr. J. S. Frame of Brown University has been promoted to an assistant professorship.

K. G. Fuller has been promoted to an assistant professorship at the Teachers College of Connecticut.

Associate Professor C. A. Garabedian of Wheaton College, Massachusetts, has been promoted to a professorship.

Assistant Professor E. C. Goldsworthy of the University of California was on leave during the spring term.

Associate Professor R. F. Graesser of the University of Arizona has been made professor and head of the department.

Dr. O. G. Harrold, Jr., of Stanford University has been appointed a National Research Fellow at the University of Virginia for 1939–40.

Associate Professor E. E. Heimann has been promoted to a professorship at the State Teachers College at Ada, Oklahoma.

Dr. Grace M. Hopper of Vassar College has been promoted to an assistant professorship.

Professor H. M. Hosford of the University of Arkansas has been appointed dean of the college of arts and sciences.

E. Marie Hove has been promoted to an associate professorship at the Teachers College at Wayne, Nebraska.

Assistant Professor J. A. Hyden of Vanderbilt University has been promoted to an associate professorship.

Assistant Professor R. D. James of the University of California has been appointed professor and head of the department at the University of Saskatchewan.

Dr. L. Louise Johnson of the University of Colorado has been appointed fellow at Reed College.

After thirty-five years of service, Professor Vladimir Karapetoff of the department of electrical engineering at Cornell University has retired with the title of professor emeritus. He is now actively engaged in research and consulting work at Leonia, New Jersey, and during the year 1939–40 will lecture at Stevens Institute of Technology.

Dr. S. H. Kimball of the University of Maine has been promoted to an assistant professorship.

Dr. C. W. MacGregor of Massachusetts Institute of Technology has been appointed associate professor of applied mechanics.

Assistant Professor S. L. Mason of the University of North Dakota has been promoted to an associate professorship.

Dr. Harriet F. Montague of the University of Buffalo has been promoted to an assistant professorship.

Assistant Professor Rufus Oldenburger of Armour Institute of Technology has been promoted to an associate professorship.

Assistant Professor R. L. O'Quinn of Louisiana State University has been promoted to an associate professorship.

Associate Professor B. C. Patterson of Hamilton College has been promoted to a professorship.

Dr. A. E. Pitcher of Lehigh University has been promoted to an assistant professorship.

Assistant Professor G. B. Price of the University of Kansas has been promoted to an associate professorship.

Assistant Professor C. B. Read has been made associate professor and acting head of the department at the University of Wichita.

Associate Professor P. R. Rider of Washington University has been promoted to a professorship.

N. S. Risley of Case School of Applied Science has been promoted to an assistant professorship.

J. E. Sandt of Marietta College has been made an assistant professor.

Associate Professor A. R. Sloan of Carson-Newman College has been promoted to a professorship.

Assistant Professor W. F. Smith of New River State College, West Virginia, has been promoted to a professorship.

Dr. D. E. South of the University of Kentucky has been promoted to an associate professorship.

Associate Professor R. C. Staley of the University of North Dakota has been promoted to a professorship.

Professor J. S. Taylor of the University of Pittsburgh has been appointed head of the department of mathematics to succeed Professor K. D. Swartzel, who recently retired with the title of professor emeritus after serving seventeen years.

Dr. H. C. Trimble has been awarded a General Education Board Fellowship in evaluation.

Assistant Professor H. L. Turrittin of the College of Mines at El Paso has been appointed assistant professor at the University of Minnesota.

P. H. Underwood of the Ball High School, Galveston, Texas, has retired after serving fifty years as a public school teacher of mathematical subjects. He is a charter member of the Mathematical Association and was a member of the National Committee on Mathematical Requirements of the Association in its work during the years 1916–1923.

Associate Professor F. W. Urban of Central Missouri State Teachers College has been promoted to a professorship.

Assistant Professor C. H. Vehse of West Virginia University has been promoted to an associate professorship.

Dr. G. P. Wadsworth of Massachusetts Institute of Technology has been promoted to an assistant professorship.

A. C. Washburne, actuary of the Berkshire Life Insurance Company since 1911, has retired with the title of actuary emeritus.

Assistant Professor G. A. Williams of Oregon State College has been promoted to an associate professorship.

Professor A. H. Wilson of Haverford College has retired with the title of professor emeritus.

Dr. Louise A. Wolf of the University of Wisconsin at Milwaukee has been promoted to an assistant professorship.

Associate Professor R. C. Yates of the University of Maryland has been appointed an associate professor at the Louisiana State University.

The following men have been appointed Benjamin Peirce Instructors at Harvard University for the Academic year 1939–40: Dr. Leon Alaoglu, Dr. B. J. Pettis; and Dr. D. T. Perkins has been reappointed.

The following appointments to instructorships have been announced:
University of Arkansas: J. R. F. Kent
Brooklyn College: Dr. Leon Gropper, Dr. A. J. Maria.
Harvard University: M. H. Heins, E. N. Nilson.
University of Michigan: Dr. E. D. Rainville
Princeton University: Dr. J. W. Tukey (Henry B. Fine instructor)
Wells College: Dr. Dorothy Manning
West Virginia University: Dr. Margaret B. Cole.

Professor T. E. Mason of Purdue University died May 25, 1939. He had been connected with the university since 1914, and had been a member of the Mathematical Association since 1923.

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## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13.

ILLINOIS, Galesburg, May 12–13.

INDIANA, Muncie, April 28–29.

IOWA, Ames, April 21–22.

KANSAS, Topeka, April 1.

KENTUCKY, Murray, April 28–29.

LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13; WASHINGTON, D. C., December.

MICHIGAN, Ann Arbor, March 18.

MINNESOTA, Northfield, May 13.

MISSOURI, Springfield, April 28.

NEBRASKA, Lincoln, May 5.

NORTHERN CALIFORNIA, San Francisco, January 28.

OHIO, Columbus, April 8.

OKLAHOMA, Tulsa, February 10.

PHILADELPHIA, Bethlehem, Pa., December 2.

ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.

SOUTHEASTERN, Charleston, S.C., March 24–25.

SOUTHERN CALIFORNIA, Whittier, March 4.

SOUTHWESTERN, Alpine, Texas, May 2–3.

TEXAS, Abilene, March 31–April 1.

WISCONSIN, Milwaukee, May 6.

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## THE APRIL MEETING OF THE IOWA SECTION

The twenty-eighth regular meeting of the Iowa Section of the Mathematical Association of America was held at Iowa State College, Ames, Iowa, on Friday and Saturday, April 21–22, 1939, in conjunction with the fifty-third regular meeting of the Iowa Academy of Science. Professor E. E. Moots, chairman of the Section, presided.

The attendance was about fifty, including the following twenty-five members of the Association: J. W. Beach, Fred Brandner, J. O. Chellevold, E. W. Chittenden, L. M. Coffin, Marian E. Daniells, R. M. Deming, C. W. Emmons, Annie W. Fleming, Gertrude A. Herr, O. C. Kreider, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Martha M. McKelvey, E. E. Moots, I. F. Neff, E. N. Oberg, Fred Robertson, W. J. Rusk, E. R. Smith, G. W. Snedecor, Henry Van Engen, L. E. Ward, J. J. Westemeier.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the Section elected for 1939–40 are as follows: Chairman, Henry Van Engen, Iowa State Teachers College; Vice-Chairman, O. C. Kreider, Ellsworth Junior College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the host, Iowa State College and the Department of Mathematics, was adopted at the business meeting. The invited address was given by Professor Erich Rothe of William Penn College at Oskaloosa. The following seventeen papers were read:

1. "Bounds for the libration points in the restricted problem of four bodies" by Professor J. J. L. Hinrichsen, Iowa State College, introduced by the Secretary.

2. "Some applications of matrices in mechanics" by Professor E. W. Anderson, Iowa State College, introduced by the Secretary.

3. "Note on a certain type of finite series" by Professor A. T. Craig, University of Iowa, by title.

4. "Sampling the 1938 wheat production of North Dakota" by A. J. King, Department of Agriculture, and E. H. Jebe, Iowa State College, introduced by Professor Snedecor.

5. "Sampling agricultural facts in Iowa" by R. J. Jessen, Iowa State College, introduced by Professor Snedecor.

6. "Use of the notion of the infinite in mathematics" by Professor Erich Rothe, William Penn College, by invitation.

7. "Thermal stresses in an isosceles right-triangular plate with pinned edges" by Professor D. L. Holl, Iowa State College, introduced by the Secretary.

8. "Combinational problems in experimental design" by Gertrude M. Cox, Iowa State College, introduced by Professor Snedecor.

9. "Some problems in the mathematics curriculum faced by the secondary schools and colleges in Iowa" by Professor Henry Van Engen, Iowa State Teachers College.

10. "Roots of quadratic equations expressed in continued fractions" by Fred Robertson, Iowa State College.

11. "On the zeros of the function defined by a certain definite integral" by Professor L. E. Ward, University of Iowa.

12. "On the solution of certain types of linear differential equations for large values of the argument" by E. N. Oberg, University of Iowa.

13. "A practical application of orthogonal polynomials to Fisher's regression integral" by F. E. Davis, Department of Agriculture, introduced by Professor Snedecor.

14. "A problem in oceanographic sampling" by Professor C. P. Winsor, Iowa State College, introduced by Professor Snedecor.

15. "Some properties of Schlömilch's function" by Professor E. R. Smith, Iowa State College.

16. "An exposition of the theory of the real number system" by Professor E. W. Chittenden, University of Iowa.

17. "Euler's 'Integral Calculus' " by Professor R. B. McClenon, Grinnell College, by title.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Hinrichsen discussed the case with three of the four bodies restricted to lie at the vertices of an equilateral triangle and moving in concentric circles about their common center of gravity. If $\Omega(x, y)$ is the potential function, the libration points may be defined as solutions of $\Omega_x(x, y) = \Omega_y(x, y) = 0$. They are the critical points of the manifold of states of motion and the points about which infinitesimal periodic orbits may exist. These points are shown to be in certain regions associated with the unit circles about the vertices of the equilateral triangle and the extended sides of the triangle.

2. Professor Anderson indicated that technical writers are now using matrices in solving circuit nets and vibration problems. Some elements of the more unfamiliar but useful matrix methods of solution were illustrated by solving a mechanical problem involving two simultaneous second-order linear differential equations with boundary conditions.

4. Mr. Jebe gave a preliminary report of the possibilities of forecasting and estimating the yield of wheat by sampling commercial fields. A summary of the data collected with the analysis of variance and covariance was presented.

5. Mr. Jessen gave a description of the objectives, the plan of the sampling procedure, and some of the statistical findings of a farm-to-farm survey of agricultural and economic items in Iowa.

6. Professor Rothe gave a most interesting discussion of the notion of the infinite in mathematics. Cantor's theory of sets was described and a survey concerning the theories of Brouwer and of Hilbert was given.

7. Dr. Holl showed that the problem of the thermal stresses in thin polygonal plates of uniform thickness having different uniform temperatures on its two faces and with pinned edges is equivalent to a membrane under constant

normal pressure and covering the same region. The isosceles right-triangular plate shows infinite shearing stresses at the right angle corner and finite stresses at the other corners. These forces prevent the natural curling of these corners.

8. A survey of some of the combinatorial problems underlying balanced incomplete block experimental designs was given by Miss Cox. Several devices used to aid in the construction of these designs were discussed.

9. Professor Van Engen discussed three main problems: (1) the underlying causes for the changes in the mathematical situation in the Iowa secondary schools, and some of their implications of interest to the teachers of college mathematics; (2) the problem of certification of teachers in Iowa and the approval of teaching combinations by the State Department of Public Instruction; and (3) the problem of organizing courses that would fit into a "general education" program on the college level. It was brought out that one of the outstanding weaknesses of the mathematical program on the college level, as now organized, is that it is too rigid; that is, it does not provide for individual differences, individual needs, and individual interests.

10. Mr. Robertson indicated the approximation of the roots of a quadratic equation by continued fractions. Then he obtained a formula for computing the real or complex roots when the continued fraction is expressed in a prescribed form.

11. Professor Ward studied the zeros of the function $f(\lambda)=\int_0^\infty e^{-\lambda t^4}\sinh \lambda t \sin \lambda t\, dt$, where $\lambda$ is a complex variable confined to the sector $-\pi/3 \leqq \arg \lambda \leqq \pi/3$. By application of the method of steepest descents an asymptotic formula for $f(x)$, valid for $0 \leqq \arg \lambda \leqq \pi/3$ was found. This formula shows that for $|\lambda|$ large and confined to the sector, the zeros of $f(\lambda)$ are real and nearly equally spaced.

12. Dr. Oberg showed that if $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the equation (1) $d^2u/dx^2+\phi(x, 0)u=0$ and if $|y_1(x)|\int_x^\infty|y_2(t)| |v(t)|dt+|(y_2(x)|\int_x^\infty|y_1(t)|\ |v(t)|dt$ grows small as $x$ increases, then the general solution of (2) $d^2u/dx^2+\phi(x,v)u=0$ approaches asymptotically the general solution of (1) as $x$ grows large. The proof is obtained by expressing the general solution of (2) by means of an integral equation, which under the above conditions approaches the general solution of (1) as $x$ grows large.

13. Mr. Davis indicated that the orthogonal polynomials tabulated by Fisher and Yates are very convenient for use in Fisher's regression integral as applied to a study of effects of weather on corn yields; such integral is a polynomial function of the time.

14. Professor Winsor showed how information as to oceanic plankton populations comes largely through the use of tow nets and is naturally subject to sampling errors. The use of the analysis of variances technique in estimating these errors was illustrated. It appears that the standard error of single observations may run from 30 to 50%.

15. The function $S_n(x)=\int_0^\infty e^{-vx}/(1+v)^n dv$ was considered by Schlömilch in *Zeitschrift für Mathematik und Physik*, vol. 4, 1859. Professor Smith showed

that for positive integral values of $n$ the functions are the coefficients of $\alpha^n$ in the expansion of

$$\phi(x, \alpha) = \frac{1}{x} + \alpha \int_0^\infty \frac{e^{-vx}}{1 - \alpha + v} dv$$

$$= \frac{1}{x} - \alpha e^{x(\alpha-1)} E(x\overline{\alpha - 1}).$$

By means of this generating function it is shown that

$$(n + 1)S_{n+2}(x) + (x - n)S_{n+1}(x) - xS_n(x) = 0$$

and

$$xS_n''(x) - (x + n - 2)S_n'(x) - S_n(x) = 0.$$

Moreover, the integral $\int_0^\infty S_n(x)\ S_m(x)dx$ is equal to the coefficient of $\alpha^n\ \beta^m$ in the expansion of $1/2(\alpha+\beta-2)\ [\pi^2+\log^2 (1-\alpha)/(1-\beta)]$.

16. Professor Chittenden indicated how the class $P$ of positive real numbers can be defined in terms of a subclass of the class of binary functions on the class $I$ of the positive and negative integers and zero. A binary function is one which assumes one of two possible values, which may be represented by the symbols 0, 1. The properties of the entire real number system are briefly and easily obtained from this definition. The present paper contains a detailed exposition of the theory of the real number system on this basis.

17. Professor McClenon gave an account of some of the outstanding results published in the first volume of Euler's *Integral Calculus*.

CORNELIUS GOUWENS, *Secretary*

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## THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-second annual meeting of the Kentucky Section of the Mathematical Association of America was held at Murray State Teachers College on Saturday, April 29, 1939, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor N. B. Allison, chairman of the Section, presided.

There were twenty-two in attendance, including the following ten members of the Association: N. B. Allison, M. G. Carman, Fritz John, W. L. Moore, Sister Charles Mary Morrison, Mabel I. Nowlan, W. F. Smith, D. E. South, Elizabeth C. Strayhorn, H. M. Yarbrough.

At the luncheon meeting the following officers were elected for next year: Chairman, H. H. Downing, University of Kentucky; Secretary, D. E. South, University of Kentucky.

The following papers were presented:

1. "The discriminant of the sextic of double point parameters of the plane rational quartic curve" by Sister Mary Charlotte Fowler, Nazareth College, introduced by Sister Charles Mary Morrison.

2. "Some remarks on curve tracing" by Dean P. P. Boyd, University of Kentucky, by title.

3. "The language of mathematics and its relation to quantitative thinking" by Professor M. E. Schell, Western Kentucky State Teachers College, introduced by Professor Yarbrough.

4. "The attraction of an ellipsoid" by Professor Fritz John, University of Kentucky.

5. "Fermat's Last Theorem" by Jerome Krumpleman, University of Louisville, introduced by Professor Moore.

6. "Application of force of mortality to certain probabilities" by V. W. Pfeiffer, University of Kentucky, introduced by Professor South.

7. "Making a reflecting telescope" by Professor W. F. Smith, New River State College, West Virginia.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

1. When two of the double point parameters of the plane rational quartic coincide the curve may have as special singularities a triple point, a cusp, or a tacnode. That the discriminant of the sextic giving the double point parameters contains these factors was shown by Sister Mary Charlotte Fowler. The triple point condition is an invariant of the second degree in $\Delta_{ijk}$, the three-row determinant formed from the coefficient matrix of the parametric equations. It is designated by $T_2$. The cusp condition and the tacnode condition are of the sixth degree in the same determinants, designated by $K_6$ and $T_6$ respectively. Then $T_2^{\alpha}$, $K_6^{\beta}$, and $T_6^{\gamma}$ are factors of the discriminant which is of the thirtieth degree in $\Delta_{ijk}$. The triple point factor enters into $T_2$ once and into the discriminant six times, so that $\alpha=6$; the cusp factor is contained in $K_6$ and in the discriminant once and $\beta=1$. The tacnode factor is contained six times in the discriminant; so $\gamma=2$.

3. The purpose of Professor Schell's discussion was to create an interest in, and emphasize the importance of, the language of mathematics and its relation to quantitative thinking, as regards the teaching of mathematics in the basic courses as offered in the elementary grades, high school, and undergraduate courses. The discussion was from the teacher's point of view, using examples to illustrate the use of the language of mathematics in checking as well as directing the mental reactions of the student in mathematics classes.

4. Professor John gave a short review of the history of the problem of determining the attraction of a homogeneous ellipsoid, and then derived the elliptic integral representing the solution.

5. Mr. Krumpleman gave an exposition of the nature of Fermat's Last Theorem and its influence on algebraic numbers.

6. In this paper Mr. Pfeiffer showed how the nominal annual rate of mortality can be expressed in terms of $l_x$ and its first differential coefficient with respect to $x$, developed a simple working approximation to the force of mortality,

and used the force of mortality in the expression of certain fundamental probabilities of survival.

7. Professor Smith explained the need for a careful selection of materials and apparatus to be employed in the making of a reflecting telescope. He demonstrated the techniques to be followed in the grinding, polishing, figuring and surfacing of the reflector and diagonal. Then the assembling and mounting of the parts were explained.

D. E. SOUTH, *Secretary*

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## THE 1939 MEETING OF THE MISSOURI SECTION

The 1939 meeting of the Missouri Section of the Mathematical Association of America was held at Drury College, Springfield, Missouri, on Friday, April 28. The meeting was presided over by the Secretary, Professor L. M. Blumenthal.

Among the thirty-two persons attending the meeting were the following nine members of the Association: S. Louise Beasley, L. M. Blumenthal, J. E. Case, Sister M. Borgia Clarke, G. M. Ewing, B. F. Finkel, W. L. Graves, W. H. Lyons, D. T. Sigley.

At the business session the following officers were elected for the coming year: Chairman, L. M. Blumenthal, University of Missouri; Secretary, G. M. Ewing, University of Missouri.

The following papers were presented:

1. "The characteristics of a system of conics" by Professor W. H. Lyons, Kansas State College of Agriculture.

2. "Sufficient conditions for an ordinary problem in the calculus of variations" by Dr. G. M. Ewing, University of Missouri.

3. "A generalization of a special class of groups" by Professor D. T. Sigley, Kansas State College of Agriculture.

4. "An *ab initio* derivation of the derivatives of $a^x$ and $\sin^{-1} x$" by Professor B. F. Finkel, Drury College.

5. "Analytic geometry without coördinates" by B. E. Gillam, University of Missouri, introduced by Professor Blumenthal.

6. "A numerical solution of quadratic congruences" by J. F. Wulftange, St. Louis University, introduced by Professor Sigley.

Abstracts of the papers follow, numbered as in the above list:

1. The characteristics $(u, v)$ of a system of conics are the number of conics of the system that pass through a given point and the number of conics of the system tangent to a given line. Professor Lyons found the characteristics of a system satisfying four conditions and from that determined the number of conics satisfying a fifth condition. The use of the characteristics in solving certain other problems was briefly sketched.

2. Dr. Ewing showed that if $C_0$: $y = y_0(x)$ is a class $C'$ curve satisfying the Euler condition together with the condition $G(x, y, y_0(x), y', y_0'(x)) > 0$ (or $\geqq 0$)

for $x_1 \leqq x \leqq x_2$, with $(x, y)$ in a neighborhood of $C_0$, and $y - y_0(x)$ and $y' - y_0'(x)$ not both zero, then $C_0$ furnishes a proper (or improper) strong relative minimum for $J = \int_{x_1}^{x_2} f(x, y, y')dx$, $G(x, y, \bar{y}, y', \bar{y}') \equiv f(x, y, y') - f(x, \bar{y}, \bar{y}') - (y - \bar{y})f_y(x, \bar{y}, \bar{y}') - (y' - \bar{y}')f_{y'}(x, \bar{y}, \bar{y}')$. Examples were given which satisfy these sufficient conditions but which do not yield to classical methods.

3. The finite abstract groups in which every subgroup is invariant (Hamiltonian groups) are well known, as are also the groups in which every subgroup is abelian. Classifying the finite abstract groups according to the number of complete sets of non-invariant conjugate subgroups contained in them, Professor Sigley discussed the class of groups which contain a single complete set of non-invariant conjugate subgroups. Necessary and sufficient conditions that a group belong to this class were obtained.

4. It was shown by Professor Finkel how the derivatives of $a^x$ and $\sin^{-1} x$ may be derived directly by the $\Delta$-process, without reverting to the derivatives of the logarithmic and sine functions.

5. Mr. Gillam sketched a development of three-dimensional analytic euclidean geometry (free of coördinates) with "point" as the single undefined element and "distance" as the only primitive relation. Lines and planes were defined as determinantal loci.

6. Mr. Wulftange presented a new method for the numerical solution of quadratic congruences which is believed to possess certain practical advantages.

L. M. BLUMENTHAL, *Secretary*

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## THE TRANSCENDENCE OF $\pi$

IVAN NIVEN,* University of Pennsylvania

Among the proofs of the transcendence of $e$, which are in general variations and simplifications of the original proof of Hermite, perhaps the simplest is that of A. Hurwitz.† His solution of the problem contains an ingenious device which we now employ to obtain a relatively simple proof of the transcendence of $\pi$.

We assume that $\pi$ is an algebraic number, and show that this leads to a contradiction. Since the product of two algebraic numbers is an algebraic number, the quantity $i\pi$ is a root of an algebraic equation with integral coefficients

$$\theta_1(x) = 0, \tag{1}$$

whose roots are $\alpha_1 = i\pi, \alpha_2, \alpha_3, \cdots, \alpha_n$. Using Euler's relation $e^{i\pi} + 1 = 0$, we have

$$(e^{\alpha_1} + 1)(e^{\alpha_2} + 1) \cdots (e^{\alpha_n} + 1) = 0. \tag{2}$$

We now construct an algebraic equation with integral coefficients whose roots are the exponents in the expansion of (2). First consider the exponents

$$\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \cdots, \alpha_{n-1} + \alpha_n. \tag{3}$$

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* Harrison Research Fellow.

† A. Hurwitz, Beweis der Transendenz der Zahl $e$, Mathematische Annalen, vol. 43, 1893, pp. 220–221 (also in his Mathematische Werke, vol. 2, pp. 134–135).

By equation (1), the elementary symmetric functions of $\alpha_1, \alpha_2, \cdots, \alpha_n$ are rational numbers. Hence the elementary symmetric functions of the quantities (3) are rational numbers. It follows that the quantities (3) are roots of

$$\theta_2(x) = 0, \tag{4}$$

an algebraic equation with integral coefficients. Similarly, the sums of the $\alpha$'s taken three at a time are the ${}_nC_3$ roots of

$$\theta_3(x) = 0. \tag{5}$$

Proceeding thus, we obtain

$$\theta_4(x) = 0,\ \theta_5(x) = 0, \cdots, \theta_n(x) = 0, \tag{6}$$

algebraic equations with integral coefficients, whose roots are the sums of the $\alpha$'s taken 4, 5, $\cdots$, $n$ at a time respectively. The product equation

$$\theta_1(x)\theta_2(x) \cdots \theta_n(x) = 0 \tag{7}$$

has roots which are precisely the exponents in the expansion of (2).

The deletion of zero roots (if any) from equation (7) gives

$$\theta(x) = cx^r + c_1x^{r-1} + \cdots + c_r = 0, \tag{8}$$

whose roots $\beta_1, \beta_2, \cdots, \beta_r$ are the non-vanishing exponents in the expansion of (2), and whose coefficients are integers. Hence (2) may be written in the form

$$e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_r} + k = 0, \tag{9}$$

where $k$ is a positive integer.

We define

$$f(x) = \frac{c^s x^{p-1}\{\theta(x)\}^p}{(p-1)!}, \tag{10}$$

where $s = rp - 1$, and $p$ is a prime to be specified. Also we define

$$F(x) = f(x) + f^{(1)}(x) + f^{(2)}(x) + \cdots + f^{(s+p+1)}(x), \tag{11}$$

noting, with thanks to Hurwitz, that the derivative of $e^{-x}F(x)$ is $-e^{-x}f(x)$. Hence we may write

$$e^{-x}F(x) - e^0F(0) = \int_0^x - e^{-\xi}f(\xi)d\xi.$$

The substitution $\xi = \tau x$ produces

$$F(x) - e^xF(0) = -x\int_0^1 e^{(1-\tau)x}f(\tau x)d\tau.$$

Let $x$ range over the values $\beta_1, \beta_2, \cdots, \beta_r$ and add the resulting equations. Using (9), we obtain

$$\sum_{j=1}^{r} F(\beta_j) + kF(0) = -\sum_{j=1}^{r} \beta_j \int_0^1 e^{(1-\tau)\beta_j} f(\tau\beta_j)d\tau. \tag{12}$$

This result gives us the contradiction we desire. For we shall choose the prime $p$ to make the left side a non-zero integer, and the right side as small as we please.

By (10), we have

$$\sum_{j=1}^{r} f^{(t)}(\beta_j) = 0, \quad \text{for} \quad 0 \leqq t < p.$$

Also by (10) the polynomial obtained by multiplying $f(x)$ by $(p-1)!$ has integral coefficients. Since the product of $p$ consecutive positive integers is divisible by $p!$, the $p$th and higher derivatives of $(p-1)!f(x)$ are polynomials in $x$ with integral coefficients divisible by $p!$. Hence the $p$th and higher derivatives of $f(x)$ are polynomials with integral coefficients each of which is divisible by $p$. That each of these coefficients is also divisible by $c^s$ is obvious from the definition (10). Thus we have shown that, for $t \geqq p$, the quantity $f^{(t)}(\beta_j)$ is a polynomial in $\beta_j$ of degree at most $s$, each of whose coefficients is divisible by $pc^s$. By (8), a symmetric function of $\beta_1, \beta_2, \cdots, \beta_r$ with integral coefficients and of degree at most $s$ is an integer provided each coefficient is divisible by $c^s$ (by the fundamental theorem on symmetric functions). Hence

$$\sum_{j=1}^{r} f^{(t)}(\beta_j) = pk_t, \qquad (t = p, p+1, \cdots, p+s),$$

where the $k_t$ are integers. It follows that

$$\sum_{j=1}^{r} F(\beta_j) = p \sum_{t=p}^{p+s} k_t.$$

In order to complete the proof that the left side of (12) is a non-zero integer, we now show that $kF(0)$ is an integer prime to $p$. From (10) it is clear that

$$\begin{aligned} f^{(t)}(0) &= 0, & (t = 0, 1, \cdots, p-2), \\ f^{(p-1)}(0) &= c^s c_r^p, \\ f^{(t)}(0) &= pK_t, & (t = p, p+1, \cdots, p+s), \end{aligned}$$

where the $K_t$ are integers. If $p$ is chosen greater than each of $k$, $c$, $c_r$ (possible since the number of primes is infinite), the desired result follows from (11).

Finally, the right side of (12) equals

$$-\sum_{j=1}^{r} \frac{1}{c} \int_0^1 \frac{\{c^r\beta_j\theta(\tau\beta_j)\}^p}{(p-1)!} e^{(1-\tau)\beta_j} d\tau.$$

This is a finite sum, each term of which may be made as small as we wish by choosing $p$ very large, because

$$\lim_{p\to\infty} \frac{\{c^r\beta_j\theta(\tau\beta_j)\}^p}{(p-1)!} = 0.$$

# ON THE CIRCLES OF CURVATURE OF THE IMAGES OF CIRCLES UNDER A CONFORMAL MAP*

J. L. WALSH, Harvard University

Under a smooth conformal map $w=f(z)$ of the region $|z|<1$ of the $z$-plane onto a region $R$ of the $w$-plane, the circles $|z|=r<1$ correspond to level curves of Green's function for $R$ with pole in the point $w=f(0)$; the lines through $z=0$ correspond to lines of flow, or orthogonal trajectories of these level curves. The geometric properties of both sets of curves in the $w$-plane have been widely studied,† including the location of their centers of curvature. But so far as the writer is aware, there has been no systematic study of the circles of curvature *with especial reference to intersection of such circles with the boundary*. It is the object of the present paper to initiate such a systematic study, by considering the circles of curvature of the images of an arbitrary circle under both the transformation $w=f(z)$ and its inverse, especially intersections with boundaries in the respective planes. The present paper does not, however, claim to be complete; in fact possible classes of regions $R$ suggest themselves in such variety that it is doubtful whether any treatment of the subject could be regarded as entirely exhaustive.

**1. Curvature of images.** As a first step in our discussion we establish

THEOREM 1. *Let the function $f(z)$ with the expansion for $|z|$ sufficiently small*

$$w = f(z) \equiv z + a_2z^2 + a_3z^3 + \cdots \tag{1}$$

*be meromorphic for $|z|<1$ and map the region $|z|<1$ onto a smooth region $R$ of the extended $w$-plane. Let $B$ denote the boundary of $R$. Let $C_\rho$ be a circle of radius $\rho$ in the $z$-plane which passes through the origin, and let $A_\rho$ be its image in the $w$-plane, or to be more exact, the image in the $w$-plane of the portion of $C_\rho$ which lies in $|z|<1$. Let $T_\rho$ denote the circle of curvature of $A_\rho$ at the origin in the $w$-plane. Then $T_\rho$ depends on $a_2$ but not on $a_k$ for $k>2$.*

In Theorem 1 and below we use the term *circle* in the extended sense, to include straight line.

Theorem 1 seems reasonable, by virtue of the fact that curvature depends on second derivatives of rectangular coördinates but not on higher derivatives.

* Presented to the American Mathematical Society, February 1939.

† See for instance the following:

Scheffers, Jahresbericht. d. deut. Math-Vereinigung, vol. 31, 1922, pp. 170–174;

Neubauer, same journal, vol. 32, 1923, p. 310;

Lilienthal, same journal, vol. 33, 1925, pp. 127–139;

Ringleb, same journal, vol. 45, 1935, pp. 57–60;

Tzitzéica, Paris Comptes Rendus, vol. 195, 1932, pp. 476–478;

Ginzel, Deutsche Mathematik, vol. 2, 1937, pp. 401–416.

Thus Theorems 1 and 2 of the present paper are not to be regarded as novel; compare especially Ginzel, *loc. cit.*, and mention there of results due to Böhmer. Introduction of the comparison transformation (3) seems to be due to Böhmer.

Nevertheless, certain expressions involving second derivatives may fail to be defined for such a function as $f(z)$, and may require the use of higher derivatives in their evaluation, so we proceed to give a formal proof of Theorem 1 based on computation of $T_\rho$.

Let $C_\rho$ be the circle

$$(2) \qquad |z - \lambda\rho| = \rho, \qquad |\lambda| = 1,$$

so that $C_\rho$ can be represented as

$$z = \lambda\rho + \rho e^{i\theta}, \qquad \theta \text{ real}.$$

Then we have

$$dz = i\rho e^{i\theta} d\theta.$$

The transformation is given by $w = f(z)$, whence

$$dw = f'(z)dz = i\rho f'(z)e^{i\theta}d\theta,$$

$$\arg(dw) = \arg[f'(z)] + \theta + \frac{\pi}{2}, \qquad |dw| = \rho|f'(z)|d\theta.$$

If $\kappa$ denotes the curvature of $T_\rho$ we have

$$\kappa = \frac{d[\arg(dw)]}{|dw|}.$$

We proceed to compute

$$d[\arg(dw)] = d[\arg f'(z)] + d\theta = \Im\{d\log[f'(z)]\} + d\theta$$
$$= \Re\left[1 + \rho e^{i\theta}\frac{f''(z)}{f'(z)}\right]d\theta.$$

The normal to $C_\rho$ at $z=0$ has the direction $\arg\lambda$, and the normal to $T_\rho$ at $w=0$ has this same direction; at $z=0$ we have $\lambda = -e^{i\theta}$, whence for $T_\rho$ at $w=0$ the center of curvature is

$$\frac{\lambda}{\kappa} = \frac{\rho\lambda}{\Re[1 - 2\rho\lambda a_2]},$$

an expression whose independence of $a_k$ for $k>2$ establishes Theorem 1. Of course the case of a line $C_\rho$ or $T_\rho$ is not properly included in the formulas given, but can be treated by an obvious limiting process. Henceforth the value $\rho=\infty$ is not excluded.

## 2. Circles of curvature under transformation.

THEOREM 2. *Under the hypothesis of Theorem* 1 *the set of all circles $T_\rho$ for a given $\rho$ is the set of circles through $w=0$ tangent to the circle $|(1+a_2w)/w| = 1/2\rho$ of the coaxal family determined by $w=0$ and $w=-1/a_2$ as null circles.*

By virtue of Theorem 1, the circles $T_\rho$ are the same whether we set $w=f(z)$ or

$$w = \frac{z}{1 - a_2 z} = z + a_2 z^2 + a_2^2 z^3 + \cdots . \tag{3}$$

For the transformation (3), the circles $A_\rho$ and $T_\rho$ are identical. Under the transformation

$$\zeta = \frac{1}{w} = \frac{1 - a_2 z}{z} \equiv \frac{1}{z} - a_2, \tag{4}$$

the points $z=0$ and $z=\infty$ correspond to $\zeta=\infty$ and $\zeta=-a_2$, respectively. The circle $|z|=2\rho$ (to which all $C_\rho$ are tangent) corresponds to the circle $|\zeta+a_2|=1/2\rho$, to which all the transforms of the $C_\rho$ are tangent. Thus all the $T_\rho$ under the transformation (1) or (3) pass through $w=0$ and are tangent to the circle

$$\left| \frac{1}{w} + a_2 \right| = \frac{1}{2\rho},$$

and the proof is complete.

The case $a_2=0$ is especially interesting, for in that case the center and radius of $T_\rho$ in the $w$-plane are the same as the center and radius of $C_\rho$ in the $z$-plane. Whenever the region $R$ is bounded, there exists* at least one point interior to it for which the mapping function (either for $R$ or for a region found from $R$ by shrinking or stretching) is of form (1) with $a_2=0$.

**3. Intersection of circles of curvature with boundary.** We shall now continue to study the circles $S_\rho$ in the $\zeta$-plane, which are simultaneously the images in the $\zeta$-plane of the circles $C_\rho$ in the $z$-plane under the transformation (4), and also the image of the circles $T_\rho$ in the $w$-plane under the transformation $\zeta=1/w$, where $w$ is given by (1); the circles $S_\rho$ are precisely the straight lines (*i.e.*, circles through the point at infinity) tangent to the circle

$$K_\rho\colon \ | \zeta + a_2 | = \frac{1}{2\rho} \cdot$$

Under the transformation $\zeta=1/w$ the region $R$ of the $w$-plane corresponds to a region $R_1$ of the $\zeta$-plane whose boundary $B_1$ is bounded. The region $R_1$ is the image of $|z|<1$ under the univalent transformation

$$\zeta = \frac{1}{w} = \frac{1}{f(z)} = \frac{1}{z} - a_2 + b_1 z + b_2 z^2 + \cdots .$$

It is consequently well known† that if the origin $\zeta=0$ is not interior to $R_1$ we have

* Walsh, Bulletin American Mathematical Society, vol. 44, 1938, pp. 520–523.

† The results are due mainly to Koebe, Bieberbach, Faber, Pick, and Löwner. See, for instance, Faber, Münchner Berichte, 1920, pp. 49–64. Or see Pólya and Szegö, Aufgaben und Lehrsätze, Berlin, 1925, vol. 2, p. 25.

$$|a_2| \leqq 2, \tag{5}$$

and that in any case $B_1$ lies in the closed region

$$|\zeta + a_2| \leqq 2; \tag{6}$$

the equality signs can be omitted in (5) and (6) unless $B_1$ is a line segment, necessarily of length 4. The point set $B_1$ has points exterior to the circle $|\zeta+a_2|=1$ unless $B_1$ is identical with that circle.

The point $\zeta=-a_2$ is called the *conformal center of gravity of* $R_1$, and is the actual center of gravity in the $\zeta$-plane of each curve corresponding to a circle $|z|=r<1$, weighted according to the weight function $|dz/d\zeta|$.

If $R$ contains the point $w=\infty$ in its interior, or if $w=\infty$ lies on $B$, or if $B$ is limited, then in the respective cases the point $\zeta=0$ is interior to $R_1$, or is on $B_1$, or does not lie on $B_1$ but is separated by $B_1$ from the point $\zeta=\infty$.

We are now in a position to prove

THEOREM 3. *Under the hypothesis of Theorem* 1 *there exist two numbers* $\rho_1$ *and* $\rho_2$ *depending on* $R$, *with* $\frac{1}{4}\leqq\rho_1\leqq\rho_2\leqq\infty$ *such that for* $\rho_2\leqq\rho\leqq\infty$ *each* $T_\rho$ *cuts* $B$; *for* $\rho_1\leqq\rho<\rho_2$ *a given* $T_\rho$ *need not cut* $B$, *but for each* $\rho$ *some* $T_\rho$ *cuts* $B$; *for* $\rho<\rho_1$ *no* $T_\rho$ *cuts* $B$. *If* $B$ *is a circle* (*of the extended plane*) *we have* $\rho_1=\rho_2=\frac{1}{2}$; *in every other case* $\rho_1<\rho_2$ *and* $\rho_1<\frac{1}{2}$.* *If* $B$ *is an arc of a circle through* 0, *we have* $\rho_1=\frac{1}{4}$, $\rho_2=\infty$; *in every other case we have* $\frac{1}{4}<\rho_1$, $\rho_2<\infty$.

If $B_1$ is not a line segment, the convex hull of $B_1$ [*i.e.*, the smallest convex set (necessarily closed) containing $B_1$] has the point $\zeta=-a_2$ as an *interior* point.† This is a consequence of the interpretation of $\zeta=-a_2$ as center of gravity of the weighted transforms of the curves $|z|=r<1$ in the $\zeta$-plane. Let

$$|\zeta + a_2| = \frac{1}{2\rho_2} > 0$$

be the largest circle with $\zeta=-a_2$ as center contained in the convex hull of $B_1$. Then any line tangent to the circle

$$|\zeta + a_2| = \frac{1}{2\rho}, \qquad \rho \geqq \rho_2, \tag{7}$$

---

* It might be conjectured that we always have $\rho_1\leqq\frac{1}{2}\leqq\rho_2$, but *this conjecture is false*, as is illustrated by the function $w=z/(1+z^8)^{1/4}$. The corresponding point set $B_1$ of the $\zeta$-plane consists of eight equally spaced line segments each of length $2^{1/4}$ radiating from $\zeta=0$. Thus we have

$$\frac{1}{2\rho_2} = 2^{1/4}\cos 22\tfrac{1}{2}^\circ = 1.0987,$$

whence $\rho_2<\frac{1}{2}$.

† This fact is intuitively obvious. The writer expects to publish a formal proof in another connection, in a paper to appear in the Bulletin of the American Mathematical Society.

cuts $B_1$, so for $\rho_2 \leqq \rho \leqq \infty$ each $T_\rho$ cuts $B$. But for every $\rho < \rho_2$ there exists a line tangent to (7) which does not cut $B_1$, so there exists a $T_\rho$ which does not cut $B$. Let the largest circle with $\zeta = -a_2$ as center passing through a point of $B_1$ be

$$|\zeta + a_2| = \frac{1}{2\rho_1} > 0;$$

we obviously have $\rho_1 \leqq \rho_2$. Any circle

$$|\zeta + a_2| = \frac{1}{2\rho}, \qquad \rho_1 \leqq \rho < \rho_2, \tag{8}$$

cuts $B_1$, but fails to lie completely in the convex hull of $B_1$. Thus some lines tangent to (8) cut $B_1$, and some lines tangent to (8) fail to cut $B_1$. There exists a $T_\rho$ which cuts $B$ and there exists a $T_\rho$ which fails to cut $B$. For $\rho < \rho_1$ the circle

$$|\zeta + a_2| = \frac{1}{2\rho}$$

contains $B_1$ in its interior, so no line tangent to this circle can cut $B_1$; no circle $T_\rho$ can cut $B$. The fact that $B_1$ lies in the region (6) shows that $\frac{1}{4} \leqq \rho_1$; unless $B_1$ is a line segment, that is to say, unless $B$ is the arc of a circle through 0, we have $\frac{1}{4} < \rho_1$.

If $\rho_1 = \rho_2$, each point of $B_1$ lies on or within the circle

$$|\zeta + a_2| = \frac{1}{2\rho_1},$$

and every line tangent to that circle cuts $B_1$. Then every point of that circle is a point of $B_1$, and $R_1$ consists of the exterior of the circle. Consequently $B_1$ is a circle, necessarily of radius unity, by the form of the mapping function $\zeta = 1/f(z)$; and we have $\rho_1 = \rho_2 = \frac{1}{2}$.

The fact that not every point of $B_1$ can lie interior to the circle $|\zeta + a_2| = 1$ and that if $B_1$ is not that circle points of $B_1$ lie exterior to the circle, shows that we have $\rho_1 < \frac{1}{2}$ unless $B_1$ is that circle, in which case $\rho_1 = \frac{1}{2}$. Hence if $B$ is a circle we have $\rho_1 = \frac{1}{2}$, otherwise $\rho_1 < \frac{1}{2}$.

We mention explicitly a result of some intrinsic interest:

COROLLARY. *Under the conditions of Theorem* 3, *no circle* $T_\rho$ *with* $\rho < \frac{1}{4}$ *can cut* $B$.

The fact that every $T_\infty$ cuts $B$ has been established previously.*

Although Theorem 3 has been established for a region $R$ which corresponds to a mapping function of form (1), it is clear that the requirements $f(0) = 0$, $f'(0) = 1$ may be ignored for the purposes of that theorem; the point of the extended $w$-plane which corresponds to $z = 0$, and which may be $w = \infty$, is denoted by 0.

* Walsh, this MONTHLY, vol. 42, 1935, pp. 1–17; Proceedings National Academy of Sciences, vol. 23, 1937, pp. 166–169.

**4. Numerical relations involving circles $T_\rho$.** Let us now proceed to determine which circles in the $w$-plane having the origin as center can be cut by a given $T_\rho$. We establish

THEOREM 4. *Under the conditions of Theorem* 1, *every* $T_\rho$ *cuts the circle* $|w| = r$ *if*

$$r \leqq \frac{2\rho}{1 + 2|a_2|\rho}; \tag{9}$$

*no* $T_\rho$ *cuts the circle* $|w| = r$ *if*

$$r > \frac{2\rho}{1 - 2|a_2|\rho}, \qquad 1 - 2|a_2|\rho > 0. \tag{10}$$

*In particular, if* $w = \infty$ *is not an interior point of* $R$, *it is sufficient for* (9) *if*

$$r \leqq \frac{2\rho}{1 + 4\rho}, \tag{11}$$

*and it is sufficient for* (10) *if*

$$r > \frac{2\rho}{1 - 4\rho}, \qquad 1 - 4\rho > 0. \tag{12}$$

*If* $w = \infty$ *is not an exterior point of* $R$, *we denote by* $M$ ($M = \infty$ *is not excluded*) *the greatest distance from* $w = 0$ *to a point of* $B$. *It is sufficient for* (9) *if*

$$r \leqq \frac{2M\rho}{M + 2(1 + 2M)\rho}, \tag{13}$$

*and it is sufficient for* (10) *if*

$$r > \frac{2M\rho}{M - 2(1 + 2M)\rho}, \qquad M - 2(1 + 2M)\rho > 0. \tag{14}$$

*If* $R$ *is not the extended plane cut along the arc of a circle through* $w = 0$, *the first sign* $>$ *in* (12) *and* (14) *may be replaced by* $\geqq$.

Under the conditions of Theorem 4, with (9) fulfilled, the circle in the $\zeta$-plane

$$|\zeta + a_2| = \frac{1}{2\rho} \tag{15}$$

is internally tangent to or is completely interior to the circle

$$|\zeta| = \frac{1}{r} \geqq |a_2| + \frac{1}{2\rho}. \tag{16}$$

Consequently every line tangent to the circle (15) must cut the circle (16).

That is to say, every $T_\rho$ must cut $|w| = r$ if (9) is fulfilled.

The circle

$$|\zeta| = \frac{1}{r} < \frac{1}{2\rho} - |a_2|, \qquad \frac{1}{2\rho} - |a_2| > 0, \tag{17}$$

lies interior to the circle (15), so no line tangent to (15) can cut the circle (17). That is to say, no $T_\rho$ can cut $|w| = r$ if (10) is fulfilled.

We remark also that if the sign $<$ is replaced by $=$ in (17), the corresponding circle can have no point exterior to the circle (15), so if the first sign $>$ in (10) is replaced by the sign $=$, no $T_\rho$ can cut the circle $|w| = r$ in more than one point. This remark applies similarly in connection with (12) and (14).

If $w = \infty$ is not an interior point of $R$, the point $\zeta = 0$ is not an interior point of $R_1$, and it follows from (5), that (11) and (12) are sufficient for (9) and (10), respectively.

If $w = \infty$ is not an exterior point of $R$, the point $\zeta = 0$ is not an exterior point of $R_1$; by virtue of the fact that $B_1$ lies in the circle (6) it follows that $|a_2|$ is not greater than 2 plus the distance from 0 to $B_1$, namely $1/M$. Consequently (13) and (14) are sufficient for (9) and (10), respectively.

If $w = \infty$ is not an interior point of $R$, the point $\zeta = 0$ is not an interior point of $R_1$, and if $B_1$ is a line segment it must be the segment of a line through 0; if $B_1$ is not such a line segment (that is to say, if $R$ is not the extended plane cut along the arc of a circle through $w = 0$), the equality sign may be omitted in (5) and hence may be inserted in the first of inequalities (12). If $w = \infty$ is not an exterior point of $R$, the point $\zeta = 0$ is not an exterior point of $R_1$. If $\zeta = 0$ is a point of $B_1$ and if $B_1$ is not a line segment we have $|a_2| < 2$. If $\zeta = 0$ is not a point of $B_1$ and if $B_1$ is not a line segment through $\zeta = 0$, either $B_1$ satisfies $|\zeta + a_2| < 2$, or the shortest distance $1/M$ from $\zeta = 0$ to $B_1$ does not lie along the line joining $\zeta = 0$ and $\zeta = -a_2$; in either case we have $|a_2| < 2 + 1/M$, and the equality sign may be inserted in the first inequality in (14).

**5. Applications concerning $T_\rho$.** It is a consequence of (10) that whenever we have

$$\rho < \frac{1}{2|a_2|}$$

the circle $T_\rho$ cannot cut $|w| = r$ provided $r$ is sufficiently large. Hence $T_\rho$ cannot be a straight line, and $A_\rho$ cannot have a point of inflection at the point $w = 0$. In particular, if we have $|a_2| \leqq 1$, any circle $C_\rho$ which lies interior to $|z| = 1$ corresponds to a curve $A_\rho$ which fails to have a point of inflection at $w = 0$. If the region $R$ is convex, the function which maps $|z| < 1$ onto $R$ with $z = 0$ corresponding to an arbitrary point $w_0$ of $R$ is of the form

$$w - w_0 = \mu[z + a_2 z^2 + a_3 z^3 + \cdots], \qquad \mu \neq 0,$$

with $|a_2| \leqq 1$. Consequently the image of every circle interior to $|z| = 1$ under a

specific map of $|z|<1$ onto $R$ can have no point of inflection in $R$, a theorem due to Study* for the case of circles $|z|=r<1$ and to Carathéodory† in the general case.

We formulate explicitly the

COROLLARY. *Under the conditions of Theorem* 4, *the curve $A_\rho$ has no point of inflection at $w=0$ if we have*

$$\rho < \frac{1}{2|a_2|};$$

*if $w=\infty$ is not an interior point of $R$ it is sufficient if we have*

$$\rho < \tfrac{1}{4};$$

*if $w=\infty$ is not an exterior point of $R$, and if $M$ denotes the greatest distance from $w=0$ to a point of $B$, it is sufficient if we have*

$$\rho < \frac{M}{2(1+2M)}.$$

We have thus far used, in addition to the obvious geometric properties of the situation in the $w$-plane, only inequalities (5) and (6). It is clear that various other results related to (5) and (6) that already occur in the literature and others that may be established in the future can be used to sharpen our results. For instance,‡ under the hypothesis of Theorem 4 let $R$ lie in the region $|w|<M$. Under the present circumstances we have the inequality§

$$|a_2| \leqq 2\left(1-\frac{1}{M}\right) \quad \text{or} \quad M \geqq \frac{2}{2-|a_2|}.$$

Consequently (9) and (10) are fulfilled if we have, respectively,

$$\begin{aligned} r &\leqq \frac{2M\rho}{M+4(M-1)\rho}, \\ r &> \frac{2M\rho}{M-4(M-1)\rho}, \qquad M-4(M-1)\rho > 0. \end{aligned} \tag{18}$$

In particular, no circle $T_\rho$ cuts $|w|=M$ provided we have either of the two inequalities

* Konforme Abbildung Einfach-Zusammenhängender Bereiche, Leipzig, 1913, p. 110.

† Mathematische Annalen, vol. 79, 1919, p. 402.

‡ Other instances are: (i) determination of the sharp lower bound for $\rho_2$ in connection with Theorem 3; (ii) determination of limits for $\rho_1$ and $\rho_2$ when $R$ lies interior to a given circle $|w|=M$; (iii) determination of limits for $\rho_1$ and $\rho_2$ when $B$ has no point interior to a given circle $|w|=M$; (iv) improvement of (6) and hence of Theorem 3 for convex $R$.

§ See Pick, Wiener Berichte, vol. 126, 1917, pp. 247–263.

Inspection of the situation in the $\zeta$-plane yields only the weaker inequality $|a_2|\leqq 2-1/M$.

$$\rho < \frac{M}{4M-2}$$

(a condition which implies the second of inequalities (18) for $r=M$), or

$$\rho < \frac{1}{2+|a_2|}$$

(a condition which implies (10) directly); it is of course sufficient if $\rho<\frac{1}{4}$.

**6. Properties of** $U_\nu$. We have studied the images in the $w$-plane of circles through the origin in the $z$-plane. We now proceed to consider the inverse problem. The results already established enable one to deduce results on the circle of curvature $U_\nu$ in the $z$-plane of the image in the $z$-plane of a circle of radius $\nu$ through the origin in the $w$-plane.

The inverse of (1) can be written

$$z = F(w) \equiv w - a_2w^2 + c_3w^3 + \cdots \tag{19}$$

which is precisely of form (1) with the sign of the second coefficient of the last member reversed. For convenience in reference we formulate the analog of Theorem 1, which is essentially merely Theorem 1 itself:

THEOREM 5. *Under the hypothesis of Theorem* 1 *let $D_\nu$ be a circle of radius $\nu$ in the w-plane which passes through the origin, and let $B_\nu$ be its image in the z-plane (or to be more exact, let $B_\nu$ be the image in the z-plane of that portion of $D_\nu$ which lies in $R$). Let $U_\nu$ denote the circle of curvature of $B_\nu$ at the origin in the z-plane. Then $U_\nu$ depends on $a_2$ but not on $a_k$ for $k>2$.*

We proceed next to the discussion of Theorem 4. The entire reasoning used in the proof of Theorem 4 is still valid if $a_2$ is replaced by $-a_2$; such substitution leaves unchanged the formal statement, so we have

THEOREM 6. *Under the conditions of Theorem* 4 *every $U_\nu$ cuts the circle $|z|=s$ if*

$$s \leqq \frac{2\nu}{1+2|a_2|\nu}; \tag{20}$$

*no $U_\nu$ cuts the circle $|z|=s$ if*

$$s > \frac{2\nu}{1-2|a_2|\nu}, \qquad 1-2|a_2|\nu > 0. \tag{21}$$

*In particular, if $w=\infty$ is not an interior point of $R$, it is sufficient for* (20) *if*

$$s \leqq \frac{2\nu}{1+4\nu}, \tag{22}$$

*and it is sufficient for* (21) *if*

$$s > \frac{2\nu}{1-4\nu}, \qquad 1-4\nu > 0. \tag{23}$$

*If $w=\infty$ is not an exterior point of $R$, we denote by $M$ ($M=\infty$ is not excluded) the greatest distance from $w=0$ to a point of $B$. It is sufficient for* (20) *if*

$$s \leqq \frac{2M\nu}{M+2(1+2M)\nu},$$

*and it is sufficient for* (21) *if*

$$s > \frac{2M\nu}{M-2(1+2M)\nu}, \qquad M-2(1+2M)\nu > 0.$$

Some special situations under Theorem 6 deserve comment. If $R$ is convex, we have $|a_2| \leqq 1$, and it follows from (20) that every $U_\infty$ cuts the circle $|z|=1$. The geometric properties are independent of the condition $f'(0)=1$, so it follows that *whenever $R$ is convex, each circle of curvature in the $z$-plane to the curve in the $z$-plane which is the image of a line segment interior to $R$ in the $w$-plane,—every such circle of curvature cuts the circle* $|z|=1$, a remark due to Carathéodory, *loc. cit.** Whether or not $R$ is convex, if $w=\infty$ is not an interior point of $R$ we have $|a_2| \leqq 2$, so it follows from (20) that every $U_\infty$ cuts the circle $|z|=\frac{1}{2}$. On the other hand if $w=\infty$ is not an interior point of $R$ it follows from (23) that no $U_\nu$ cuts $|z|=1$ if we have $\nu<\frac{1}{6}$.

The test analogous to (18), and similarly proved, is: *Let $R$ lie in the region $|w|<M$; then sufficient conditions for* (20) *and* (21) *respectively are*:

$$s \leqq \frac{2M\nu}{M+4(M-1)\nu},$$

$$s > \frac{2M\nu}{M-4(M-1)\nu}, \qquad M-4(M-1)\nu > 0.$$

**7. Applications concerning $U_\nu$.** A corollary of Theorem 6 is analogous to the corollary to Theorem 4:

COROLLARY. *Under the conditions of Theorem 4 no $U_\nu$ is orthogonal to* $|z|=1$ *provided we have*

$$\nu < \frac{1}{2|a_2|}. \tag{24}$$

*In particular if the diameter of $D_\nu$ is less than the distance from $w=0$ to $B$, this condition* (24) *is fulfilled.*

*If $w=\infty$ is not an interior point of $R$ it is sufficient if we have $\nu<\frac{1}{4}$.*

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* Even if $R$ is not convex, every $|z|=r<r_1<1$ for sufficiently small $r_1$ is transformed into a convex curve; see Theorem 8 below. Hence at every point of $|z|\leqq r_1$, a circle of curvature to the image in the $z$-plane of a line segment in $R$ must cut $|z|=r_1$; this remark also is due to Carathéodory.

Whether $R$ is convex or not, it is a consequence of Theorem 6 that every $U_\infty$ cuts $|z|=1$ provided $|a_2|\leqq 1$.

Of course $U_\nu$ is orthogonal to $|z|=1$ when and only when $U_\nu$ cuts every $|z|=s$. It follows from (21) that (24) is sufficient to ensure that $U_\nu$ shall not cut every $|z|=s$.

Let $m$ denote the distance in the $w$-plane from 0 to $B$. In the $\zeta$-plane as used in the proof of Theorem 4, the point set $B_1$ then lies in the closed region $|\zeta| \leqq 1/m$, and for every $\epsilon>0$ lies interior to $|\zeta|=\epsilon+1/m$. The transform in the $\zeta$-plane of some circumference $|z|=r<1$ lies also interior to $|\zeta|=\epsilon+1/m$; hence the center of gravity $\zeta=-a_2$ of that weighted transform also lies interior to $|\zeta|=\epsilon+1/m$. Consequently we have*

$$|a_2| \leqq \frac{1}{m}, \tag{25}$$

and the condition

$$2\nu < m$$

implies (24).

The last part of the corollary follows from (5).

We formulate another consequence of Theorem 6 as follows:

THEOREM 7. (i) *Under the conditions of Theorem* 4, *suppose*

$$2\nu = \frac{M}{\lambda}, \qquad \lambda > 0,$$

*where $M$ is defined as in Theorem* 4 *and where $w=\infty$ is not an interior point of $R$. Then every $U_\nu$ cuts the circle*

$$|z| = \frac{M}{\lambda + 2M - 2}; \tag{26}$$

*no $U_\nu$ cuts the circle $|z|=s$ if*

$$s > \frac{M}{\lambda - 2M + 2}, \qquad \lambda - 2M + 2 > 0. \tag{27}$$

(ii) *Under the conditions of Theorem* 4, *suppose*

$$2\nu = \frac{m}{\lambda}, \qquad \lambda > 0,$$

*where $m$ is the distance in the $w$-plane from $w=0$ to $B$. Then every $U_\nu$ cuts the circle*

$$|z| = \frac{m}{1+\lambda}; \tag{28}$$

*no $U_\nu$ cuts the circle $|z|=s$ if*

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* Walsh, Proceedings National Academy of Sciences, vol. 23, 1937, pp. 166–169.

$$s > \frac{m}{\lambda - 1}, \qquad \lambda > 1. \tag{29}$$

In connection with (18) we have used the inequality

$$|a_2| \leqq 2\left(1 - \frac{1}{M}\right).$$

Under the hypothesis of (i), the second member of (20) is not less than the second member of (26); the second member of the first inequality in (27) is not less than the second member of (21); then (i) is established.

Under the hypothesis of (ii), it follows from (25) that the second member of (20) is not less than the second member of (28); the second member of (29) is not less than the second member of (21); hence (ii) is established.

Some numerical cases of Theorem 7 are of interest. Thus if $2\nu \geqq M/2$, that is to say if the diameter of $D_\nu$ is at least half the greatest distance from $w=0$ to a point of $B$, and if $w=\infty$ is not an interior point of $R$, then every $U_\nu$ cuts $|z| = \frac{1}{2}$. If $2\nu \geqq m$, that is to say if the diameter of $D_\nu$ is at least the distance from $w=0$ to $B$, whether or not $w=\infty$ is an interior point of $R$, then every $U_\nu$ cuts $|z| = m/2$; if $w=\infty$ is not an interior point of $R$, and if $D_\nu$ cuts $B$ (or more generally if the diameter of $D_\nu$ is no less than the distance from 0 to $B$) we have $2\nu \geqq m \geqq \frac{1}{4}$, so every $U_\nu$ cuts $|z| = \frac{1}{8}$; a more favorable result is obtained by taking $2\nu \geqq \frac{1}{4}$, for then it follows from (22) that every $U_\nu$ cuts $|z| = \frac{1}{6}$. Also a part of the corollary to Theorem 6 follows from (ii).

**8. Circles not passing through origin.** Thus far we have obtained results on circles of curvature at the origin $w=0$ or $z=0$ of the images of circles. These results can be applied in the study of circles of curvature of such images at points other than the origin, by the aid of a transformation of the region $|z| < 1$ into itself. But of course the number $a_2$ that figures so prominently in our results must be adjusted to accord with the new configuration. In certain cases the number $a_2$ enters only implicitly by means of a standard inequality, in which case the new results may be of relatively simple form. Let us proceed to give several illustrations of this method.

LEMMA. *Let the circle* $|z-r_1| = r_1 < \frac{1}{2}$ *be the transform of the circle* $|z| = r < 1$ *under the transformation*

$$z_1 = \frac{z + r}{1 + rz} \tag{30}$$

*which leaves the region* $|z| < 1$ *invariant. Then we have*

$$r_1 = \frac{r}{1 + r^2}, \tag{31}$$

$$r = \frac{1 - \sqrt{1 - 4r_1^2}}{2r_1}. \tag{32}$$

The transformation (30) carries the points $z = -r$, 0, $r$ into the respective points $z_1 = 0$, $r$, $2r_1$, where $r_1$ is given by (31); hence (30) carries the circle $|z| = r$ into the circle $|z - r_1| = r_1$. Equation (32) is merely the solution of (31) for $r$, with the restriction $r < 1$.

The value $r_1 = \frac{1}{4}$ corresponds by means of (32) to the value $r = 2 - \sqrt{3} = 0.26+$. Thus we have from Theorem 3 and its corollary:

THEOREM 8. *Under a smooth conformal map of the region $|z| < 1$ onto a region $R$ of the extended plane with boundary $B$, no circle of curvature of the image of a circle $|z| = r < 2 - \sqrt{3}$ can cut $B$.*

*Unless $B$ is the arc of a circle through the image of $z = 0$, the requirement $r < 2 - \sqrt{3}$ can be replaced by $r \leqq 2 - \sqrt{3}$.*

*If $B$ is the arc of a circle through the image of $z = 0$, one and only one circle of curvature of the image of the circle $|z| = 2 - \sqrt{3}$ cuts $B$, and that in but a single point of $B$.*

A consequence of the first part of Theorem 8 is the well known fact that if $R$ does not contain the point at infinity the image of every circle $|z| = r < 2 - \sqrt{3}$ is convex; for no circle of curvature can cut $B$, hence no circle of curvature can pass through the point at infinity, hence the image of $|z| = r$ can have no point of inflection.

Parts of Theorem 7 are independent of the conditions $f(0) = 0$, $f'(0) = 1$. We shall prove:

THEOREM 9. *Let the region $|z| < 1$ be mapped smoothly onto the region $R$ with boundary $B$ of the $w$-plane by the function $w = f(z)$, analytic for $|z| < 1$. If $w_0$ is a point of $R$, with $w_0 = f(z_0)$, $|z_0| < 1$, if $D_\nu$ is a circle through $w_0$ whose diameter is at least half the greatest distance from $w_0$ to a point of $B$, if $D_\nu$ is transformed into a curve $B_\nu$ of the $z$-plane, and if $U_\nu$ is the circle of curvature of $B_\nu$ at $z_0$, then under a transformation of the form*

$$z_1 = \frac{z - \alpha}{1 - \bar{\alpha} z}$$

*which carries $U_\nu$ into a circle whose center is the origin $z_1 = 0$, $U_\nu$ corresponds to a circle of radius at least $2 - \sqrt{3}$.*

It follows from Theorem 7 that the radius of $U_\nu$ is at least $\frac{1}{4}$, if we have $f(0) = w_0$, and the value $r_1 = \frac{1}{4}$ in the lemma corresponds to $r = 2 - \sqrt{3}$.

THEOREM 10. *Let the region $|z| < 1$ be mapped smoothly onto the region $R$ with boundary $B$ of the $w$-plane, by the function $w = f(z)$, analytic for $|z| < 1$. If $w_0$ is a point of $R$, with $w_0 = f(z_0)$, $|z_0| < 1$, if $D_\nu$ is a circle through $w_0$ which cuts $B$ (or more generally whose diameter is at least as great as the distance from $w_0$ to $B$),*

*if the transform of $D_\nu$ is the curve $B_\nu$ of the z-plane, and if $U_\nu$ is the circle of curvature of $B_\nu$ at $z_0$, then under a transformation of the form*

$$z_1 = \frac{z - \alpha}{1 - \bar{\alpha} z}$$

*which carries $U_\nu$ into a circle whose center is the origin $z_1 = 0$, $U_\nu$ corresponds to a circle of radius at least* $6 - \sqrt{35}$.

It follows from Theorem 7, as we have already remarked, that the radius of $U_\nu$ is at least $1/12$, if we have $f(0) = w_0$; and the value $r_1 = 1/12$ in the lemma corresponds to $r = 6 - \sqrt{35}$.

**9. Invariant formulation.** Throughout the present paper we have studied the circles of curvature of the transforms of circles. It is of course true, as the reader may prove, that if two curves have the same circle of curvature at $z = 0$, then their images under the transformation (1) have the same circle of curvature at $w = 0$. This remark can be applied in connection with virtually all of the preceding results.

Thanks to this remark, such results as Theorems 8, 9, 10 are conveniently expressed in general form in terms of non-euclidean geometry, for this formulation is independent of the center of the circle in the $z$-plane. Thus Theorems 8, 9, 10 are the respective parts of

THEOREM 11. *Let $R$ be an arbitrary simply-connected region of the w-plane whose boundary $B$ has more than one point, and let $R$ be provided with the usual non-euclidean metric by a conformal map onto the region $|z| < 1$ of the z-plane.*

*If the non-euclidean curvature* of a curve at a point of $R$ is numerically greater than 4, the euclidean circle of curvature at that point cannot cut $B$.*

*If $w = \infty$ is not interior to $R$, and if the euclidean circle of curvature of a curve at a point $w_0$ of $R$ has as diameter at least half the greatest distance from $w_0$ to a point of $B$, then the non-euclidean curvature of the curve at $w_0$ is at most* 4.

*If $w = \infty$ is not interior to $R$, and if the euclidean circle of curvature of a curve at a point $w_0$ of $R$ has a diameter at least as great as the distance from $w_0$ to $B$, then the non-euclidean curvature of the curve at $w_0$ is at most* 12.

Various other results of the present paper are conveniently expressed in terms of non-euclidean geometry; for instance, it follows from Theorem 3 that under the conditions of Theorem 11 if the non-euclidean curvature of a curve at a point of $R$ is zero, then the euclidean circle of curvature at that point must cut $B$. Further similar formulations should present no difficulty to the reader.

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* Defined as in Carathéodory, Conformal Representation (Cambridge Tract No. 28), §44.

# MODIFIED SERIES

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**1. Introduction.** Certain of the most familiar series such as $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ and $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$ converge very slowly, so that it would be necessary to take a large number of terms to obtain even a rough approximation to their sums, say with an error not exceeding 1%. Means of increasing the rapidity of convergence of such series have interested mathematicians for a century or more. In many of the studies dealing with this problem the object has been to replace the given series by equivalent series that converge more rapidly. This paper attempts to make a contribution to the solution of the problem by calling attention to the notion of a "modified" series. The idea is simply this, that having added a certain number of terms, say $n$, of the given series, a better approximation may be obtained by adding an appropriate function of $n$. The procedure here set forth resembles the method of Kummer, in that the first step involves summing a small number of terms; its elementary nature and the simplicity of its formulation and application would seem to justify reference to it in early courses in analysis. The notion is applicable also to the corresponding problems as they arise in connection with infinite products and infinite continued fractions. These applications it is proposed to discuss in a later paper.

A simple illustration will serve to bring out the underlying ideas and clarify the distinction between "modifying" a series, in the sense here used, and replacing it by a new one.

**2. A modified series for log 2.** A recent paper on a means of increasing the rapidity of the convergence of certain series is one by Shohat* appearing in this MONTHLY in 1933. The author remarks that his method does not apply to the series for the logarithm of 2. For that reason this series may furnish a particularly interesting illustration for our purpose. The rule in this case may be strikingly formulated as follows: "Having taken any number of terms of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, pass over an equal number, and add in the next term, giving it, however, the sign opposite to that of the last term taken." The result is a much better approximation than would be obtained by including all the intervening terms and giving the last term its proper sign. Thus for example,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{11}=.6924,$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{11}=.7365,$$

while $\log 2=.693147$. This seems like a mere trick until the procedure is formulated in algebraic language.

* J. A. Shohat, On a certain transformation of infinite series, this MONTHLY, vol. 40, 1933, pp. 226–229. See also an article by the present author, Modified continued fractions for certain series, this MONTHLY, vol. 45, 1938, pp. 352–362, where numerous references are given.

Denote by $s_n$ the sum of the first $n$ terms of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$

and by $t_n$ the sum of the same terms with the addition of $(-1)^n/(2n+1)$. Then obviously $s_n$ and $t_n$ represent two sequences of numbers both converging to the same limit, since the difference between $s_n$ and $t_n$ is numerically $1/(2n+1)$, and this approaches zero as $n$ tends to infinity. But the difference between successive terms in the $s_n$ sequence is very different from that between successive values of $t_n$. For $s_n - s_{n-1} = (-1)^{n-1}/n$, while

$$\begin{aligned} t_n - t_{n-1} &= (-1)^{n-1}\frac{1}{n} + (-1)^n\frac{1}{2n+1} - (-1)^{n-1}\frac{1}{2n-1} \\ &= (-1)^n\frac{1}{n(2n-1)(2n+1)}. \end{aligned}$$

Thus successive values in the $t_n$ sequence, like those in the $s_n$ sequence, are on opposite sides of the limit, log 2, but the order of magnitude of the difference is smaller. This explains why $t_n$ is a better approximation than $s_n$ or even than $s_{2n+1}$.

But this result immediately suggests another step; we can readily make a new series out of the $t$ sequence. For

$$t_n = t_1 + \sum_{k=2}^{n}(t_k - t_{k-1}) = \frac{2}{3} + \sum_{k=2}^{n}(-1)^k\frac{1}{k(2k-1)(2k+1)}.$$

If we make the summation include $k=1$ and change the preceding numerical term accordingly, we may consider our original series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ as replaced by the more rapidly convergent one

$$1 + \sum_{n=1}^{\infty}(-1)^n\frac{1}{n(2n-1)(2n+1)}.$$

If, however, we retain only this end result and forget about the modified series, we shall have lost a valuable consideration, the modified series is better adapted to computation than the new one. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{11}$$

and

$$1 - \frac{1}{1\cdot 1\cdot 3} + \frac{1}{2\cdot 3\cdot 5} - \frac{1}{3\cdot 5\cdot 7} + \frac{1}{4\cdot 7\cdot 9} - \frac{1}{5\cdot 9\cdot 11}$$

yield the same result, but certainly the former may be more readily converted into a decimal than the latter.

**3. A second modified series for log 2.** We may now go a step further and modify this new series. If to the $n$th term of this series, which is $(-1)^n/[n(2n-1)(2n+1)]$, we add $(-1)^{n+1}/[8n(n^2+2)+6(2n^2+1)]$, thereby obtaining a modified series, and denote the modified $t_n$ by $u_n$, then as before $t_n$ and $u_n$ approach the same limit. But whereas the difference between $t_n$ and $t_{n-1}$ is numerically equal to $1/[n(2n-1)(2n+1)]$, the difference between $u_n$ and $u_{n-1}$ is

$$u_n - u_{n-1} = (-1)^n \frac{9}{n(4n^3 - 6n^2 + 8n - 3)(4n^3 + 6n^2 + 8n + 3)}.$$

The doubly modified series which we now have is

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} + (-1)^n \frac{1}{2n+1} + (-1)^{n+1} \frac{1}{8n(n^2+2) + 6(2n^2+1)}.$$

Using $n=5$ this yields the approximation to log 2,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{11} + \frac{1}{1386} = .6931457,$$

with an error of about 2 in the sixth decimal place. From $u_n$ a new series might be obtained and a third modification, and the process might be continued, but with modifications more and more complicated.

**4. A doubly modified series for $\pi/4$.** The result of applying the same process to a well known series for $\pi/4$ is the doubly modified series

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{2k-1} + (-1)^n \frac{1}{4n} + (-1)^{n+1} \frac{1}{4n(4n^2+5)},$$

which yields for $n=5$ the approximation 3.141587 for $\pi$.

**5. General formulation for modification.** The problem of finding modified series adapted to computation may be given the following general formulation. Given the convergent series $\sum_{n=1}^{\infty} a_n$, let $s_n$ denote the sum of the first $n$ terms. We seek a function of $n$ denoted by $b_n$, such that $\lim_{n\to\infty} b_n = 0$, and such that, if $t_n = s_n + b_n$, then $t_n - t_{n-1}$ is of a lower degree of magnitude than $s_n - s_{n-1}$. This may be expressed by the formula,

$$\lim_{n\to\infty} n^\alpha \left\{ 1 + \frac{b_n - b_{n-1}}{a_n} \right\} = K \neq 0$$

for some positive value of $\alpha$.

**6. The general term a rational function of $n$.** That series whose general terms are rational functions of the index may be "modified" in accordance with this condition is at once apparent. The possible effectiveness of such modification

in improving approximation may be seen by considering the two series $\sum_{n=1}^{\infty}(-1)^{n-1}/x_n$, where $x_n$ is a polynomial in $n$ of degree $\rho \geqq 1$, and $\sum_{n=1}^{\infty} 1/y_n$, where $y_n$ is a polynomial in $n$ of degree $\rho \geqq 2$.

In the first case we take for $b_n$ a rational fraction $(-1)^n u_n/v_n$, so chosen that the numerator of

$$\begin{aligned} t_n - t_{n-1} &= s_n - s_{n-1} + b_n - b_{n-1} \\ &= (-1)^{n-1}\frac{v_n v_{n-1} - x_n(u_n v_{n-1} + u_{n-1}v_n)}{x_n v_n v_{n-1}} \end{aligned}$$

shall be of low degree in $n$. In the first place the degree of $v_n$ must exceed that of $u_n$ by $\rho$. There is then a sufficient number of coefficients in $u_n$ and $v_n$ to reduce the degree of this numerator to $\rho - 1$, the degree of the denominator being at least $3\rho$.

In the second case $b_n$ is taken equal to the rational fraction $u_n/v_n$ and the numerator whose degree is to be made low is $v_n v_{n-1} + y_n(u_n v_{n-1} - u_{n-1}v_n)$, the degree of $v_n$ must exceed that of $u_n$ by $\rho - 1$, and the number of coefficients is sufficient to reduce the degree of this numerator to $\rho - 2$, while the degree of the denominator is at least $3\rho - 2$.

In either case, if in using the method of arbitrary coefficients the degree of $u_n$ is taken in succession $0, 1, 2, \cdots$, a sequence of modifications can be obtained which in some cases exhibits the simple law of formation of the continued fraction.

**7. The continued fraction as modification.** The way in which a continued fraction may appear as an effective modification is well illustrated by the series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$, with which we have dealt above. Let

$$\begin{aligned} v_n &= p_0 n^t + p_1 n^{t-1} + \cdots + p_t \\ u_n &= q_0 n^{t-1} + q_1 n^{t-2} + \cdots + q_{t-1}. \end{aligned}$$

Then, equating to zero the successive coefficients in $v_n v_{n-1} - n(u_n v_{n-1} + u_{n-1}v_n)$, we obtain a sequence of equations of the form

$$p_k - 2q_k = \text{a linear function of } p_{k-1}, \cdots, p_0.$$

For $t = 1, 2, 3$, and 4, these yield the modifications

$$(-1)^n\frac{1}{2n+1},$$

$$(-1)^n\frac{2n+1}{4n^2+4n+2} = (-1)^n\left\{\frac{1|}{|2n+1} + \frac{1|}{|2n+1}\right\},$$

$$(-1)^n\frac{4n^2+4n+5}{8n^3+12n^2+16n+6} = (-1)^n\left\{\frac{1|}{|2n+1} + \frac{1|}{|2n+1} + \frac{4|}{|2n+1}\right\},$$

$$(-1)^n \frac{8n^3 + 12n^2 + 32n + 14}{16n^4 + 32n^3 + 80n^2 + 64n + 24}$$

$$= (-1)^n \left\{ \frac{1|}{|2n+1} + \frac{1|}{|2n+1} + \frac{4|}{|2n+1} + \frac{9|}{|2n+1} \right\}.$$

The correctness of the surmise that succeeding modifications would follow the law of the continued fraction here readily suggested can be verified by mathematical induction.

Let $A_s(n)$ and $B_s(n)$ denote the numerator and the denominator of the $(s+1)$th convergent of the continued fraction $b_0 = 2n+1$, $a_s = s^2$, $b_s = 2n+1$. The proposed modification is then $(-1)^n/(A_s/B_s)$, and

$$t_n - t_{n-1} = (-1)^{n-1}\left\{\frac{1}{n} - \frac{B_s(n)}{A_s(n)} - \frac{B_s(n-1)}{A_s(n-1)}\right\}$$

$$= (-1)^{n-1}\frac{A_s(n)A_s(n-1) - nB_s(n)A_s(n-1) - nA_s(n)B_s(n-1)}{nA_s(n)A_s(n-1)}.$$

We denote the numerator of this fraction by $L_s$ and proceed to show that it is independent of $n$. Using the formulas

$$A_s = b_sA_{s-1} + a_sA_{s-2}, \qquad B_s = b_sB_{s-1} + a_sB_{s-2},$$

we obtain

$$L_s = (4n^2 - 1)L_{s-1} + s^4L_{s-2} + s^2[(2n+1)K_{s-1} + (2n-1)K'_{s-1}],$$

where

$$K_{s-1} = A_{s-1}(n)A_{s-2}(n-1) - nB_{s-1}(n)A_{s-2}(n-1) - nA_{s-1}(n)B_{s-2}(n-1),$$

and

$$K'_{s-1} = A_{s-1}(n-1)A_{s-2}(n) - nB_{s-1}(n-1)A_{s-2}(n) - nA_{s-1}(n-1)B_{s-2}(n).$$

Applying the reduction formulas twice to $K_{s-1}$ and $K'_{s-1}$, we have

$$K_{s-1} = (2n+1)L_{s-2} + (s-1)^2(2n-1)L_{s-3} + (s-1)^2(s-2)^2K_{s-3},$$

$$K'_{s-1} = (2n-1)L_{s-2} + (s-1)^2(2n+1)L_{s-3} + (s-1)^2(s-2)^2K'_{s-3}.$$

By means of these relations $L_s$ is expressed in terms of $L_{s-1}$, $L_{s-2}$, $L_{s-3}$ and $(2n+1)K_{s-3}+(2n-1)K'_{s-3}$. If from this equation and

$$L_{s-2} = (4n^2 - 1)L_{s-3} + (s-2)^4L_{s-4} + (s-2)^2[(2n+1)K_{s-3} + (2n-1)K'_{s-3}]$$

the expression $(2n+1)K_{s-3}+(2n-1)K'_{s-3}$ is eliminated, we have finally

$$L_s = (4n^2 - 1)L_{s-1} + (2s^4 - 2s^3 + 3s^2 + 8n^2s^2)\,L_{s-2}$$
$$+ s^2(s-1)^2(4n^2 - 1)L_{s-3} - s^2(s-1)^2(s-2)^4L_{s-4}.$$

Now $L_0 = -1$, $L_1 = 2^2$, $L_2 = -6^2$, $L_3 = 24^2$, $L_4 = -120^2$. Assuming then that

$L_{s-1}=(-1)^s(s!)^2$, $L_{s-2}=(-1)^{s-1}[(s-1)!]^2$, $L_{s-3}=(-1)^{s-2}[(s-2)!]^2$, $L_{s-4}=(-1)^{s-3}[(s-3)!]^2$, it follows from the relation derived above that $L_s=(-1)^{s+1}[(s+1)!]^2$. This completes the proof.

Since

$$t_n - t_{n-1} = \frac{(-1)^{n-1}L_s}{nA_s(n)A_s(n-1)},$$

for fixed $s$, is alternately positive and negative as $n$ increases, the difference between $t_n$ and its limit $t_\infty$, the sum of the series, is numerically less than this quantity. Inasmuch as, further,

$$A_s(0) = (s+1)!, \qquad A_s(1) > (s+2)!,$$

it follows that $|t_n - t_\infty| < 1/(s+2)$, and, for fixed $n \geqq 1$, approaches zero as $s$ increases. The conclusion can then be drawn that the remainder of the series after $n$ terms is represented by the infinite continued fraction, or in other words that we may write

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = \sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k} + (-1)^n \bigg/ \left\{2n+1+\underset{s=1}{\overset{\infty}{K}} \frac{s^2}{2n+1}\right\}.$$

For $n=5$, $s=5$ the resulting approximation is .693,147,184,9 with an error of 4 in the ninth decimal place.

**8. Continued fraction modification of $\sum 1/n^2$.** Illustrating the case of the non-alternating series, the continued fraction whose convergents furnish effective modification of the series $\sum_{n=1}^{\infty} 1/n^2$ is given by $b_0=2n+1$, $a_s=s^4$, $b_s=(2s+1)(2n+1)$ and the modifications are $2/[A_s(n)/B_s(n)]$. In this case $t_n-t_{n-1}$ is the fraction

$$\frac{A_s(n)A_s(n-1) + 2n^2B_s(n)A_s(n-1) - 2n^2A_s(n)B_s(n-1)}{n^2A_s(n)A_s(n-1)},$$

of which the numerator $L_s=(-1)^{s+1}[(s+1)!]^4$. It may now be shown that for $s \geqq 1$, $A_s(0)=[(s+1)!]^2$, $A_s(1) > \frac{1}{2}[(s+2)!]^2$. Hence $|t_n - t_{n-1}| < 2/[n^2(s+2)^2]$. Here, however, for fixed $s$ this difference between $t_n$ and $t_{n-1}$ has a constant sign, and all that we can say is that $t_n$ differs from its limit $t_\infty$ by the sum of such differences, that is, by less than the product of the sum of the convergent series $\sum 1/n^2$ and $2/(s+2)^2$, a quantity which approaches zero with increasing $s$. Hence again we are justified in saying that the infinite continued fraction represents the remainder of the series after $n$ terms:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=1}^{n} \frac{1}{k^2} + 2 \bigg/ \left\{2n+1+\underset{s=1}{\overset{\infty}{K}} \frac{s^4}{(2s+1)(2n+1)}\right\}.$$

This for $n=5$, $s=3$ yields 1.644,934,064,4, an approximation with an error of 3 in the ninth decimal place.

**9. Two more general series.** To make the formal work of §7, including the mathematical induction, immediately applicable to the more general series

$$\sum_{n=1}^{\infty}(-1)^{n-1}/(nx+y)=\frac{1}{x}\sum_{n=1}^{\infty}(-1)^{n-1}/(n+y/x)$$

all that is necessary is to put $n+y/x=n'$ and express $u_n$ and $v_n$ as polynomials in $n'$. The result is

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{nx+y}=\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{kx+y}$$
$$+(-1)^n\bigg/\left\{(2n+1)x+2y+\underset{s=1}{\overset{\infty}{K}}\frac{s^2x^2}{(2n+1)x+2y}\right\}.$$

In a similar manner the modification of $\sum 1/n^2$ may be generalized to yield

$$\sum_{n=1}^{\infty}\frac{1}{(nx+y)^2}=\sum_{k=1}^{n}\frac{1}{(kx+y)^2}$$
$$+\frac{2}{x}\bigg/\left\{(2n+1)x+2y+\underset{s=1}{\overset{\infty}{K}}\frac{s^4x^2}{(2s+1)[(2n+1)x+2y]}\right\}.$$

The series discussed are, perhaps, a sufficient illustration of the method to be used in "modification" and the results which may be obtained by it. It yields comparable results, somewhat less simple, for the series $\sum(-1)^{n-1}/n^2$ and $\sum 1/n^3$. The continued fractions cited belong to the class treated by Perron in Chapter 11 of his book,* those in which $a_s$ and $b_s$ are polynomials in $s$, and, in fact, to that subclass in which the degree of $a_s$ exceeds twice that of $b_s$ by 2.

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* Die Lehre von den Kettenbrüchen, 2d edition.

# A NEW CLASS OF ORTHOGONAL POLYNOMIALS*

DUNHAM JACKSON, University of Minnesota

**1. Introduction.** The ordinary construction of a system of orthogonal polynomials may be varied by requiring the polynomials to satisfy one or more linear homogeneous auxiliary conditions. The simplicity or difficulty of the resulting theory naturally depends on the choice of these conditions. If the requirement is for example that each polynomial vanish for $x=0$, the interval being $(-1, 1)$, the result at the outset is merely a superficial reformulation of the theory of polynomials orthogonal on the interval with respect to $x^2$ as weight function and without supplementary condition. Other conditions, however, lead to results possessing a greater degree of novelty.

This paper is concerned primarily with a single illustration, the definition and properties of a system of polynomials orthogonal on the interval $(-1, 1)$ with unit weight function, each polynomial $p_n(x)$ of the system being subject to the boundary condition

$$p_n(1) = p_n(-1). \tag{1}$$

The discussion is similar in general outline to that of an orthogonal system of the usual type,† even with regard to some of the finer properties of convergence, but shows differences in detail which seem worthy of attention. A second illustration is treated briefly in the concluding section.

**2. Definition of the orthogonal system.** The monomials $1, x^2, x^4, \cdots$, of even degree, and the binomials

$$x^3 - x,\ x^5 - x^3 = (x^3 - x)x^2,\ x^7 - x^5 = (x^3 - x)x^4, \cdots, \tag{2}$$

of odd degree, satisfy the condition of taking on the same value at both ends of the interval. It is clear that this condition is not satisfied by any polynomial of the first degree; applied to the expression $a_1x+a_0$, it gives immediately $a_1=0$.

Let $p_0(x), p_2(x), p_4(x), \cdots$ be the normalized Legendre polynomials of even degree. Each satisfies the condition (1). Application of the Schmidt process‡ to the sequence (2) is equivalent to the construction of the even polynomials of the orthogonal system corresponding to the weight function $(x^3-x)^2$. Let these be denoted by $q_0(x), q_2(x), q_4(x), \cdots$, when normalized so that

$$\int_{-1}^{1} (x^3 - x)^2 [q_k(x)]^2 dx = 1,$$

* Presented to the Rocky Mountain Section of the Association, by invitation, at Laramie, Wyoming, April 29, 1939 (with the exception of §5, the substance of which was presented to the Minnesota Section at Northfield, Minnesota, May 13, 1939).

† See *e.g.*, D. Jackson, Series of orthogonal polynomials, Annals of Mathematics, (2), vol. 34, 1933, pp. 527–545.

‡ See *e.g.*, D. Jackson, The Theory of Approximation, American Mathematical Society Colloquium Publications, vol. 11, New York, 1930, pp. 89–90, 95.

the relation of orthogonality being expressed by

$$\int_{-1}^{1} (x^3 - x)^2 q_k(x) q_l(x) dx = 0, \qquad k \neq l,$$

and let $p_n(x) = (x^3 - x)q_{n-3}(x)$ for $n = 3, 5, 7, \cdots$. Polynomials $p_n(x)$ satisfying (1) have thus been defined for all non-negative integral values of $n$ except $n = 1$. They are even or odd, *i.e.*, consist exclusively of even powers or of odd powers, according as $n$ is even or odd. Since* any two even $p$'s are orthogonal to each other and any two odd $p$'s are orthogonal to each other by construction, while any even polynomial whatever is orthogonal to any odd polynomial over the interval $(-1, 1)$, the relation

$$\int_{-1}^{1} p_m(x) p_n(x) dx = 0$$

is satisfied whenever $m \neq n$, while

$$\int_{-1}^{1} [p_n(x)]^2 dx = 1$$

for each value of $n$, and the $p$'s constitute the desired orthogonal system.

Any polynomial $P_n(x)$ of the $n$th degree for which $P_n(1) = P_n(-1)$ can be expressed as a linear combination of $p_0, p_2, p_3, \cdots, p_n$ with constant coefficients. For terms of degrees $n, n-1, \cdots, 2$ can be removed successively by subtraction of suitable multiples of $p_n, p_{n-1}, \cdots, p_2$, and the remainder, being a polynomial of degree not higher than the first which takes on the same value at both ends of the interval, must be a constant.

**3. Recursion formula and Christoffel-Darboux identity.** For each $p_n(x)$ the product $x^2 p_n(x)$, as a polynomial of degree $n+2$ satisfying the boundary condition, is expressible in the form

$$x^2 p_n(x) = c_{n,n+2} p_{n+2}(x) + c_{n,n+1} p_{n+1}(x) + \cdots.$$

On multiplication of the identity by $p_k(x)$, $k \leqq n+2$, and integration it appears that

$$c_{nk} = \int_{-1}^{1} x^2 p_n(x) p_k(x) dx, \tag{3}$$

by reason of the orthogonality and normalization of the $p$'s. But $p_n(x)$ is orthogonal to every polynomial of lower degree which satisfies the boundary condition, and $x^2 p_k(x)$ is such a polynomial for $k < n-2$. So $c_{nk} = 0$ if $k < n-2$. Furthermore, the integrand contains only odd powers of $x$ if $k = n \pm 1$, and $c_{nk}$

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* A similar observation has been used effectively in more complicated situations by Mr. Fulton Koehler; see Bulletin of the American Mathematical Society, vol. 43, 1937, p. 772, Abstract No. 402.

therefore vanishes for these values of $k$ also. Consequently

$$x^2 p_n(x) = c_{n,n+2} p_{n+2}(x) + c_{nn} p_n(x) + c_{n,n-2} p_{n-2}(x). \tag{4}$$

This identity corresponds to the recursion formula satisfied by ordinary orthogonal polynomials. If $p_1(x)$, $p_{-1}(x)$, $p_{-2}(x)$ are defined as identically zero, the relation (4) holds formally, with $c_{nk}$ given by (3), for $n=3, 1, 0$, as well as for $n \geqq 4$ and $n=2$.

Analogous reasoning shows that

$$(x^3 - x) p_n(x) = b_{n,n+3} p_{n+3}(x) + b_{n,n+1} p_{n+1}(x) + b_{n,n-1} p_{n-1}(x) + b_{n,n-3} p_{n-3}(x),$$

with

$$b_{nk} = \int_{-1}^{1} (x^3 - x) p_n(x) p_k(x) dx,$$

and, on occasion, $p_{-3}(x) = 0$. More generally, if $\pi(x) = Ax^2 + B(x^3 - x)$,

$$\pi(x) p_n(x) = \sum_{k=n-3}^{n+3} a_{nk} p_k(x), \qquad a_{nk} = \int_{-1}^{1} \pi(x) p_n(x) p_k(x) dx. \tag{5}$$

Still more general relations of similar character could obviously be written down, but are less important for subsequent application.

Let (4) be multiplied by $p_n(t)$, and subtracted from the corresponding identity with $t$ and $x$ interchanged. If the combination

$$p_k(t) p_l(x) - p_k(x) p_l(t)$$

is denoted by $p_{k,l}(t, x)$, it is seen that

$$(t^2 - x^2) p_n(t) p_n(x) = c_{n,n+2} p_{n+2,n}(t, x) - c_{n,n-2} p_{n,n-2}(t, x).$$

Let this relation be written with $n$ replaced by $n-1$, $n-2$, $\cdots$, and let the various identities be added to give an expression for $(t^2 - x^2) K_n(t, x)$, where

$$K_n(t, x) = \sum_{k=0}^{n} p_k(t) p_k(x).$$

With the use of the fact that $c_{kl} = c_{lk}$ and the fact that $p_{k,l}(t, x)$ is identically zero whenever one of the subscripts is negative, it is found that

$$(t^2 - x^2) K_n(t, x) = c_{n+2,n} p_{n+2,n}(t, x) + c_{n+1,n-1} p_{n+1,n-1}(t, x). \tag{6}$$

This has the character of the Christoffel-Darboux identity,* which is familiar in connection with other orthogonal systems.

If an "arbitrary" function $f(x)$ on the interval $(-1, 1)$ is formally expanded in a series of the form $\sum A_k p_k(x)$, with

$$A_k = \int_{-1}^{1} f(t) p_k(t) dt,$$

* See, *e.g.*, Series of orthogonal polynomials, *loc. cit.*, pp. 529–530.

the partial sum

$$s_n(x) = \sum_{k=0}^{n} A_k p_k(x)$$

is represented by

$$s_n(x) = \int_{-1}^{1} f(t) K_n(t, x) dt.$$

The identity for $K_n(t, x)$ is therefore fundamental, as in other, similar cases, for the theory of convergence of the series.

Repetition of the above calculation with (5) in place of (4) gives at first

$$[\pi(t) - \pi(x)] p_n(t) p_n(x) = \sum_{k=n-3}^{n+3} a_{kn} p_{kn}(t, x), \tag{7}$$

in connection with which it is to be noted that for all values of the subscripts $a_{kl} = a_{lk}$, $p_{kl}(t, x) = -p_{lk}(t, x)$, $p_{kk}(t, x) = 0$. When the summation is performed over the values $n, n-1, n-2, \cdots$ of the index in the left-hand member of (7), each term $a_{kl} p_{kl}(t, x)$ which does not vanish identically is cancelled by a term $a_{kl} p_{lk}(t, x)$, except for those terms in which one subscript of the pair exceeds $n$. There are just six such pairs of subscripts: $(n+3, n)$, $(n+2, n)$, $(n+1, n)$, $(n+2, n-1)$, $(n+1, n-1)$, and $(n+1, n-2)$. It follows that

$$[\pi(t) - \pi(x)] K_n(t, x) = \sum a_{kl} p_{kl}(t, x), \tag{8}$$

the summation being extended over these six subscript pairs.

**4. Convergence.** In the case of systems of orthogonal polynomials without auxiliary condition, convergence theorems of considerable generality can be readily proved* if the normalized polynomials are bounded for all values of $n$ at the point where convergence is to be demonstrated, or are uniformly bounded throughout an arbitrary closed interval interior to the interval of orthogonality. Similarly procedures are effective here. *The polynomials $p_n(x)$ possess the requisite properties of boundedness.* This is recognized immediately in the case of the $p$'s of even degree, since they are merely normalized Legendre polynomials. The $q$'s mentioned above were introduced as the even orthogonal polynomials for weight function $(x^3 - x)^2$. The products $xq_{2i}(x)$ are the normalized orthogonal polynomials of odd degree for weight $(x^2-1)^2$. As such they are uniformly bounded† throughout any closed interval interior to $(-1, 1)$. Hence the polynomials

* See Series of orthogonal polynomials, *loc. cit.*, pp. 531–538; D. Jackson, Orthogonal trigonometric sums, Annals of Mathematics, (2), vol. 34, 1933, pp. 799–814; pp. 807–812; see also D. Jackson, On the degree of convergence of the development of a continuous function according to Legendre's polynomials, Transactions of the American Mathematical Society, vol. 13, 1912, pp. 305–318; J. A. Shohat, On the development of continuous functions in series of Tchebycheff polynomials, Transactions of the American Mathematical Society, vol. 27, 1925, pp. 537–550; pp. 541–545.

† See, *e.g.*, Series of orthogonal polynomials, *loc. cit.*, pp. 534–535.

$$p_{2j+1}(x) = (x^2 - 1)xq_{2j-2}(x)$$

are similarly bounded.

If a discussion of convergence is based on the Christoffel-Darboux identity in the form (6), restrictions must be placed on the function $f(t)$ not only in the neighborhood of the point $t=x$ at which convergence is to be proved, but also near the point $t=-x$. The specialization in the latter neighborhood, which is irrelevant to the essence of the problem, can be removed by supplementary use of the identity with $x^3-x$ in place of $x^2$, or, what amounts to the same thing, by taking the identity at the outset in the more general form (8), with arbitrary values of $A$ and $B$ in the definition of $\pi(x)$. The conclusions with regard to convergence in the interior of the interval $(-1, 1)$ are then of the same degree of generality as in the corresponding discussion of ordinary systems of orthogonal polynomials. It is to be remarked in particular that $f(x)$ is not subjected to any boundary condition; on the other hand, the proof of convergence does not apply at the ends of the interval.

If it be objected that there has been so far not much more than a combination of two orthogonal systems, each of which by itself comes within the scope of theories already well known, one immediate answer is that a complete foundation has been laid for a substantial generalization, without appreciable increase in the difficulty of the reasoning, by the introduction of a weight function in the definition of orthogonality. The generalization extends to the discussion of convergence, if the weight function is for example a polynomial* positive throughout the interior of the interval and taking on the same value at both end points.

**5. Modified boundary condition.** Similar results are obtained, with slight differences which are perhaps worth noting, if the boundary condition (1) is replaced by the requirement that

$$p_n(-1) = -p_n(1)$$

for each $n$. This condition is of course not satisfied by any non-vanishing constant; it is satisfied by $x, x^3, x^5, \cdots$, and by $x^2-1, (x^2-1)x^2, (x^2-1)x^4, \cdots$. The orthogonal system consists of the normalized Legendre polynomials of odd degree and the polynomials $(x^2-1)q_{n-2}(x)$, $n=2, 4, \cdots$, if $q_0, q_2, \cdots$ are the normalized orthogonal polynomials of even degree for weight $(x^2-1)^2$. The polynomials $p_n(x)$ thus defined with degrees $1, 2, 3, \cdots$ are uniformly bounded in any closed interval interior to $(-1, 1)$. It is true again that the product of any $p_n(x)$ by $x^2$ or by $x^3-x$ is a polynomial satisfying the boundary condition. Recursion formulas, Christoffel-Darboux identities, and convergence theorems are then derived in essentially the same way as in the case previously considered.

* *Cf.* preceding footnote.

## QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

*The department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

### WHAT IS AN ASYMPTOTE?

C. B. READ, University of Wichita

Students in the writer's classes have brought out the fact that depending upon the particular text or reference book used, the line $y=0$ is or is not an asymptote of the curve $y=e^{-cx}\sin kx$. Rarely will two books give identically the same definition, but an analysis of approximately one hundred text and reference books shows essentially three types of definition of an asymptote.

(A) If the distance from a fixed line to a point $P$ on a curve approaches zero as a limit as $P$ moves toward infinity, then the fixed line is an asymptote of the curve. (Variation: An asymptote is a tangent whose point of contact is at infinity, the tangent itself not lying entirely at infinity.)

(B) If the distance from a fixed straight line and a point on a curve continually decreases and becomes indefinitely small, as at least one of the coördinates of the point becomes indefinitely large, the fixed line is called an asymptote of the curve.

(C) If $P$ moves to an infinite distance along a branch of the curve, and the tangent at $P$ tends to a limiting position, this limiting position is called an asymptote to the curve.

If one excludes those books which restrict the definition of an asymptote to certain types of curves, such as algebraic curves, about 45 per cent of the definitions found were of type (A), about 20 per cent of type (B), and the balance of type (C). It seems rather unusual in the field of elementary mathematics to encounter contradictory definitions; yet certainly, if we consider the curves of damped vibration mentioned above, the line $y=0$ is or is not at asymptote depending upon which definition is used.

*Editorial Note.* Professor Read's note raises the question as to just what properties an asymptote should have. We certainly want the curve to approach the line (in the sense of definition A). Do we also wish that (D) the slope of the curve approach the slope of the line? And (E) the curvature of the curve approach the curvature of the line? Probably (E) is too restrictive, but we might well require (D), since it is weaker than (C). We observe that (C) implies (D), which in turn implies (A), but not conversely, as is shown by the examples

$$y = \frac{1}{x}\sin x^2, \tag{1}$$

$$y = \frac{1}{x}\sin x, \tag{2}$$

$$y = \frac{1}{x^2} \sin x. \tag{3}$$

None of these satisfy (B), in which the emphasis is on the *continual decrease* of the distance, that is, the limit is approached from one side. This idea can also be incorporated in definitions (C), (D), and (E).

There seems, then, to be a considerable amount of latitude on the choice of a definition of an asymptote, and one would have to consider the uses to which an asymptote is put before making a selection. It should be noticed that for algebraic curves all the above definitions are equivalent.—R. J. W.

### THE BEST (?) FORMULA FOR COMPUTING $\pi$ TO A THOUSAND PLACES

J. P. Ballantine, University of Washington

In the December 1938 issue of this Monthly, D. H. Lehmer gave a very comprehensive list of formulas for computing $\pi$. He rightly chose formulas (23) and (32) as the best self checking pair, with (18) a good substitute for (23).

I shall give some suggestions on the use of formula (32) which will strengthen Lehmer's conclusion.

If [18], which means arc tan 1/18, and [57] are evaluated by the Gregory series, their measures 0.7966 and 0.5695, are correctly stated by Lehmer. However, Euler also developed a formula for $[x]$, namely

$$\text{arc cot } x = x\left(\frac{1}{S} + \frac{2}{3S^2} + \frac{2\cdot 4}{3\cdot 5S^3} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7S^4} + \cdots\right),$$

where $S=x^2+1$. Lehmer's remarks about the best way of computing the terms of the Gregory series apply equally well to the Euler series, because

$$u_{n+1}(x) = \frac{2nu_n(x)}{(2n+1)S},$$

and it is apparent that for machine computation one series is about as good as the other. The Euler series has the slight advantage that all the terms are positive.

Now, to return to formula (32), we have seen that nothing can be lost by evaluating [18] and [57] by the Euler series instead of the Gregory series. In fact there is a great saving, due to a lucky accident, for we have

$$[18] = 18\left(\frac{1}{325} + \frac{2}{3\cdot 325^2} + \frac{2\cdot 4}{3\cdot 5\cdot 325^3} + \cdots\right),$$

$$[57] = 57\left(\frac{1}{3250} + \frac{2}{3\cdot 3250^2} + \frac{2\cdot 4}{3\cdot 5\cdot 3250^3} + \cdots\right).$$

Thus the successive terms of the series for [57] can be obtained from those of [18] by shifting the decimal point. In fact, they do not even have to be copied, because they can be added on the bias.

Thus, the measure of formula (32) is reduced from 1.7866 to 1.2171 by the practical elimination of [57]. This leaves formulas (32) and (23) about tied for first place, and certainly making the best self checking pair, with Machin's formula (18) close behind. Moreover, all the other low measure formulas mentioned in Lehmer's paper are found on closer examination to have higher measure.

Before once and for all eliminating formula (18), one important feature should be mentioned. Suppose it is desired to compute both $\pi$ and $M=\log_{10}e$ $=.43429\cdots$ to 1000 decimal places. Then formula (18) is better than (23), because

$$\log\frac{3}{2}=2\left[\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\cdots\right],$$

and the terms of the series are the same as of [5]. Furthermore, log 120/119 will be useful to find log 17.

For the readers convenience, I restate formulas (18), (23), and (32) discussed here, with revised measures:

$$(18)\qquad [1]=4[5]-[239]\quad (1.3511).$$

$$(23)\qquad [1]=8[10]-[239]-4[515]\quad (1.2892).$$

$$(32)\qquad [1]=12[18]+8[57]-5[239]\quad (1.2171).$$

In summary, the best formulas now known for computing $\pi$ to a large number of places are:

$$\frac{[18]}{18}=\left(\frac{1}{325}+\frac{2}{3\cdot 325^2}+\frac{2\cdot 4}{3\cdot 5\cdot 325^3}+\cdots\right),$$

$$\frac{[57]}{57}=\left(\frac{1}{3250}+\frac{2}{3\cdot 3250^2}+\frac{2\cdot 4}{3\cdot 5\cdot 3250^3}+\cdots\right),$$

$$[239]=\left(\frac{1}{239}-\frac{1}{3\cdot 239^3}+\frac{1}{5\cdot 239^5}-\cdots\right),$$

$$\pi=864\,\frac{[18]}{18}+1824\,\frac{[57]}{57}-5[239].$$

For checking, we may use

$$[10]=\frac{1}{10}-\frac{1}{3\cdot 10^3}+\frac{1}{5\cdot 10^5}\cdots,$$

$$[515]=\frac{1}{515}-\frac{1}{3\cdot 515^3}+\frac{1}{5\cdot 515^5}\cdots,$$

$$\pi=32[10]-4[239]-16[515].$$

*Editorial Note.* The following observation is due to Dr. J. B. Rosser. The labor of computing the reciprocals of the first thousand integers can be very much expedited by the use of either the table given in vol. ii, pp. 412–434, of Gauss's Werke or the much more comprehensive table given in *A Table of the Circles arising from the Division of a Unit or any other Whole Number by all the Integers from 1 to 1024*, by Henry Goodwyn, London, 1823. (A description of this table is given by Glaisher, Proceedings of the Cambridge Philosophical Society, 1878, 3, p. 185.) For this reason, the measure of [10] should probably be taken as much less than 0.5; possibly 0.2 or 0.1.—R. J. W.

## A NOTE ON THE "PROBLÈME DES RENCONTRES"

MAURICE FRÉCHET, University of Paris

In the March 1939 issue of this MONTHLY there appeared an interesting note by I. Kaplansky on a generalization of the famous *problème des rencontres*. Calling $P_{[r]}$ the probability of exactly $r$ *rencontres*, and $P_r$ the probability of at least $r$ *rencontres*, it may be said that Kaplansky's result consists in expressing $P_{[0]}$ symbolically. It would be interesting to see whether his proof may be extended to the computation of $P_{[r]}$ and $P_r$ for $r=1, 2, \cdots$.

In the meantime, it may interest the readers of the MONTHLY to hear that through a different method, starting with the computation of some other probabilities generally associated with the same problem, it is possible to prove that

$$P_{[r]} = F(a_1, p_1) \cdots F(a_n, p_n)\Psi_r(0),$$
$$P_r = 1 + F(a_1, p_1) \cdots F(a_n, p_n)\Theta_r(0),^*$$

where $F(a, p)$ is the operator defined by Kaplansky and

$$\Psi_r(x) = (-1)^r \frac{x!}{m!} \frac{(m-x)(m-x-1)\cdots(m-x-r+1)}{r!},$$

$$\Theta_r(x) = (-1)^r \frac{x!}{m!} \frac{(m-x-1)(m-x-2)\cdots(m-x-r+1)}{(r-1)!},$$

where $m = a_1 + a_2 + \cdots + a_n$. The proof will be given in a booklet, *Les probabilités des systèmes d'événements compatibles et dépendants*, to appear next year in the series, *Exposés d'Analyse générale*, Hermann, Paris.

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* My thanks are due to Mr. Kaplansky for pointing out a slip in my manuscript concerning this formula.

## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department at the Mathematical Association of America, 513 West 116th St. New York, N. Y., and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Coordinate Geometry*. By L. P. Eisenhart. Boston, Ginn and Co., 1939. 11+298 pages. $2.50.

*Engineering Descriptive Geometry*. The Direct Method for Students, Draftsmen, Architects, and Engineers. By C. E. Rowe. New York, D. Van Nostrand Co., 1939. 8+299 pages. $2.50.

*Theory of Evaluation*. By J. Dewey. (International Encyclopedia of Unified Sciences, vol. 2, no. 4.) Chicago University Press, 1939. 7+67 pages. $1.00.

*Notice sur les Fondaments de la Géomêtrie*. By J. Malengreau. Brussels, A. De Boeck, 1939. 48 pages.

*Ist es wahr dass* $2\times 2=4$ *ist*? Zweiter Band: Von den Kriterien der Wahrheit. By Fred Bon. Leipzig, 1939. 83 pages.

*Ist es wahr dass* $2\times 2=4$ *ist*? Dritter Band: Von den mathematischen Grundbegriffen. By Fred Bon. Leipzig, 1939. 80 pages.

*Business Arithmetic for College Students*. By William S. Schlauch. New York, F. S. Crofts and Co., 1939. 7+299 pages. $2.80.

*Differentialgeometrie der Kurven und Flächen und Tensorrechnung*. Autorisierte Ubersetzung von Dr. Max Pinl. By Hlavaty, Vaclav. Groningen-Batavia, P. Noordhoff N. V., 1939. 11+569 pages. fl 14.00; Geb. fl 15.50.

### REVIEWS

*Seis Conferencias*. By J. Barinaga, Madrid, 1938. 79 pages (including card-board cover title). ("Six Conferences on Mathematics" published by the "Junta for the Amplification of Scientific Studies and Investigations.")

This work consists of six popular lectures given in Madrid, between July 1932 and February 1938. The subjects treated are as follows: the concept of the solvability of equations in the evolution of algebra; the concept of metrical space; Albert Lista as mathematician; truth in pure and applied mathematics; aptitude for mathematics; and concept of measure.

It is interesting to note the varied types of audiences in this war period in Madrid to whom such scientific lectures appealed, including the teachers in normal schools, those preparing to teach science, and the "popular university." The authorities cited are, in general, those given in our popular expositions given by competent mathematicians. In Spain works designed for the elementary teachers set a high standard as to what constitutes "authority" in such publications.

An assertion is made (p. 13) that "from 1543 to 1770 (Lagrange) no effective progress was made in the algebraic resolution of equations." It is hardly worth

while to enumerate the host of notables, such as Tschirnhausen and Euler, who contributed to the "algebraic resolution of equations" in this period; of this time also, Vieta, whose name does not seem to appear in the American *Source Book in Mathematics*, is quite certainly the most fundamental contributor of all in this field, as well as one of the most fundamental in analytic geometry and even the calculus.

L. C. KARPINSKI

*Wahrscheinlichkeitsrechnung für Nichtmathematiker*. By Karl Dörge, with the aid of Hans Klein. Berlin, Walter de Gruyter and Co., 1939. 113 pages.

Dörge has written an excellent little book, introducing probability to "non-mathematicians"; it grew, he says, out of lectures to statisticians. The book avoids the use of calculus, restricting prerequisites to topics "taught in every secondary school." Yet the author warns: "Complete understanding of the book demands the necessary measure of intensive mental cooperation. Without that, no such relatively profound mathematical knowledge can be gained as is contained in the law of large numbers."

After a brief introduction, wherein discussion of the birth register's tale of the sex-ratio hints at the idea of probability, the first chapter gives the necessary concepts on sequences and their limits. Leniency relegates to the appendix most proofs, such as that of the uniqueness of the limit.

Probability appears in the second chapter, as limit of relative frequency; it is an attribute of a direction for experiment (Versuchsvorschrift). That this limit may not exist is clearly recognized, and emphasized by an example. Yet, for the sake of application, the hypothesis is set up that a particular result has such a limit if an experiment is repeated. A second hypothesis is that physically symmetrical results are also statistically symmetrical; that, if physics leads us to deny preference to one result over another, the two have the same probability. The dicta of physics themselves of course result from induction acting on observations. This hypothesis may, then, mean that, if particular experiences and general confidence in the world's regularity—perhaps with the intervention of deductive reasoning—lead us to expect two probabilities to be equal, they will be equal. (Thus we may expect a die to show two faces with equal frequency, not because we have counted the results with just this die, but because of century-old observations on falling bodies, centers of gravity, and so on—it is still an inference from one less-than-certain statement to another.) On the other hand the hypothesis may mean: the universe obeys fixed laws, even though we can never be certain what they are; if those laws give no preference as between two results, their probabilities are equal. In neither case, in the end, are we really sure of equal probability. Dörge does not trouble his readers' consciences with such worries; he says, "in many cases physicists believe that they can decide that none of the outcomes considered is to be preferred to another." In fact, he seems to think such worries at times superfluous—for he speaks of "such parts of physics as have firmly established theories." A third empirical hypoth-

esis states that physically independent directions for experiment are also statistically independent. These considerations come at the end of a chapter treating, simply and adequately, the elementary manners of combining probabilities to obtain other probabilities—for instance, the general and special multiplication theorems.

"Series of events" is the title of the third chapter—such convenient series as have complete independence of their members. Permutation and combination formulas are developed at leisure, and so the binomial coefficients find their usual application. Next comes mathematical expectation, with applications to such gambling games as insurance and roulette; the gambler's ruin (even if the bank plays fair) is neatly prophesied.

The final, most substantial chapter bears the title "Mean value, variance and law of large numbers." Avoiding continuous probability distributions (which would have brought in calculus) Dörge defines the measurement of a magnitude as an experiment whose results constitute a complete system of finitely many mutually exclusive members. The symbol is $\mu[\mathfrak{M}(x)]$, $x$ being a chance variable, $\mathfrak{M}$ its fixation by measurement, $\mu$ the mean value. Clearly the "magnitude" measured need not be the attribute of anything in the physical world—a fact which the use of a noun rather disguises.

The climax of the treatise is a justification of the use of statistics for prediction. Variance ("Streuungsquadrat" for Dörge) is used as the measure of inexactness of a measurement—its inexactness, of course, merely as representation of the limit of averages, not as representation of a physical reality. Steps toward the law of large numbers are the decrease of the variance as more data are averaged and the Tshebysheff inequality (Gauss and Tshebysheff are the only persons mentioned in the whole book). Thus there results the law of large numbers in this form: given $\epsilon>0$, the probability, $w_{a,k}(\epsilon)$, that the discrepancy between $\mu[\mathfrak{M}(x)]$ and the average of $k$ measurements exceed $\epsilon$ in absolute value approaches zero as $k$ becomes infinite. More precisely: if the probability of each admissible value of $x$—and hence its mean value and variance—be known, then $w_{a,k}(\epsilon)<C/K$, where $C=\sigma^2[\mathfrak{M}(x)]/\epsilon^2$. If merely the extreme values of $x$, $x_1$ and $x_n$, are known, the corresponding, weaker, inequality is $w_{a,k}(\epsilon)\leqq(x_n-x_1)^2/(k\epsilon^2)$. Dörge does not, however, mention that the experimenter can but rarely be sure that he has reached the extreme values of $x$, and that even this inequality must be used cautiously.

This is, then, a book which presents only a few results, but chooses important ones, and discusses them carefully, clearly, and with unusual penetration.

E. S. ALLEN

*Addenda to Rara Arithmetica*. By David Eugene Smith. Ginn and Company, 1939, Boston and London. First edition. 10+52 pages. \$2.00.

As indicated by the title, this pamphlet is a supplement to the *Rara Arithmetica* by the same author and publisher, first issued in 1908. The purpose and value of the *Addenda* can be made clear only by reference to the character

of the latter publication. The original work is a most significant contribution to the history of mathematics treating specifically of the evolution of arithmetic during the sixteenth century. Its extensive illustrations and reproductions of rare mathematical books are taken from the George A. Plimpton collection now in possession of the Low Memorial Library of Columbia University. In fact, it presents in its bibliographical scope a complete illustrated catalogue of the Plimpton collection of mathematical publications between the dates 1472 and 1601. It was eminently suitable to make these books the core of the *Rara Arithmetica* because, after the dispersion of the Libri, Boncompagni, and the De Morgan libraries, no other collector has made so exhaustive an accumulation. Of the four hundred fifty arithmetics known to have been printed in the sixteenth century and of sufficient importance to have appeared in two or more editions, the Plimpton library lacked less than two dozen in 1908. Besides this source material of over four hundred original copies of printed arithmetics, Professor Smith used in the preparation of the *Rara Arithmetica* the results of his own exhaustive researches, thus extending its scope to a list of five hundred fifty titles which, including the various editions, makes a catalogue of nearly twelve hundred books. In order to make the *Rara* exhaustive as well an authoritative, the author drew upon not only the original sources consulted by himself but also upon the studies of others—for example, the bibliographies of Graesse, Hain-Copinger, the lists of Riccardi and Murhard, and the catalogues of Libri, the Bibliotheque Nationale, and the British Museum.

With this background it is now possible briefly to state the significance of the *Addenda* in relation to the *Rara*. Its fifty-two pages are devoted to descriptions of the additional books and further editions of the same books as noted in the original volume that have come to notice during the thirty-one years since the publication of the *Rara*. There are one hundred thirty-four items keyed to the pages of the parent work, of which only seven refer to mere typographical errors, omissions and textual changes. Some thirty new titles of works have come to light in the interval between the dates of the two publications, and these are identified and described in the *Addenda*. During this period there have been discovered not only publications hitherto unknown, but also rare editions already on record and catalogued in the *Rara*. Copies of many of these have been procured and are now in the Plimpton library. The *Addenda* reproduces parts of twenty such copies. For example:

Puerbach, Opus Algorithmic—8 leaves. 15.3×21.4 cm. (copy of title page is given).

Conradus Noricus, Commentatio Arithmetice—First edition. Leipzig, 1503.

Rabbi Elias Misrachi, Arithmetic, 1534. This book is printed in Hebrew with a parallel commentary by Schreckenfuchs.

Anonymous, Arithmetique, Lyons 1594. Rare copy of a book arranged in meter; the only known book of its kind appearing before the 17th century.

Approximately one hundred items identifying further editions of books listed in the *Rara* are given their chronological place in that work. For example:

Other editions of "tractatus" of Albert of Saxony. Padua, 1477 and 1487; Paris, 1485; Rouen, 1493, 1494, and 1500; Venice, 1476 and 1494; Trent, 1475.

Seventy editions of Anianus before 1600 (34 being before 1501) with or without the added name of Sacrobosco.

An edition of Landshut (Lanzut), Cracow, 1538.

Add to the editions of Johann Albert's Rechenbuechlein: Wittenberg, 1534, 1541, 1553, 1556; Frankfort, 1541 (colophon 1542).

Recorde's "Ground of Arts" carried into the 17th century. The following have come to the author's attention: 16th century, 1540, 1561, 1570, 1575, 1583; 17th century, 1600, 1607, 1610, 1648, 1652, 1654, 1658, 1662, and ten others.

Many sources, including such bibliographies as those of Hain-Copinger, Proctor, and Clebs, and such catalogues as those of Tregaskis, Goldschmidt, Sotheran, Quaritch, Stechert, Rosenthal, and Maggs have been searched and supplemented by personal investigations of private and public collections both in Europe and America.

It thus appears that the *Addenda*, taking the same form of presentation as the *Rara*, contains a large amount of technical information; in fact, it is an exhaustive summary of the findings of research during the last thirty years in the history of sixteenth-century arithmetic. It will serve all scholars in this field, particularly lovers and collectors of rare books, cataloguers, bibliographers, and dealers interested in early publications.

L. L. JACKSON

*Coordinate Geometry*. By L. P. Eisenhart. Boston, Ginn and Company, 1939. 12+298 pp. $2.50.

The contents of this book were used as a text-book in manuscript for two years at Princeton University and subjected to frequent criticisms from both students and teachers before being published.

An unusual feature of the procedure is the presentation of linear geometry of three dimensions immediately after the corresponding parts of plane geometry have been given. The essential point of view is that of the directed line, related to the sign of the coefficient of $y$ in the plane, and of $z$ in space; associated with the right-hand side if the line and plane do not contain these terms, respectively.

The style is brief and direct; it requires close cooperation on the part of a reader. Previous instruction in determinants is so varied that the book plays safe by introducing the subject from the beginning, as if it were a tool peculiar to coördinate geometry, but the concepts are naturally broadened to provide a good working knowledge of the elements of the subject. In the plane there is a short chapter on oblique coördinates and one on polar coördinates, but little use is made of either. Similarly for cylindrical coördinates in space. At the end of the chapter on linear equations follows an excellent treatment of the circle and sphere, respectively, which provides an inclusive discussion in a small space.

Throughout the book, the concept is first introduced algebraically, and later

applied to geometry, making little use of intuition, and not attempting to develop a rigorously logical system; but at the end of the book is appended an enumeration and a historical description of Hilbert's axioms, followed by an explanation of the steps necessary to apply coördinate geometry to the same treatment to show that any problem of euclidean plane geometry can be solved (or be shown to be insolvable) by methods of coördinate geometry.

The geometry of the quadratic equation treats the subject as a whole, and features the various conic sections as incidents. Tangents and normals, poles and polars, conjugate diameters, *etc.* are all discussed adequately without any knowledge of the calculus, and only a very direct and natural use of any limiting process.

A very important feature of the book is the frequent and generous lists of exercises. When a new idea is first mentioned, it is carefully and fully explained but when that has been done the same idea may repeatedly bob up again among the exercises, not infrequently with rather novel settings.

The press-work and the proof-reading are both excellent.

VIRGIL SNYDER

*An Introduction to Modern Geometry.* By L. S. Shively. New York, John Wiley & Sons; London, Chapman & Hall, 1939. 11+167 pp. 122 figures. $2.00.

The book is designed for those students of mathematics who wish to become teachers in the high schools, but who do not include a course in projective geometry in their mathematical studies; and also for general culture.

No knowledge of analytic geometry nor the calculus is presupposed, nor any use made of imaginary elements in algebra. Very few bibliographical or historical notes are included, and those supplied are associated frequently with less important topics.

The first five chapters are concerned with metrical particularizations of general theorems of one-dimensional projective geometry, as are also the chapters on cross-ratio and on involution. Three chapters are given respectively to coaxial circles, inversion, and poles and polars, restricted to the circle. The chapter on construction by ruler and compass deals in detail with the Mascheroni theorem, but is sketchy in the discussion of the division of a circle, on account of the limited algebraic prerequisites. A final chapter is devoted to a considerable collection of problems which are not particular cases of more general theorems of projective geometry.

Each chapter is supplied with a goodly number of exercises, and at the end of the volume a hundred more are added. These are well selected; they serve as a measure of the thoroughness with which the text was mastered.

Within the rather narrow limits assigned as to previous courses assumed, the book is well written, and should stimulate further interest in the subject. The press-work and the proof-reading are excellent. There is appended a list of books on the subject, and a good index.

VIRGIL SNYDER

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.*

## AN ALIGNMENT CHART FOR THE QUADRATIC EQUATION*

L. R. FORD, Armour Institute of Technology

The accompanying nomographic chart, here much reduced, was made for distribution at the 1939 "Open House" of the Armour Institute of Technology. The actual drawing was made by Stephen Kroll, a student. So many requests have been made for the theory which underlies the chart that it seems desirable to write it out in detail.

**1. Scales.** Examples of scales are the foot rule with marks to indicate inches and fractions thereof, the marks on a thermometer to denote degrees, the minute marks around the face of a clock, and the marks on a protractor. Essentially a scale is a curve expressed in parametric form with the values of the parameter marked at intervals along the curve.

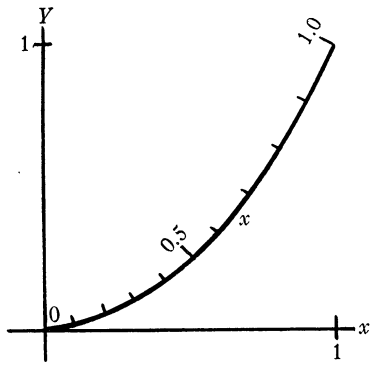



FIG. 1.

We use rectangular coördinates in the plane with $X$ and $Y$ as variables. Consider, for example,

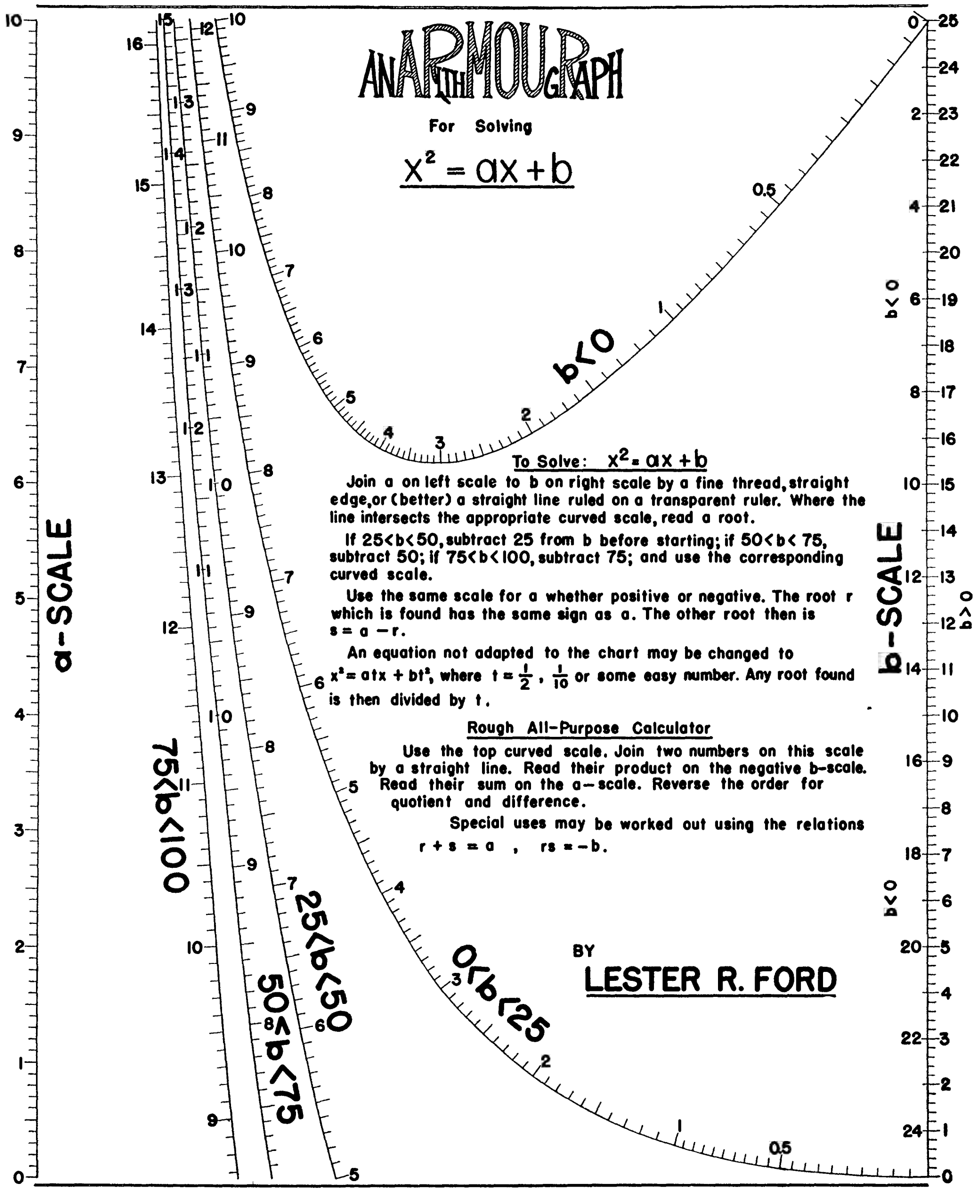
$$X = x, \qquad Y = x^2.$$

This is a parabola, $Y = X^2$. Let us mark on the curve a set of values of the parameter $x$, as $x = 0, 0.1, 0.2, \cdots, 0.9, 1$. This gives us the scale in Figure 1. Conversely, if such a scale is given then the $X$ and $Y$ of a point on the curve depend upon (that is, are functions of) the scale number at the point.

**2. The alignment chart.** Suppose we have three curves in parametric form

$$X = X_1(x), \qquad X = X_2(y), \qquad X = X_3(z),$$
$$Y = Y_1(x), \qquad Y = Y_2(y), \qquad Y = Y_3(z).$$

* The material in this article was presented before the Men's Mathematics Club of the Chicago and Metropolitan Area, April 21, 1939.




NOMOGRAPHIC CHART FOR SOLVING QUADRATICS.

Three points, one from each scale, lie on a straight line, if and only if

$$(1)\qquad \begin{vmatrix} X_1(x), & Y_1(x), & 1 \\ X_2(y), & Y_2(y), & 1 \\ X_3(z), & Y_3(z), & 1 \end{vmatrix} = 0.$$

We use the chart to solve (1) for one parameter when the other two are given. The line joining a point bearing a given $x$ with a point bearing a given $y$ will have at its intersection with the third scale a $z$ to satisfy (1).

It will be observed that each variable in (1) is restricted to a single row. If we wish to construct an alignment chart to solve an equation in three variables, our first task is to write it in determinant form with each variable appearing in a single row. It is not always possible to do this. There is the further requirement that a column of 1's appear, but this is easily managed.

**3. The quadratic equation.** We propose to solve

$$(2)\qquad x^2 = ax + b.$$

Here the variables are $x$, $a$, and $b$. We write (2) without difficulty in the form

$$(3)\qquad \begin{vmatrix} x^2, & x, & 1 \\ a, & 1, & 0 \\ b, & 0, & 1 \end{vmatrix} = 0.$$

Let us now multiply through by a non-vanishing determinant whose elements are constants

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Applying the usual rule for the multiplication of two determinants, we get

$$\begin{vmatrix} a_1x^2 + a_2x + a_3, & b_1x^2 + b_2x + b_3, & c_1x^2 + c_2x + c_3 \\ a_1a + a_2, & b_1a + b_2, & c_1a + c_2, \\ a_1b + a_3, & b_1b + b_3, & c_1b + c_3 \end{vmatrix} = 0.$$

Dividing each row by its element in the third column we get the required 1's there. We have then the curves in terms of the parameters $x$, $a$, $b$ respectively,

$$C_x:\quad X = \frac{a_1x^2 + a_2x + a_3}{c_1x^2 + c_2x + c_3}, \qquad Y = \frac{b_1x^2 + b_2x + b_3}{c_1x^2 + c_2x + c_3},$$

$$L_a:\quad X = \frac{a_1a + a_2}{c_1a + c_2}, \qquad Y = \frac{b_1a + b_2}{c_1a + c_2},$$

$$L_b:\quad X = \frac{a_1b + a_3}{c_1b + c_3}, \qquad Y = \frac{b_1b + b_3}{c_1b + c_3}.$$

Here $C_x$ is a conic and $L_a$, $L_b$ are straight lines. There is much variety possible owing to the constants at our disposal, but these curves have the general form shown in Figure 2. Here $P$ is the point $X = a_1/c_1$, $Y = b_1/c_1$, which is on all the curves for the values $x = \infty$, $a = \infty$, $b = \infty$. $Q$ is $X = a_3/c_3$, $Y = b_3/c_3$, which lies on $C_x$ and $L_b$ for $x = 0$, $b = 0$. That $L_a$, $C_x$ are tangent at $P$ is shown readily. Differentiating, we find the slopes of $C_x$ and $L_a$ to be

$$\frac{dY}{dX} = \frac{(b_1c_2 - b_2c_1)x^2 + 2(b_1c_3 - b_3c_1)x + b_2c_3 - b_3c_2}{(a_1c_2 - a_2c_1)x^2 + 2(a_1c_3 - a_3c_1)x + a_2c_3 - a_3c_2},$$

$$\frac{dY}{dX} = \frac{b_1c_2 - b_2c_1}{a_1c_2 - a_2c_1},$$

and the former is equal to the latter when $x = \infty$.

Owing to the constants at our disposal the chart may be put into many forms. We may require that $L_a$, $L_b$ be the coördinate axes, that $C_x$ be a circle,* and so on.

In our chart we shall require that $L_a$ and $L_b$ be the parallel lines

$$X_a = 0, \qquad X_b = 1.$$

This gives immediately

$$a_1 = a_2 = 0, \qquad c_1 = 0, \qquad a_3 = c_3 = \lambda, \qquad \text{say.}$$

Make the $a$-scale have zero at the origin by setting $b_2 = 0$. Put $b_1 = c_2 = 1$ and $b_3 = \beta$. Our equations now take the form

$$\begin{aligned} C_x:&\quad X = \frac{\lambda}{x+\lambda}, &\quad Y &= \frac{x^2+\beta}{x+\lambda}, \\ L_a:&\quad X = 0 &\quad Y &= a, \\ L_b:&\quad X = 1, &\quad Y &= \frac{b+\beta}{\lambda}. \end{aligned}$$

In the accompanying chart the units are chosen so that $X$ may run from 0 to 1 and $Y$ from 0 to 10 on the page. We take $\lambda = 5/2$, so that 25 $b$-units appear.

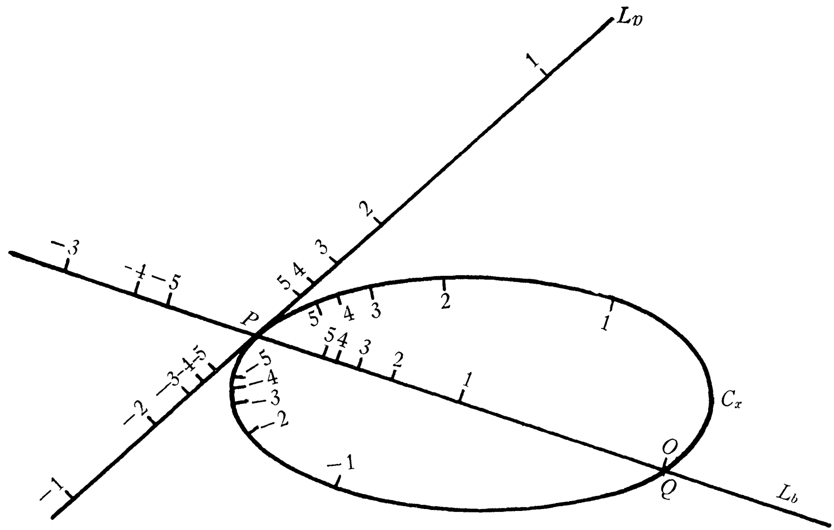



FIG. 2.

If $\beta = 0$ we get the $b$-scale on the right and the hyperbola $C_x$ through the lower right-hand corner of the figure. If $\beta = 25$ we get the hyperbola $C_x$ at the top; $b = 0$ appears at the top of the $b$-scale and negative values of $b$ run down the scale. For $\beta = -25, -50, -75$, the $b$'s that appear on the figure are those at the right, increased by 25, 50 or 75. The hyperbolas for these three cases are drawn on the chart. Since the equations of $L_a$ are independent of $\beta$, we use the same $a$-scale throughout. The object of breaking the scales in this way is to increase the range of the scales without a corresponding increase in the dimensions of the chart or a decrease in its accuracy.

**4. Remarks.** We have in these few paragraphs the essentials of the theory of nomography. The determinant (1) is the basic notion of the theory. The use of the determinant (4) as a multiplier provides the transformation theory. We thus achieve the results of the general collineation in a very elementary and comparatively painless way.

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* See Whittaker and Robinson, Calculus of Observation, p. 129, where both these requirements are met.

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND H. S. M. COXETER

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems. (Please note change of editorship.)

### PROBLEMS FOR SOLUTION

E 387. *Proposed by Thorold Gosset, Cambridge, England.*

A rabbit runs straight at a constant speed. A dog (not directly behind him) runs on an ever-changing course directly towards him, at a constant (greater) speed, until he catches him. Show that the distance traversed by the dog is the same as it would have been if he had run straight half-way to the rabbit's initial position and then straight to the point of capture.

E 388. *Proposed by V. Thébault, Le Mans, France.*

On the lateral surface of any right prism, find the length of the shortest route from end to end of one lateral edge, winding $n$ times round the prism on the way.

E 389. *Proposed by C. W. Trigg, Los Angeles City College.*

Show that there is but one five-digit integer whose last three digits are alike and whose square contains no duplicate digits.

E 390. *Proposed by H. T. R. Aude, Colgate University.*

Two freshmen work on a problem. At one point in their progress they compare their calculations and find that they have the same number $N$ which is greater than 1. They agree to four significant figures, which are all they use. Thereupon one takes the cube root, while the other freshman divides by 3. Again they compare and find that their results agree—reading from the left—in the first, second, and fourth figures, but differ by 1 in the third figure. Find the number $N$.

### SOLUTIONS

E 349 [1938, 629]. *Proposed by R. H. Bardell, University of Wisconsin.*

Show that for all positive integral values of $k$,

$$\sum_{n=0}^{k} (-1)^n C_n^k/(2n+1) = \frac{2^k \cdot k!}{(2k+1)(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}.$$

*Solution by T. C. Fry, Bell Telephone Laboratories, New York.*

Introduce powers of $x$ into the expression, so as to create a Taylor's series

$$f(x) = \sum_{n=0}^{k} (-1)^n C_n^k x^{2n+1}/(2n+1),$$

which can be differentiated term by term to give

$$\frac{df}{dx} = \sum_{n=0}^{k} C_n^k (-x^2)^n = (1-x^2)^k.$$

The desired sum is obviously

$$f(1) = \int_0^1 (1-x^2)^k dx = \int_0^{\pi/2} \cos^{2k+1} y\, dy = \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\frac{2}{3},$$

(a familiar special case of the beta function).

Also solved by A. G. Clark, V. W. Graham, J. A. Greenwood, H. D. Larsen, E. P. Starke, and the proposer.

E 350 [1938, 629]. *Proposed by V. Thébault, Le Mans, France.*

If from the feet of the bisectors of the interior angles of a triangle perpendiculars erected to the respective sides are concurrent, prove that the triangle is isosceles.

I. *Solution by C. W. Trigg, Los Angeles City College.*

The bisector of an interior angle of a triangle divides the opposite side into segments proportional to the adjacent sides. If perpendiculars are dropped from a point to the sides of the triangle, then the sums of the squares of the alternate segments of the sides are equal. Hence in a triangle with sides $a$, $b$, $c$, if the perpendiculars to the sides at the feet of the internal angle-bisectors are concurrent,

$$\left(\frac{ab}{b+c}\right)^2+\left(\frac{bc}{c+a}\right)^2+\left(\frac{ca}{a+b}\right)^2=\left(\frac{ca}{b+c}\right)^2+\left(\frac{ab}{c+a}\right)^2+\left(\frac{bc}{a+b}\right)^2.$$

Cancelling $b+c$, *etc.*, we deduce

$$a^2(b-c)/(b+c) + b^2(c-a)/(c+a) + c^2(a-b)/(a+b) = 0,$$

and then

$$(b-c)(c-a)(a-b)(a+b+c)^2 = 0.$$

Since at least one of the first three factors must vanish, the triangle is isosceles.

II. *Solution by G. A. Yanosik, New York University.*

Let the triangle be $ABC$. Using trilinear coördinates, let the side $BC$ be $\alpha=0$. The interior bisector of angle $A$ is $\beta-\gamma=0$, and it meets $BC$ at the point $(0, 1, 1)$. The line perpendicular to $\alpha=0$ at $(0, 1, 1)$ is

$$L\alpha + M\beta + N\gamma = 0,$$

where $L-N\cos B-M\cos C=0$ and $M+N=0$; *i.e.*, it is

$$(\cos B - \cos C)\alpha - \beta + \gamma = 0.$$

Three such lines are concurrent if and only if

$$\begin{vmatrix} \cos B - \cos C & -1 & 1 \\ 1 & \cos C - \cos A & -1 \\ -1 & 1 & \cos A - \cos B \end{vmatrix} = 0;$$

*i.e.*, if and only if

$$(\cos B - \cos C)(\cos C - \cos A)(\cos A - \cos B) = 0.$$

Also solved by V. W. Graham, D. L. MacKay, and E. P. Starke.

*Note.* MacKay (whose solution resembles Trigg's) remarks that the same conclusion follows if we take symmedians instead of angle-bisectors, the condition then being

$$a^2(b^2 - c^2)/(b^2 + c^2) + b^2(c^2 - a^2)/(c^2 + a^2) + c^2(a^2 - b^2)/(a^2 + b^2) = 0.$$

E 351 [1938, 629]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Determine $b$ as a function of $a$ such that 4 may be the sum of the squares of the roots of the equation $x^3+ax^2+bx=2a$, and discuss the equation, considering $a$ as a variable parameter.

*Solution by E. P. Starke, Rutgers University.*

Let $\sum x_1$ be the sum of the roots, $\sum x_1^2$ the sum of their squares, and $\sum x_1x_2$ the sum of their products by twos. Then $\sum x_1^2 = (\sum x_1)^2 - 2\sum x_1x_2$ and we have $4=(-a)^2-2b$ or

$$(1) \qquad b = (a^2 - 4)/2.$$

Thus the given cubic becomes

$$(2) \qquad 2x^3 + 2ax^2 + (a^2 - 4)x - 4a = 0.$$

If we set the $a$-discriminant of (2) equal to the square of a rational number, we have the values of $x$ corresponding to rational $a$. But the discriminant is $16-4x^4$, and $16-4x^4=s^2$ is not possible in rational numbers all different from zero. (See Carmichael, *Diophantine Analysis*, p. 19.) Thus, apart from zero (corresponding to $x=0$) there is no rational value of $a$ for which (2) is reducible.

The $x$-discriminant of (2) is $-4(a^6+24a^4+224a^2-128)$, which vanishes for two real values of $a$; namely,

$$a = \pm 2(h + k - 2)^{1/2} \qquad \text{where} \qquad h^3,\ k^3 = 7 \pm (11/3)^{3/2},$$

or approximately $a=\pm 0.7345$. Thus the given equation has all roots real for $-2(h+k-2)^{1/2} \leqq a \leqq 2(h+k-2)^{1/2}$, and one real root and two conjugate imaginary roots for other values of $a$.

Also solved by C. W. Trigg.

E 352 [1938, 691]. *Proposed by F. A. Alfieri, New York, N. Y.*

Show that the product of the six segments, cut by the interior angle-bisectors from the sides of any rational triangle, must be the square of a rational number.

*Solution by S. H. Gunn, University of Alabama.*

Let the sides of the triangle be denoted by $a$, $b$, $c$. Then since the bisectors of the angles divide the opposite sides into segments proportional to the adjacent sides, the six segments are $ac/(b+c)$, $ab/(b+c)$, $ba/(c+a)$, $bc/(c+a)$, $cb/(a+b)$, $ca/(a+b)$, and their product is obviously the square of

$$a^2b^2c^2/(b + c)(c + a)(a + b),$$

which is rational if $a$, $b$, $c$ are rational.

Also solved by H. T. R. Aude, W. E. Buker, W. B. Clarke, Fred Discepoli, William Douglas, V. W. Graham, E. R. Heinemann, L. M. Kelly, Herman Levy, D. L. MacKay, K. B. Patterson, Herbert Tate, C. W. Trigg, and the proposer.

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3924. *Proposed by M. T. Bird, Utah State Agricultural College.*

Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{b \cos x \sinh y + \sinh 2y}{b \sin x \cosh y + \sin 2x}.$$

3925. *Proposed by V. Thêbault, Le Mans, France.*

The triangle $ABC$ is right angled at $A$, and has the inscribed and escribed circles $(I)$, $(I_a)$, $(I_b)$, $(I_c)$ with the radii $r$, $r_a$, $r_b$, $r_c$. The parallel to $BC$ through $I_a$, the center of $(I_a)$, cuts $AB$ in $N$ and $AC$ in $M$; the orthogonal projections of $M$ and $N$ on $BC$ are $P$ and $Q$. Show that: (1) $MQ-MN=r$, $MQ-MP=(BC)^2/2r$; (2) the circumcircle $(\omega)$ of rectangle $MPQN$ is tangent to $(I_a)$, $(I_b)$, $(I_c)$; and (3) if $D$ and $E$ are the other intersections of $(\omega)$ with $AB$ and $AC$, then $MP=DE=QN=r_a$, and the lines $MP$, $DE$, $QN$ are tangent to a circle with the center $\omega$.

3926. *Proposed by V. Thêbault, Le Mans, France.*

With the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 form a number with ten figures with no repetition of digits such that, if this number is increased by unity, the sum is a perfect square.

## SOLUTIONS

3833 [1937, 394]. *Proposed by Emma Lehmer, Bethlehem, Pa.*

If $U_n$, $n=0, 1, 2, \cdots$, is a $k$th order recurring series defined by

$$U_{n+k} = a_1U_{n+k-1} + a_2U_{n+k-2} + \cdots + a_kU_n,$$

$$U_0 = U_1 = \cdots = U_{k-2} = 0, \qquad U_{k-1} = 1,$$

prove that for $n>0$

$$U_{n+k-1} = \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)!}{\alpha_1!\alpha_2! \cdots \alpha_k!},$$

where the summation extends over all integral solutions of

$$\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n.$$

*Editorial Note.* The proposer stated: "This can be proved by induction using well known properties of multinomial coefficients. The case $n=2$ is well known and was discussed by Lucas in his *Théorie des Nombres*, p. 313. As far as I know it has not been stated for $n>2$. Problem E230 [1937, 246] is a special case with $n=2$, $a_1=1$, $a_2=-1$ for example."

The formula can be derived without the use of induction. Set $f(x)=x^k-(a_1x^{k-1}+a_2x^{k-2}+\cdots+a_k)$, where the $a_i$'s are such that $f(x)$ has no multiple roots, $a_k\neq 0$, and denote the roots by $r_1, r_2, \cdots, r_k$. Then

$$u_n = \sum_{i=1}^{k} A_i r_i^n, \tag{1}$$

where the $A_i$'s are arbitrary constants, is a solution of the difference equation

$$u_n - (a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k}) = 0. \tag{2}$$

For the given problem the $A_i$'s are to be determined so that $u_0=u_1=\cdots=u_{k-2}=0$, $u_{k-1}=1$. It is easily seen that the determination is unique, and the values of the $A_i$'s will appear later. In what follows $x$ is not less than a fixed positive quantity which is large enough to secure the absolute convergence of the series used. We then have

$$\begin{aligned} \frac{1}{f(x)} &= \sum_{i=1}^{k} \frac{1}{f'(r_i)} \frac{1}{(x-r_i)} = \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \sum_{i=1}^{k} \frac{r_i^n}{f'(r_i)} \\ &= \frac{1}{x^k\left[1 - \left(\dfrac{a_1}{x} + \dfrac{a_2}{x^2} + \cdots + \dfrac{a_k}{x^k}\right)\right]} \\ &= \frac{1}{x^k} \sum_{m=0}^{\infty} \left(\frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_k}{x^k}\right)^m. \end{aligned} \tag{3}$$

Hence we have

$$\sum_{i=1}^{k} \frac{r_i^n}{f'(r_i)} = 0, \qquad (n = 0, 1, \cdots, k-2), \tag{4}$$
$$= 1, \qquad n = k-1;$$

and, consequently, $A_i = 1/f'(r_i)$; and we may now rewrite (3) as

$$\sum_{n=0}^{\infty} \frac{u_n}{x^{n+1}} = \frac{1}{x^k} \sum_{m=0}^{\infty} \left(\frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_k}{x^k}\right)^m,$$
$$= \frac{1}{x^k} \sum_{m=0}^{\infty} \sum \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)!}{\alpha_1! \alpha_2! \cdots \alpha_k!} \frac{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}}{x^{\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k}}, \tag{5}$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = m.$$

Hence

$$u_n = \sum \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_k)!}{\alpha_1! \alpha_2! \cdots \alpha_k!} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \tag{6}$$
$$\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n - k + 1 \geqq 0.$$

The equation (2) with the initial values of $u_i$ determines in succession the values of $u_n$ as polynomials in the $a_i$'s without any restriction upon the values of the $a_i$'s. The above development with certain limitations on the $a_i$'s gives the form of the coefficients for the above polynomials; and, after the determination of these coefficients, we may drop the limitation on the range of the $a_i$'s in (6).

3836 [1937, 394]. *Proposed by H. P. Thielman, College of St. Thomas, St. Paul, Minn.*

Given Kelvin's function

$$\text{bei } x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x/2)^{4n-2}}{[(2n-1)!]^2},$$

evaluate

$$\int_0^{\infty} \frac{\text{bei } x}{x} \, dx.$$

*Note by Hsien-yü Hsü, Washington University, Saint Louis, Mo.*

Given that Kelvin function

$$\text{bei } (x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\frac{1}{2}x)^{4n-2}}{\{(2n-1)!\}^2}, \tag{1}$$

which series is easily seen to be convergent for all values of $x$. For large values of $x$, we have the asymptotic expansion of bei $(x)$.* We shall prove that the integral

* Cf. H. B. Dwight, Tables of Integrals and other Mathematical Data, p. 162 (821.2).

$$(2) \qquad \int_0^\infty \frac{1}{x} \operatorname{bei}(x)dx$$

does not exist. For if it did exist, then, for every $\epsilon > 0$, there would exist a fixed $\omega(\epsilon)$, such that for every pair of values $(\omega_1, \omega_2)$, $\omega(\epsilon) < \omega_1 < \omega_2$, the inequality

$$(3) \qquad \left| \int_{\omega_1}^{\omega_2} \frac{1}{x} \operatorname{bei}(x)dx \right| < \epsilon$$

would hold.

Now, for sufficiently large values of $x$, we have

$$(4) \qquad \frac{1}{x} \operatorname{bei}(x) \cong \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi}\, x^{3/2}} \sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right),$$

and*

$$(5) \qquad \begin{aligned} \int_{\omega_1}^{\omega_2} \frac{1}{x} \operatorname{bei}(x)dx &\cong \int_{\omega_1}^{\omega_2} \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi}\, x^{3/2}} \sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) dx \\ &= \frac{1}{2^{3/4}\sqrt{\pi}} \int_{\Omega_1}^{\Omega_2} e^u u^{-3/2} \sin\left(u - \frac{\pi}{8}\right) du, \end{aligned}$$

where $u = x/\sqrt{2}$ and $\Omega = \omega/\sqrt{2}$. Take $\Omega_1 = (2n + 1/8)\pi$ and $\Omega_2 = (2n + 1/8 + 1/2)\pi$, here $n$ being a large positive integer; then, by the second mean value theorem, the right member of formula (5) becomes, for $\Omega_1 < \xi < \Omega_2$,

$$(5') \qquad \begin{aligned} &\frac{1}{2^{3/4}\sqrt{\pi}} \left\{ e^{\Omega_1}\Omega_1^{-3/2} \int_{\Omega_1}^{\xi} \sin\left(u - \frac{\pi}{8}\right) du + e^{\Omega_2}\Omega_2^{-3/2} \int_{\xi}^{\Omega_2} \sin\left(u - \frac{\pi}{8}\right) du \right\} \\ &\qquad \cong e^{2n\pi} n^{-3/2} 0(1) \to \infty, \end{aligned}$$

when $n \to \infty$. Hence the inequality (3) does not hold, and thus the integral (2) does not exist.

It is, however, of interest to note that, under the arbitrary (*illegitimate*) assumption of its existence, the integral (2) would turn out to have the value $\pi/4$. The first part of this evaluation, concerning the Laplace-transform of the function bei $(x)$, $x = 2\sqrt{t}$, may be demonstrated; the second part, however, is only formally valid.

(i) The Laplace-transform of bei $(2\sqrt{t})$ is $(1/s)\sin(1/s)$, for $s > 0$.

Set $x = 2\sqrt{t}$; we obtain

$$(1') \qquad \operatorname{bei}(2\sqrt{t}) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{\{(2n-1)!\}^2},$$

and for sufficiently large values of $t$,

$$(6) \qquad |\operatorname{bei}(2\sqrt{t})| \leqq \frac{e^{\sqrt{2t}}}{2\sqrt{\pi}\, t^{1/4}} \{1 + 0(1)\}.$$

* Cf. Whittaker-Watson, Modern Analysis, Chap. VIII, Asymptotic Expansion.

Consider the Laplace-integral

$$(7)\qquad I \equiv \int_0^\infty e^{-st}\,\mathrm{bei}\,(2\sqrt{t})dt = \left(\int_0^\omega + \int_\omega^\infty\right) e^{-st}\,\mathrm{bei}\,(2\sqrt{t})dt \equiv I_1 + I_2,$$

where $s>0$, a constant parameter, and $\omega$ a fixed value but arbitrarily large. For $I_2$, by making use of (6), we have

$$(8)\qquad |I_2| \leqq \int_\omega^\infty e^{-st}\,|\,\mathrm{bei}\,(2\sqrt{t})\,|\,dt \leqq \int_\omega^\infty e^{-st}\frac{e^{\sqrt{2t}}}{2\sqrt{\pi}\,t^{1/4}}\{1+0(1)\}dt.$$

But for $t$ being sufficiently large, $t\exp(-st+\sqrt{2t}) \leqq K$, a positive constant; thus, formula (8) becomes

$$(8')\qquad |I_2| \leqq \frac{K}{2\sqrt{\pi}}\{1+0(1)\}\int_\omega^\infty t^{-5/4}dt = \frac{2K}{\sqrt{\pi}}\{1+0(1)\}\omega^{-1/4}\to 0,$$

as $\omega\to\infty$. For $I_1$, we obtain, by means of (1′),

$$(9)\qquad \begin{aligned} I_1 &= \int_0^\omega e^{-st}\,\mathrm{bei}\,(2\sqrt{t})dt = \int_0^\omega e^{-st}\sum_{n=1}^\infty(-1)^{n-1}\frac{t^{2n-1}}{\{(2n-1)!\}^2}dt \\ &= \sum_{n=1}^\infty\frac{(-1)^{n-1}}{\{(2n-1)!\}^2}\int_0^\omega e^{-st}t^{2n-1}dt = \sum_{n=1}^\infty\frac{(-1)^{n-1}\left(\frac{1}{s}\right)^{2n}}{\{(2n-1)!\}^2}\int_0^{s\omega}e^{-u}u^{2n-1}du, \end{aligned}$$

where the interchange of the infinite summation and the integral sign is permissible, since the power series (1′) is uniformly convergent in any fixed radius of convergence $\omega$. Now, as $\omega\to\infty$,

$$(9')\qquad I_1\to\sum_{n=1}^\infty\frac{(-1)^{n-1}s^{-2n}}{\{(2n-1)!\}^2}\Gamma(2n) = \frac{1}{s}\sum_{n=1}^\infty(-1)^{n-1}\frac{\left(\frac{1}{s}\right)^{2n-1}}{(2n-1)!} = \frac{1}{s}\sin\frac{1}{s}\cdot$$

Hence, the Laplace-transform of bei $(2\sqrt{t})$, denoted by $f(s)$, is

$$(10)\qquad f(s)\equiv\int_0^\infty e^{-st}\,\mathrm{bei}\,(2\sqrt{t})dt = \frac{1}{s}\sin\frac{1}{s},$$

which holds continuously for $s>0$.

(ii) The integral (2) would be $\pi/4$, if it existed.

Suppose we define for $s=0+$,

$$(11)\qquad f(s) = \frac{1}{s}\sin\frac{1}{s}\cdot$$

Then, if we integrate with respect to the parameter $s$ from 0 to $\infty$ for both sides of equation (10), we formally obtain without difficulty by interchanging integral signs

$$\int_0^\infty \frac{1}{x} \operatorname{bei}(x)dx = \pi/4, \tag{12}$$

provided that the integral in (12) exists.*

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## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Ill.*

The University of Michigan will soon publish a *Bibliography of American Mathematical Publications to 1850*, by Professor L. C. Karpinski. Over 1200 pages of texts and 850 titles are reproduced. Particular attention is given to Spanish American, including South American, writings. The book will contain about 750 pages, and will be sold for $6.00.

Dr. J. M. Barbour of Ithaca College has been appointed to an assistant professorship in the music department of Michigan State College.

Assistant Professor Clifford Bell of the University of California at Los Angeles has been promoted to an associate professorship.

Assistant Professor L. M. Blumenthal of the University of Missouri has been promoted to an associate professorship.

Assistant Professor J. C. Brixey of the University of Oklahoma has been promoted to an associate professorship.

Assistant Professor Ann S. Buchanan of Southwestern Louisiana Institute has been promoted to an associate professorship.

Professor J. H. Butchart of Phillips University, Oklahoma, has been appointed to a professorship at William Woods College, Fulton, Missouri.

Associate Professor J. A. Cooley of the University of Tennessee has been promoted to a professorship.

Dr. W. A. Cordrey has been appointed a professor at Southeastern Louisiana College.

Associate Professor A. R. Crathorne of the University of Illinois has been promoted to a professorship.

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* Cf. H. P. Thielman, A Note on the Use of the Laplace-transformation, this MONTHLY, No. 8, vol. 45, 1938, pp. 508–510.

Dr. A. E. Currier of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. G. M. Ewing of the University of Missouri has been promoted to an assistant professorship.

Dr. C. H. Graves has been promoted to an assistant professorship at Pennsylvania State College.

Dr. E. A. Hazlewood of State Teachers College, Mansfield, Pennsylvania, has been appointed an assistant professor at Texas Technological College.

Assistant Professor E. R. Heineman of Texas Technological College has been promoted to an associate professorship.

Associate Professor C. M. Jensen of Kansas Wesleyan University has been promoted to a professorship.

Assistant Professor Fritz John of the University of Kentucky has been promoted to an associate professorship.

Associate Professor R. C. Lamb of the U. S. Naval Academy has been promoted to a professorship.

Dr. V. V. Latshaw of Lehigh University has been promoted to an assistant professorship.

Dr. Caroline A. Lester of New York State College for Teachers has been promoted to an assistant professorship.

Dr. D. C. Lewis, Jr., of Cornell University has been appointed an assistant professor at the University of New Hamsphire.

Assistant Professor H. B. MacDougal of South Dakota State College has been promoted to an associate professorship.

Dr. Roy MacKay of Eastern New Mexico Junior College has been promoted to a professorship.

Assistant Professor Dora McFarland of the University of Oklahoma has been promoted to an associate professorship.

Assistant Professor Morris Marden of the University of Wisconsin Extension Division has been promoted to an associate professorship.

Dr. A. L. O'Toole has been appointed professor of mathematics and statistics at Mundelein College, Chicago.

Dr. H. R. Pyle of Earlham College has been made dean and professor.

Dr. H. W. Raudenbush, Jr., of Queens College has been promoted to an assistant professorship.

Dr. P. K. Rees of New Mexico State College has been appointed an assistant professor at Southern Methodist University.

Dr. Nathan Schwid has been appointed an assistant professor at the College of Mines and Metallurgy, El Paso.

Assistant Professor H. W. Smith of Oklahoma A. and M. College has been promoted to an associate professorship.

Associate Professor E. L. Thompson of Texas Technological College has been promoted to a professorship.

Dr. T. L. Wade, Jr., of Mercer University has been appointed an assistant professor at the University of Alabama.

Dr. D. L. Webb of Georgia School of Technology has been promoted to an assistant professorship.

The following appointments to instructorships have been announced:

Adelphi College: Dr. H. J. Riblet
University of Alabama: Dr. C. L. Seebeck, Jr.
University of California at Los Angeles: Dr. W. T. Puckett, Jr.
Duke University: Dr. R. P. Boas, Jr.
Emory University: Dr. G. B. Lang
Hunter College: Dr. L. A. Aroian
University of Michigan: Dr. H. H. Goldstine
University of Rochester: Dr. J. W. Green
Tennessee Polytechnic Institute: Dr. J. A. Ward
Texas Technological College: Dr. L. F. Ollmann, Dr. R. K. Wakerling

## MATHEMATICAL EXPOSITIONS

The following statement comes from the University of Toronto:

There are many books dealing with the elementary aspects of calculus, geometry, algebra, *etc.* and an ever increasing number of more advanced treatises on topics from all branches of mathematics. In English, however, there is a definite lack of books emphasizing fundamental principles and treating the various topics in a less elaborate manner. The University of Toronto has decided to sponsor a series of such books, under the title of *Mathematical Expositions.* The first concern of each author will be to present his subject matter in readable form, while supplying the natural background and calling attention to significant ideas. Each volume will run to about 125 pages. The first two will deal with the *Foundations of Geometry* and the *Infinite in Mathematics.* It is expected that one of these will appear by the end of the present year. Arrangements have been made for the preparation of certain other volumes. It is hoped that the price will not be more than $2.00 with a reduction of 25% to members of the M.A.A.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.


## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13; California, Pa., October 7.
ILLINOIS, Galesburg, May 12–13.
INDIANA, Muncie, April 28–29.
IOWA, Ames, April 21–22.
KANSAS, Topeka, April 1.
KENTUCKY, Murray, April 28–29.
LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13; WASHINGTON, D. C., December.
MICHIGAN, Ann Arbor, March 18; Kalamazoo, November 18.
MINNESOTA, Northfield, May 13.
MISSOURI, Springfield, April 28.
NEBRASKA, Lincoln, May 5.
NORTHERN CALIFORNIA, San Francisco, January 28.
OHIO, Columbus, April 8.
OKLAHOMA, Tulsa, February 10.
PHILADELPHIA, Bethlehem, Pa., December 2.
ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.
SOUTHEASTERN, Charleston, S.C., March 24–25.
SOUTHERN CALIFORNIA, Whittier, March 4.
SOUTHWESTERN, Alpine, Texas, May 2–3.
TEXAS, Abilene, March 31–April 1.
WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.


Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

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## THE JACOB HOUCK MEMORIAL FUND OF THE MATHEMATICAL ASSOCIATION OF AMERICA

In the year 1923 Mr. C. C. Carter of Bluffs, Illinois, called the attention of Secretary Cairns to his endeavor to interest Miss Bessie Houck of Virginia, Illinois, in establishing a fund for the Mathematical Association in memory of her father, Jacob Houck. Professor H. E. Slaught was called into consultation, and Miss Houck made her will in 1924 bequeathing her estate to the Association, and appointing Mr. Carter and Mr. Slaught, or the survivor, as executors. Mr. Carter was made a life member of the Association in 1929. Following the death of Miss Houck in April 1936, the will was contested; but through the efforts of Mr. Carter and a lawyer authorized by the Association Trustees, a settlement has brought over seven thousand dollars to the Association. The income from the fund is to "be used under the direction from time to time of the Trustees of the Mathematical Association of America for the promotion of mathematical science or its teaching." The will further states that "The Mathematical Association shall cause due credit to be given to said fund in the announcement of every undertaking, act, or thing whatsoever which shall receive financial support from said fund."

At the Madison meeting the Trustees voted to express their appreciation of the services of Mr. Carter in initiating and conserving this significant bequest and to appropriate seven hundred fifty dollars from this fund to Mr. Carter.

The Mathematical Association of America gratefully accepts the custody of this unique fund. May it prove to inspire other large bequests for the cause of mathematics in America.

W. D. C.

## MATHEMATICAL REVIEWS

On page 614 of this number of this MONTHLY there appears a detailed announcement concerning "Mathematical Reviews," a new publication expected to be of supreme importance to mathematics and mathematicians. In it will appear short accounts of current mathematical publications throughout the world. It will provide a readily accessible view of mathematics as a living, growing science.

As a supplementary publication to this MONTHLY (which presents accounts of activities of the Mathematical Association), and the BULLETIN (which records the activities of the Mathematical Society), the new MATHEMATICAL REVIEWS should be found in every mathematics library.

The Mathematical Association has contributed $1,000.00 for its initial year of publication, as a result of which members of the Association may subscribe for it at an especially low rate, $6.50 for the first volume.

E. J. M.

# THE TWENTY-SECOND SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-second summer meeting of the Mathematical Association of America was held at the University of Wisconsin, Madison, Wisconsin, on Monday and Tuesday, September 4–5, 1939, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Three hundred seventy-one were in attendance at the meetings, including the following one hundred eighty-one members of the Association:

L. K. ADKINS, State Teachers College, La Crosse, Wis.
R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
G. E. ALBERT, Ohio State University
O. W. ALBERT, University of Redlands
E. F. ALLEN, Oklahoma A. and M. College
FLORENCE E. ALLEN, University of Wisconsin
R. C. ARCHIBALD, Brown University
W. L. AYRES, University of Michigan

FRANCES E. BAKER, Mount Holyoke College
R. H. BARDELL, University of Wisconsin
J. W. BEACH, University of Dubuque
A. A. BENNETT, Brown University
S. F. BIBB, Armour Institute of Technology
G. A. BLISS, University of Chicago
R. W. BRINK, University of Minnesota
FOSTER BROOKS, Kent State University
R. E. BRUCE, Boston University
H. E. BUCHANAN, Tulane University
J. HOBART BUSHEY, Hunter College
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E. W. CHITTENDEN, University of Iowa
A. H. CLIFFORD, Massachusetts Institute of Technology
J. B. COLEMAN, University of South Carolina
E. G. H. COMFORT, University of Arkansas
LENNIE P. COPELAND, Wellesley College
J. J. CORLISS, De Paul University
RICHARD COURANT, New York University
H. S. M. COXETER, University of Toronto
A. R. CRATHORNE, University of Illinois
D. R. CURTISS, Northwestern University
H. T. DAVIS, Northwestern University
W. M. DAVIS, Armour Institute of Technology
F. F. DECKER, Syracuse University
L. A. V. DECLEENE, St. Norbert College
BERNARD DIMSDALE, University of Idaho
L. L. DINES, Carnegie Institute of Technology
Sister MARIOLA DOBBIN, Rosary College
B. F. DOSTAL, University of Florida
R. H. DOWNING, Purdue University
ARNOLD DRESDEN, Swarthmore College
L. T. DUNLAP, Pennsylvania State College

J. M. EARL, University of Omaha
MARGARET C. EIDE, State Teachers College, River Falls, Wis.
G. C. EVANS, University of California
G. W. EVANS, Swampscott,.Mass.
H. P. EVANS, University of Wisconsin

L. R. FORD, Armour Institute of Technology
TOMLINSON FORT, Lehigh University
J. S. FRAME, Brown University
THORNTON C. FRY, Bell Telephone Laboratories

M. G. GABA, University of Nebraska
B. C. GETCHELL, Butler University
B. P. GILL, College of the City of New York
W. O. GORDON, Pennsylvania State College
W. C. GRAUSTEIN, Harvard University
L. M. GRAVES, University of Chicago
L. J. GREEN, Georgia School of Technology
V. G. GROVE, Michigan State College

BEATRICE L. HAGEN, Pennsylvania State College
D. W. HALL, University of Virginia
CLARA L. HANCOCK, Junior College, Virginia, Minn.
W. L. HART, University of Minnesota
W. W. HART, Kenilworth, Ill.
E. R. HEDRICK, University of California at Los Angeles

M. R. HESTENES, University of Chicago
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
R. C. HUFFER, Beloit College
M. GWENETH HUMPHREYS, Sophie Newcomb College
W. A. HURWITZ, Cornell University

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
C. M. JENSEN, Kansas Wesleyan University
FRITZ JOHN, University of Kentucky
MARIE M. JOHNSON, Oberlin College

WILFRED KAPLAN, Scituate Center, Mass.
A. J. KEMPNER, University of Colorado
P. W. KETCHUM, University of Illinois
H. R. KINGSTON, University of Western Ontario
ELIZABETH E. KNIGHT, State Teachers College, Milwaukee, Wis.
FULTON KOEHLER, University of Minnesota

R. E. LANGER, University of Wisconsin
C. G. LATIMER, University of Kentucky
V. V. LATSHAW, Lehigh University
D. H. LEHMER, Lehigh University
CAROLINE A. LESTER, New York State College for Teachers
G. H. LING, University of Saskatchewan
NEIL LITTLE, Purdue University

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MACLANE, Harvard University
H. F. MAC NEISH, Brooklyn College
DOROTHY MCCOY, Belhaven College
N. H. MCCOY, Smith College
J. V. MCKELVEY, Iowa State College
E. J. MCSHANE, University of Virginia
H. W. MARCH, University of Wisconsin
MORRIS MARDEN, University of Wisconsin Extension Division
Sister MARY FELICE, Mount Mary College
L. C. MATHEWSON, Dartmouth College
J. R. MAYOR, Southern Illinois State Normal University
L. E. MEHLENBACHER, State Teachers College, Flagstaff, Ariz.
D. D. MILLER, Ohio University
VIRGINIA MODESITT, Danville, Ind.
G. E. MOORE, University of Illinois
W. L. MOORE, University of Louisville
C. W. MORAN, Lane Technical School, Chicago, Ill.
D. S. MORSE, Union College
MARSTON MORSE, Institute for Advanced Study
DAVID MOSKOVITZ, Carnegie Institute of Technology
E. J. MOULTON, Northwestern University

J. L. NAGLE, National Park Service, St. Louis, Mo.

E. N. OBERG, University of Iowa
RUFUS OLDENBURGER, Armour Institute of Technology
L. F. OLLMANN, Texas Technological College
ALEXANDER OPPENHEIM, Raffles College, Singapore
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, State College, Pa.

G. A. PARKINSON, University of Wisconsin Extension Division
I. E. PERLIN, Armour Institute of Technology
G. W. PETRIE, III, South Dakota State School of Mines
H. P. PETTIT, Marquette University
H. R. PHALEN, Brown University
A. E. PITCHER, Lehigh University
F. J. POLANSKY, Macalester College
J. E. POWELL, Michigan State College
G. B. PRICE, University of Kansas

J. W. QUERRY, Sam Houston State Teachers College

TIBOR RADÓ, Ohio State University
W. T. REID, University of Chicago
R. G. D. RICHARDSON, Brown University
C. C. RICHTMEYER, State Teachers College, Mount Pleasant, Mich.
R. F. RINEHART, Case School of Applied Science
FRED ROBERTSON, Iowa State College
G. DEB. ROBINSON, University of Toronto
S. G. ROTH, New York University
W. E. ROTH, University of Wisconsin Extension Division
LULU L. RUNGE, University of Nebraska

R. B. SAUNDERS, University of Minnesota
M. G. SCHERBERG, University of Minnesota
W. T. SCOTT, University of Michigan
C. GRACE SHOVER, Carleton College
MARY EMILY SINCLAIR, Oberlin College
E. R. SLEIGHT, Albion College

CLARA E. SMITH, Wellesley College
E. R. SMITH, Iowa State College
G. W. SMITH, University of Kansas
VIRGIL SNYDER, Cornell University
M. H. STONE, Harvard University
E. B. STOUFFER, University of Kansas
A. G. SWANSON, General Motors Institute
OTTO SZÁSZ, University of Cincinnati
GABOR SZEGÖ, Stanford University

MILDRED E. TAYLOR, Mary Baldwin College
H. C. TRIMBLE, University of Chicago
P. L. TRUMP, University of Wisconsin High School
A. W. TUCKER, Princeton University
H. L. TURRITTIN, University of Minnesota

HENRY VAN ENGEN, Iowa State Teachers College
E. B. VAN VLECK, University of Wisconsin
J. I. VASS, University of Wisconsin Extension Division
OSWALD VEBLEN, Institute for Advanced Study

G. E. WAHLIN, University of Missouri
R. J. WALKER, Cornell University
A. D. WALLACE, University of Virginia
J. A. WARD, Tennessee Polytechnic Institute
J. H. WEAVER, Ohio State University
M. S. WEBSTER, Purdue University
MARIE J. WEISS, Sophie Newcomb College
E. T. WELMERS, Michigan State College
P. M. WHITMAN, Harvard University
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, University of California at Los Angeles
MARGARETE C. WOLF, Wayne University
F. E. WOOD, Northwestern University
H. A. WOOD, University of Connecticut

The mathematics group stayed in Tripp Hall, one of the men's dormitories of the University of Wisconsin, and had their meals at the refectory close by. Due to the expert attention of the manager, Mr. Wentworth, and Mrs. Sokolnikoff of the local committee, the meals, in cafeteria style, were attractive and generous.

Tea was served Monday and Tuesday afternoons by the ladies of the department of mathematics. Social rooms in each part of Tripp Quadrangle and in the refectory furnished adequate quarters for constant visiting, for bridge, and for other diversions. Much use was made of the bathing facilities in Lake Mendota immediately back of Tripp Hall.

On Wednesday afternoon a large number went for a boat trip on the lake and a supper on Picnic Point; and a considerable number went on a complimentary bus trip, arranged by the George Banta Publishing Company, to their plant in Menasha, with a tour through the plant and a dinner at the North Shore Country Club.

All these and many other courtesies were recognized in a resolution offered by Professor E. B. Stouffer at the dinner and adopted enthusiastically by a rising vote.

Two hundred fifty-four attended the joint dinner in the banquet hall at Hotel Loraine. After a bountiful meal President Evans of the Society introduced Dean Sellery, who welcomed the visitors to the University and spoke of the peculiar and notable characteristics of mathematics. Speeches followed by Professor Graustein on the postponement of the International Congress with a description of the plans as already formed; by Professor Veblen on the forthcoming abstract journal, *Mathematical Reviews*; and by Secretary Cairns, who emphasized the great importance to all members of the Association of the *Mathematical Reviews* and of the accompanying plan for securing microfilms or film prints of

any desired article at a low cost. Full details with regard to the abstract journal and the microfilms have been extensively circulated by the American Mathematical Society.

The American Mathematical Society held meetings for the reading of short papers Wednesday, Thursday and Friday mornings, and Tuesday and Thursday afternoons. Colloquium lectures were given during the week by Professor A. A. Albert on "Structure of algebras" and by Professor M. H. Stone on "Convex bodies."

The latter part of Monday afternoon a sound film "Rectilinear coördinates" by J. H. Lewis of Princeton, New Jersey, was shown and inquiries concerning this were answered by Professor A. W. Tucker. Mr. Lewis's purpose is to utilize the advantages of films for showing the essentials of coördinate representation in two and three dimensions.

Sessions of the Association were held Monday afternoon and Tuesday morning. The program was arranged by a committee consisting of Professors H. P. Evans, R. G. Sanger, and W. H. Bussey, chairman. President Carver presided at the Monday session and the latter part of the Tuesday session, Vice-President Hart having presided for the first part. The program follows, together with abstracts of some of the papers.

## First Session of the Association

1. "Algebra for the undergraduate" by Professor Marie J. Weiss, Sophie Newcomb College.

2. "Over and under functions as related to differential equations" by Professor W. M. Whyburn, University of California at Los Angeles.

3. "Soap films and the calculus of variations" by Professor W. T. Reid, University of Chicago.

1. Miss Weiss's paper will appear in an early issue of the Monthly.

2. Professor Whyburn defined "over" and "under" functions and discussed a number of important developments in the theory of differential equations that have resulted from their use. The Perron integral for a real function of a single real variable, a fundamental existence theorem for non-linear ordinary differential equations, the Dirichlet problem, and sub-harmonic functions were discussed. The paper emphasized the usefulness of the over and under functions in approximation of the solutions of the various problems considered.

3. The calculus of variations is a branch of mathematics which is concerned with the determination of curves and surfaces possessing prescribed maximizing or minimizing properties. One such problem is that of finding a surface whose boundary is a given closed curve, or sets of such closed curves, in space and which has the least possible area; surfaces satisfying this minimum area property are called minimal surfaces. If the bounding curve lies in a plane the problem is trivial, the solution being merely the part of its plane limited by the curve; for twisted boundaries, however, the mathematical determination of minimal surfaces bounded by the given curves involves considerable difficulty.

Nature, however, solves this problem in a simple fashion. The given boundary may be realized by pieces of wire fastened together in the prescribed manner. If such a wire frame is dipped in a soap solution, a soap film is obtained with the wire as its boundary. The Belgian scientist Plateau (1801–83) conducted extensive experiments with soap films, and because of his experimental work the problem of passing a minimal surface through a given boundary is called the problem of Plateau. Within recent years the mathematical solution of the problem of Plateau has been carried to a rather advanced stage, principally through the researches of Douglas, Radó, Courant, and Morse.

Professor Reid's paper was concerned with a general discussion of the problem of Plateau. The lecture was illustrated by soap film models and by lantern slides of such models.

## Second Session of the Association

1. "The role of isomorphism in mathematics and its applications" by Professor A. J. Kempner, University ot Colorado, retiring president of the Association.

2. Report of the Committee on Tests, by the chairman, Professor E. W. Chittenden, University of Iowa.

3. Progress report of the committee on the Association's activities, by the chairman, Professor R. E. Langer, University of Wisconsin.

1. Professor Kempner's address will appear in an early issue of the Monthly.

2. The important report of the Committee on Tests will be printed in an early issue of the Monthly.

3. Professor Langer gave a brief summary of the recommendations as to the Association's organization and policies which the committee is considering. As the plans develop and become definite, a report will be made to the Trustees and to the Association.

## Meetings of the Board of Trustees

Eleven trustees were present at their meetings Monday evening and Tuesday noon.

The following twenty-five persons and one institution were elected to membership on applications duly certified:

### *To Individual Membership*

G. M. Bloom, A.M.(Northwestern) Teacher, Ballard Mem. School, Louisville, Ky.

A. J. D'Atri, B.S.(C.C.N.Y.), C.E.(Brooklyn Poly. Inst.) Structural designer, Dept. of Pub. Works, New York, N. Y.

W. H. Glenn, Jr., A.M.(U.C.L.A.) Instr., Pasadena Jr. Coll., Pasadena, Calif.

W. W. Hartshorn, A.M.(S. Calif.) Teacher, Union High School and Jr. Coll., Brawley, Calif.

J. F. Hubbard, A.B.(Harvard), Ed.M.(T. C. of Boston) Intermed. Asst., W. B. Rogers School, Boston, Mass.

M. L. Jautz, A.B.(Marquette) Teacher, Marquette Univ. High School, Milwaukee, Wis.

D. F. Johnson, M.S.(Middlebury) Grad. student, Boston Univ., Boston, Mass.

Fulton Koehler, Ph.D.(Minnesota) Instr., Univ. of Minnesota, Minneapolis, Minn.

J. H. LEWIS, A.M.(Washington and Jefferson) 359 Nassau St., Princeton, N. J.

A. W. MACDONALD, A.M.(N. J. State T. C., Montclair) Teacher, Essex County Boys Voc. School, Newark, N. J.

H. J. MCCONNELL. Cartographic Engr. in charge Eng. Dept., Western Div., A.A.A. Photo. Lab., Salt Lake City, Utah

W. H. MCCREA, Ph.D.(Cambridge) Prof., Queen's Univ., Belfast, Northern Ireland

W. S. MCCULLEY, M.S.(A. and M. Coll. of Texas) Instr., A. and M. Coll. of Texas, College Station, Tex.

RUTH S. MCKEE (Mrs. GEORGE W.), Ph.D. (Bryn Mawr) 618 N. Second St., Harrisburg, Pa.

LEONARD MILLER, A.B.(Brooklyn Coll.) Annex Attendant, Library of New York City Bar Assn., New York, N. Y.; Grad. student, Brooklyn Coll.

E. B. ROESSLER, Ph.D.(California) Asst. Prof., Asst. Statistician, Exper. Sta., Univ. of California, Davis, Calif.

R. G. M. SABEL, B.S.(Mass. Inst. of Tech.) Teacher, High School, Bristol, Conn.

M. A. SADOWSKY, Dr.Ing.(Berlin) Instr., Armour Inst. of Tech., Chicago, Ill.

SARA GRACE SMYTH, A.M.(Columbia) Dean of Women; Instr., Knox Coll., Galesburg, Ill.

P. M. SWINGLE, Ph.D.(Michigan) Asst. Prof., New Mexico State Coll., State College, N. M.

L. J. TEMPLIN, A.M.(Loyola Univ.) Instr., Loyola Univ., Chicago, Ill.

H. C. TRIMBLE, Ph.D.(Wisconsin) Fellow in Evaluation, General Educ. Board, Chicago, Ill.

B. R. ULLSVIK, A.M.(Wisconsin) Teacher, Wisconsin High School, Madison, Wis.

ALAN WAYNE, M.S. in Educ.(C.C.N.Y.) Teacher, Rhodes School, New York, N. Y.

MARY E. WILLIAMS, A.M.(Kentucky) Teacher, Senior High School, Ashland, Ky.

*To Institutional Membership*

YALE UNIVERSITY, New Haven, Conn.

It was voted (1) to appoint Professor H. S. M. Coxeter an associate editor of the MONTHLY in the place of Professor W. F. Cheney, resigned; (2) to approve the letter of Secretary Hollcroft to Secretary F. R. Moulton expressing disapproval of the A.A.A.S. plan to hold meetings cyclically in Atlantic City, St. Louis and Cleveland, and approving, instead, such a division of meetings between the East, middle region, and West as obtains at present, the number of meetings in the East being approximately equal to those in the other two regions combined; (3) to accept the report of the Committee on Tests which had been presented to the Association on Tuesday morning by the chairman, Professor E. W. Chittenden, and to discharge the Committee with the thanks of the Trustees for their valuable work.

The Trustees discussed the relation of the Association to the new abstract journal, and the potential value of the journal and the microfilms of original articles to non-Society members of the Association. Professor Langer presented at great length a tentative report of the committee on the Association's activities. The Trustees voted a general approval of the lines along which the committee is working, and requested the committee to continue its studies along these lines. They adopted the committee's recommendation that a small committee confer with the leaders of the Junior College teaching personnel in mathematics with the purpose of ascertaining what this group regards as its essential needs and what the conditions are under which it would be ready to join in the membership of the Association.

W. D. CAIRNS, *Secretary-Treasurer*

## THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-third annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Wyoming, Laramie, on Friday and Saturday, April 28–29, 1939. There were three sessions. Professor C. F. Barr, chairman of the Section, presided at each.

The attendance was forty-one, including the following eighteen members of the Association: C. F. Barr, J. R. Britton, I. M. DeLong, G. W. Gorrell, D. F. Gunder, C. A. Hutchinson, Dunham Jackson, L. Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, W. V. Lovitt, S. L. Macdonald, W. K. Nelson, Greta Neubauer, O. H. Rechard, A. W. Recht, W. M. Stewart.

The following officers were elected for the coming year Chairman, D. F. Gunder, Colorado State College; Vice-Chairman, W. V. Lovitt, Colorado College; Secretary-Treasurer, A. J. Lewis, University of Denver. The 1940 meeting is to be held at Colorado State College, Fort Collins, Colorado.

The Section was honored in having Professor Dunham Jackson of the University of Minnesota as its guest speaker. The Saturday morning session was a joint meeting with the Mathematics Section of the Eastern Division of the Colorado Educational Association.

The following papers were presented:

1. "Teaching college mathematics to large classes" by Professor O. H. Rechard, University of Wyoming.
2. "On the Diophantine equation $x(x+1)\cdots(x+n-1)=y^k$" by Dr. L. Louise Johnson, University of Colorado.
3. "Leading differences for backward interpolation" by Professor W. V. Lovitt, Colorado College.
4. "Operational calculus and the theory of numbers" by Professor C. A. Hutchinson, University of Colorado.
5. "Evaluating shear stresses without finding flexure function" by Professor D. F. Gunder, Colorado State College.
6. "On foci and branch points" by Professor A. J. Kempner, University of Colorado.
7. "Orthogonality" by Professor Dunham Jackson, University of Minnesota.
8. "Morley triangles" by Professor Claribel Kendall, University of Colorado.
9. "A new class of orthogonal polynomials" by Professor Dunham Jackson, University of Minnesota.
10. Report: "College-secondary mathematics coordinating committee" by Professor D. F. Gunder, Colorado State College.
11. Discussion: "The report of the Joint Commission" by Professor C. A. Hutchinson, University of Colorado; Professor G. W. Gorrell, University of Denver; and Miss Glennie Bacon, University High School, Laramie.
12. "Are we ready?" by Wendall Wolf, Morey Junior High School, Denver, introduced by the Secretary.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

1. Professor Rechard continued the report given the Section at its last meeting on the teaching of large classes in mathematics at the University of Wyoming. In addition to the two reported on last year, three other classes were divided into large (over 50) and small (under 30) sections. On the basis of percentile rank in the Ohio College Ability Test two pairs of sections, those in analytic geometry and college algebra, were found to be not comparable. In the third sections which were comparable, no significant difference was found in their mathematical attainments as measured by three one-hour tests and the final two-hour examination. Thus of five classes divided into large and small groups, the three in which comparability on the P.R. basis was established have shown no significant difference in mathematical attainment between the large and small groups.

2. Let the product of $n$ consecutive positive integers be $P_n=x(x+1)\cdots(x+n-1)$. The equation $P_n=y^k$, $y$ and $k$ integers greater than 1, is known to be impossible if $n=2$, 3, or $k$; if $n\leqq 203$, $k=2$; if $n\leqq 13$, $k=3$, or 5; if $n=4$, $k$ one of a certain infinite set of primes; if one of the integers is $ap$, $p$ prime, $a\leqq 8$; or if $x<n^2$. Dr. Johnson showed that the equation is impossible also if $n=2k$; if $x\leqq(n+1)^k-n$; if one of the integers is $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $\cdots$, $p_\mu^{\alpha_\mu}$, the $p$'s prime, $\sum_{i=1}^{\mu}\alpha_i<k$; or if one of the integers is $a^k$, $a<p$, $p$ the greatest prime which divides $P_n$; and that the equation has at most one solution for $n=4$, 5, or 6, $k$ a prime $\geqq 7$. There are a few other closely related results.

3. Leading difference formulas for subtabulation are derived from Newton's formula for negative interpolation. These leading differences at the end of a table being computed, the subtabulations can be made by a series of subtractions. Professor Lovitt has hence exposed the leading differences for the subtabulation in terms of the leading differences at the end of a table of differences instead of the leading differences at the beginning of the table.

4. Professor Hutchinson gave a critique of an article by B. van den Pol, in the December 1938 *Philosophical Magazine*.

5. The complete solution of the well known flexure problem requires the determination of a function $x$ satisfying $\nabla^2 x=0$ within the section and the condition that the derivative shall be a specified function on the boundary. However, for the actual determination of the stresses within the beam only the partial derivatives of $x$ are needed. Professor Gunder found an accurate and relatively simple method of finding these derivatives by transforming the given boundary onto the unit circle and setting up the solution for $x$ in terms of the usual integral involving the transformed value of the normal derivative. This expression is then differentiated under the integral sign and the resulting integral evaluated mechanically to give the required stresses.

7. Professor Jackson emphasized the fact that mathematical study is largely concerned with the acquisition of fundamental general ideas which recur in varied and progressively more complicated situations. Familiarity with such

fundamental principles is often vitally reinforced, and their significance clarified, by acquaintance with less elementary applications in which they are viewed under diverse aspects. This is illustrated by an outline of the development of the notion of orthogonality from the perpendicularity of elementary geometry through the algebraic and analytical formulations which appear in statistics, mathematical physics, and current research in pure mathematics.

8. Professor Kendall reported on an article, *Morley's Triangle*, given in the Mathematical Gazette of February, 1938. There W. J. Dobbs defines the Morley Triangle as one obtained by the intersections of certain trisectors of the angles of a given triangle. He shows that this triangle is isosceles, and extends the discussion to show that there are 27 such triangles, 18 of which are equilateral.

9. Professor Jackson's paper appeared in full in the October issue of the MONTHLY.

A. J. LEWIS, *Secretary*

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## THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Carleton College, Northfield, Minnesota, on Saturday, May 13, 1939. A morning session was held at 10:30 o'clock and was followed by luncheon and an afternoon session at 2:15 o'clock. Professor W. H. Bussey of the University of Minnesota presided at each session.

Seventy-three persons attended the meeting, including the following twenty-eight members of the Association: C. J. Blackall, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, S. Elizabeth Carlson, Sister M. Claudette, H. H. Dalaker, J. H. Daoust, Brother Louis De La Salle, Margaret C. Eide, C. H. Gingrich, H. E. Hartig, J. S. Hickman, Dunham Jackson, Margaret P. Martin, W. R. McEwen, Sigurd Mundhjeld, F. J. Polansky, Inez Rundstrom, M. G. Scherberg, C. Grace Shover, F. J. Taylor, H. P. Thielman, Ella Thorp, A. L. Underhill, K. W. Wegner, Marion A. Wilder; and Sister Thomas à Kempis, institutional member representative.

At the business session officers were elected for the coming year as follows: Chairman, C. S. Carlson, St. Olaf College; Secretary, A. L. Underhill, University of Minnesota; Executive Committee, H. P. Thielman, College of St. Thomas, and Sister Thomas à Kempis, College of St. Teresa.

The following nine papers were presented:

1. "The stability of an unsymmetric top" by Professor E. J. Camp, Macalester College.

2. "A number problem" by Professor L. E. Bush, College of St. Thomas.

3. "An appreciation of Sophie Germain" by Sister M. Thomas à Kempis, College of St. Teresa.

4. "On the use of the complex exponential in the solution of differential equations" by Professor H. E. Hartig, University of Minnesota.

5. "The osculating conics to a point on a plane curve" by Dr. M. G. Scherberg, University of Minnesota.

6. "A system of orthogonal polynomials satisfying an auxiliary condition" by Professor Dunham Jackson, University of Minnesota.

7. "An alternate derivation of the equation of the director sphere" by Michael Norris, College of St. Thomas, introduced by Professor Bush.

8. "On roots of unity" by Dr. C. Grace Shover, Carleton College.

9. "Note on the integro-differential equations satisfied by certain generalized trigonometric functions" by Professor H. P. Thielman, College of St. Thomas.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Camp showed that a study of the motion of a rigid body with one point fixed can be reduced to a study of the Euler differential equations for a rigid body. If gravity is the only force acting on the body, there is a special solution in which one of the principal axes of inertia through the center of gravity is in the vertical position. A study of the solutions in the neighborhood of this special one is accomplished by making a transformation of variables involving a parameter $\epsilon$ in such a way that for $\epsilon=0$ the original variables reduce to the special solution mentioned above. The Euler equations are then solvable as a power series in the parameter $\epsilon$, and the coefficients of the first degree terms in $\epsilon$ furnish the basis for the discussion of stability. The conditions for stability depend on the moments of inertia and the angular velocity of the body.

2. Let $S(r, N)$ be the sum of the digits of all numbers less than $N$ when these numbers are expressed in the scale of notation of radix $r$. Then $S(r, N)$ is asymptotic to $(r-1)N(\log N)/(2 \log r)$, in the sense that as $N\rightarrow\infty$ the limit of the quotient of $S(r, N)$ by the latter function is unity. Professor Bush showed that for $N$ sufficiently large, the average sum of the digits of all numbers less that $N$ is least when these numbers are expressed in the binary scale. In fact, if $r_1$ and $r_2$ are integers such that $2\leqq r_1<r_2$, and if $N$ is sufficiently large, the average sum of the digits of all numbers less that $N$ is less when the numbers are written in the scale of radix $r_1$ than when they are written in that of radix $r_2$.

3. In this paper Sister Thomas à Kempis gave the result of her investigation of letters in the Sophie Germain collection in the Bibliothèque Nationale. These letters reveal the unusual influence which Mme. Germain exerted on mathematical physics, especially on the theory of the vibrations of elastic surfaces. In particular they show her close association with contemporary savants, such as Biot, Legendre, Poisson, Lagrange, Laplace, Fourier, Delambre, Cauchy, and others, and contribute facts to substantiate Mozan's conclusion that "all things considered, she was the most profoundly intellectual woman France has produced."

4. The so-called complex number method of obtaining the particular solution of certain ordinary linear differential equations was discussed by Professor Hartig. By the introduction of an operator, the effect of which was to convert a

vector into its projection on a fixed reference line, the simple scalar character of the differential equation was retained, without sacrificing the simplicity and the interpretability inhering in the complex number method of solution.

5. On the hypothesis that the plane curve $y=f(x)$ is differentiable in an open interval containing $x=x_0$ and that the second derivative $f''(x_0)$ is not zero, Dr. Scherberg proved that each point in the plane not on the tangent line to the curve at $(x_0, y_0)$ is the focus of a unique osculating conic. Further he proved that if $C$ is a circle tangent to the curve $y=f(x)$ on the concave side at $(x_0, y_0)$ and of radius one-fourth of the radius of curvature at $(x_0, y_0)$, then each point of $C$ is the focus of an osculating parabola; each point inside $C$ is the focus of an osculating hyperbola; each point not in $C$ but on the same side of the tangent line as $C$ is the focus of an osculating ellipse; and finally each point not in $C$ but on the opposite side of the tangent line from $C$ is the focus of an osculating hyperbola.

6. At a recent meeting of another Section, Professor Jackson reported on a system of orthogonal polynomials $p_n(x)$, each satisfying the condition that $p_n(1)=p_n(-1)$. The present paper gave a corresponding discussion, similar in general outline but with some differences in detail, for a set of orthogonal polynomials with the boundary condition $p_n(-1)=-p_n(1)$.

7. Given a quadric surface, which is non-singular,

$$F \equiv \sum_{i,j=1}^{4} a_{ij}X_iX_j = 0, \qquad a_{ij} = a_{ji},$$

it is desired to find the locus of points from which three mutually perpendicular planes can be drawn tangent to the surface. Three such planes are assumed to meet in a point $(x_1, x_2, x_3, x_4)$ of the locus. By using the equation of the quadric in plane coördinates, Mr. Norris obtained equations on the coördinates of the three planes. On elimination of these coördinates we get the necessary condition on $(x_1, x_2, x_3, x_4)$. This is the usual $A_{44}(X_1^2+X_2^2+X_3^2)-2(A_{41}X_1+A_{42}X_2+A_{43}X_3)X_4+(A_{11}+A_{22}+A_{33})X_4^2=0$, where $A_{ij}$ is the cofactor of $a_{ij}$ in the matrix of the quadric.

8. The product of the $n$ $n$th roots of unity is 1 or $-1$ according as $n$ is odd or even, the product of the imprimitive $n$th roots of unity is 1 or $-1$ according as $n$ is odd or even, and the product of the primitive $n$th roots of unity is 1. These results were proved by Dr. Shover by using simple lemmas from the theory of rational integers.

9. This paper is an extension of the article *A Generalization of Trigonometry*, H. P. Thielman, National Mathematics Magazine, vol. 11, 1937, pp. 349–351. Professor Thielman showed that in a general theory certain functions can be defined analogous to the trigonometric functions. These functions satisfy equations which in the theory of Volterra's functions of composition reduce to integro-differential equations, in the analytic function theory to differential equations. The results of L. E. Ward, this MONTHLY, vol. 34, 1927, and those

of V. B. Temple, National Mathematics Magazine, vol. 13, 1939, p. 263, are shown to be special cases of this general theory.

A. L. UNDERHILL, *Secretary*

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## THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twelfth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Thiel College, Greenville, Pennsylvania, on Saturday, May 13, 1939. Professor H. L. Black, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was seventy-two, including the following twenty-two members of the Association: C. S. Atchison, B. R. Beisel, O. F. H. Bert, H. L. Black, A. M. Bryson, P. N. Carpenter, Elizabeth B. Cowley, L. L. Dines, H. L. Dorwart, V. V. Johnston, G. R. Kraus, David Moskovitz, Frederick Mosteller, L. T. Moston, F. W. Owens, Helen B. Owens, Elizabeth Renwick, R. G. Sturm, Sister M. Clotilda Sullivan, J. S. Taylor, E. D. Wells, R. T. Zimmerman.

It was decided to hold the fall meeting of the Allegheny Mountain Section at California, Pennsylvania, on October 7, 1939.

Following an opening address by Dean Luther Malmberg of Thiel College the first three speakers of the morning session were introduced by Mr. F. R. Layng, Chief Engineer of the Bessemer and Lake Erie Railroad Company. The following eight papers were read:

1. "Use of mathematics in the study of grade and line for steam railroads" by W. S. McFetridge, Principal Assistant Engineer, Bessemer and Lake Erie Railroad Company, introduced by the Secretary.

2. "The use of mathematical formulas in railroad maintenance and operation" by M. F. Mannion, Office Assistant to Chief Engineer, Bessemer and Lake Erie Railroad Company, introduced by the Secretary.

3. "Railroad tracks without joints; welded rail" by H. H. Harman, Engineer of Track, Bessemer and Lake Erie Railroad Company, introduced by the Secretary.

4. "Types of curvature of curves and surfaces" by Dr. Mary T. Speer, University of Pittsburgh, introduced by Professor Taylor.

5. "Euclid in a present-day workshop" by Professor O. F. H. Bert, Washington and Jefferson College.

6. "A tribute to the memory of Professor N. C. Grimes" by Professor P. N. Carpenter, Grove City College.

7. "Quartic surface invariant under the quaternary collineation group of order 168" by Free Jamison, University of Pittsburgh, introduced by Professor Taylor.

8. "Measures of rank correlation" by Frederick Mosteller, Carnegie Institute of Technology.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. McFetridge pointed out that the amount of traffic and the speed at which it can be operated over a given grade and line is affected not only by the power of the locomotives, but by certain resistances to movement that must be overcome. The steps necessary to determine the tonnage that may be handled by a locomotive on a certain grade are: (1) determine the tractive effort of the locomotive at the minimum speed decided on; (2) determine the resistance in pounds per ton for the cars to be handled; (3) determine the resistance due to grade and line in pounds per ton. With these data, the tonnage handled by a given locomotive on the line under consideration, or the necessary power to handle a given tonnage, or the maximum grade over which a given tonnage may be handled, can each be determined.

2. Mr. Mannion showed the methods and results of studies of the renewals that have been made for the past twenty to thirty years in an effort to establish the law of probability governing the rate of failure. Numerical data and graphs showing the estimated renewals and actual installations of railroad ties for the past several years were exhibited, also charts which illustrated the similarity between the typical train hour diagram and the probability curve. From this train hour diagram it is possible to compute wage costs and the cost of unproductive time. A study of the effects on operations and of the estimated savings that would result if present grades were altered was shown on several graphs. From these, the efficiency of operation and the wage cost for the revised grades can be computed and the savings for the different grades estimated.

3. Mr. Harman used motion pictures to illustrate the method of welding rails, and explained the reasons why railroads have been led to this experiment. In the usual type of track, with relatively short pieces, the ends of the pieces are worn due to hammering at the ends. Longer pieces of track have been used but there is a limit to the length of single piece that can be used due to excessive cost of fabrication. Charts were shown depicting a study of the longitudinal movement in the welded track and comparative temperatures of air and rail in an experimental mile of welded rail.

4. The various types of curvature associated with curves and surfaces were developed briefly by Dr. Speer, using vector methods. These types included: (a) circular, spherical, and screw curvature for curves in space, and vector, relative, and geodesic curvature for curves on a surface, and (b) normal curvature, and first and second, or mean and specific, curvature for surfaces. Particular reference was made to values for such special types of curves as geodesics, asymptotes, and lines of curvature, and to minimal and developable surfaces.

5. Professor Bert cited the theorem that the locus of the vertex of a right-angled triangle with constant length of hypotenuse is a circle. He then showed a tool, which he called a core box plane, that is of use to pattern makers in cutting circular cylindrical cores, and whose operation depends on the above theorem. The tool is handled like a plane, and the cutting edge moves along the elements of a circular cylinder.

6. This paper was read as a tribute to the memory of the late Nathan Cesna

Grimes, professor of mathematics in Grove City College from 1926 to 1938, by Professor Carpenter, his student, friend, and colleague. The brief chronological sketch of the education, activities, and interests of Professor Grimes indicated the great breadth and depth of these interests and activities. Before his teaching at Grove City College he had been head of the mathematics department and registrar at the University of Arizona, assistant to the president at the University of Oregon, and field employment manager for the Goodrich Rubber Company over the entire United States during the World War. The paper concluded with the resolutions of respect as recorded in the minutes of the faculty of Grove City College.

7. Mr. Jamison discussed the quartic surface

$$2x_0^4 + x_1x_3^3 + x_2x_1^3 + x_3x_2^3 + 6x_0x_1x_2x_3 = 0,$$

which is invariant under the transformation

$$x_n' = \frac{n^3 - 6n^2 + 11n + 6}{12\sqrt{-7}} \left[2x_0 + (\omega^n + \omega^{6n})x_1 + (\omega^{2n} + \omega^{5n})x_2 + (\omega^{3n} + \omega^{4n})x_3\right]$$

and $x_n' = \omega^{n^2}x_n$, where $\omega^7 = 1$, $n = 0, 1, 2, 3$; the generators of the group. There is a set $S_1$ of 8 planes, of which $x_0 = 0$ is one, whose 28 intersections form a set of bitangents to the quartic. The other bitangents in $S_1$ form a set of 168. There is a set of 8 triangular pyramids, whose lateral edges, of which $x_1 = x_2 = 0$ is one, each meet the quartic in 4 coincident points at a base vertex; and whose base edges lie in $S_1$ and are inflexional tangents. There is a set of 56 triangular pyramids, whose base edges lie in $S_1$ and are inflexional tangents, and whose lateral faces are tangent planes. The 24 base vertices of pyramids in each plane of $S_1$ are points of inflexion of a plane quartic and may be grouped in 8 triples, so that for each of 28 possible choices of two triples there is a conic which passes through all six points of the two triples and the points of contact of one of the bitangents to the quartic.

8. For variates not functionally related, it is often necessary to test for the existence of correlation. Mr. Mosteller dealt with several such tests by means of various coefficients of rank correlation. Illustrative examples were provided showing the use of these coefficients both for populations with correlation, and for independent variates. From the behavior of these coefficients in sampling known populations, inferences were drawn about the estimation of correlation by means of samples in populations whose character is unknown.

DAVID MOSKOVITZ, *Secretary*

# DEFINITE INTEGRALS AND RIEMANN SUMS

J. A. SHOHAT, University of Pennsylvania

**1. Introduction.** The definition of the proper $R$-integral $\int_a^b f(x)dx$ as the limit of a certain sum offers an inexhaustible source of applications: (a) to evaluating limits of certain sums,* (b) to evaluating definite integrals. The main object of this paper is to evaluate in this way a certain general definite integral and to derive from this single source various other definite integrals (some of importance in the theory of interpolation) whose treatment usually requires artificial methods invented *ad hoc*. Our method is quite elementary and may be offered even to beginners in Definite Integrals. Next we discuss the approximation of a definite integral by its Riemann sum. We treat from this point of view the Euler-Maclaurin Summation Formula in its relation to the Trapezoidal Rule and derive a similar formula related to Simpson's Rule.

**2. Points asymptotically equidistributed on an interval.** The points

$$(a \leqq)x_{1,n} < x_{2,n} < \cdots < x_{n,n}(\leqq b)$$

are said to be "$\phi$-asymptotically equidistributed" on the finite interval $(a, b)$, if

$$\phi(x_{s,n})(x_{s+1,n} - x_{s,n}) = \frac{b-a}{n} + \frac{\epsilon_{n,s}}{n}, \quad |\epsilon_{n,s}| < \epsilon_n. \tag{1}$$

where $\phi(x)$ is $R$-integrable in $(a, b)$, $\epsilon_n$ independent of $s$, $\lim_{n\to\infty} \epsilon_n = 0$, (1) holding for $1 \leqq s \leqq n$, with the possible exception of a finite number of points

$$x_{1,n},\ x_{2,n},\ \cdots,\ x_{k,n};\ x_{n-l,n},\ x_{n-l+1,n},\ \cdots,\ x_{n,n}, \tag{2}$$

($k$, $l$ fixed, independent of $n$), where (1) is replaced by

$$\lim_{n\to\infty} (x_{j,n} - x_{j-1,n}) = 0, \tag{3}$$

($j = 1, 2, \cdots, k, n-l+1, \cdots, n+1$; $x_{0,n} = a$; $x_{n+1,n} = b$). In case of $\phi(x) \equiv 1$,

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* The following is an illustration. In order to find

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \qquad (k > 0)$$

rewrite this expression as

$$\frac{1}{n}\left[\left(\frac{1}{n}\right)^k + \left(\frac{2}{n}\right)^k + \cdots + \left(\frac{n}{n}\right)^k\right],$$

and identify with the Riemann sum

$$\frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right],$$

where $f(x) \equiv x^k$. Hence, the required limit is

$$\int_0^1 x^k dx = \frac{1}{k+1}.$$

we speak of the points $x_{i,n}$ as being "asymptotically equidistributed" on $(a, b)$. For brevity we write $x_s$, in place of $x_{s,n}$, if there is no fear of ambiguity.

LEMMA I. *If the points $x_i$ are $\phi$-asymptotically equidistributed on $(a, b)$ then, for $f(x)$ R-integrable on $(a, b)$,**

$$\lim_{n\to\infty} \frac{1}{n} [f(x_1) + f(x_2) + \cdot \quad \cdot + f(x_n)] \equiv \lim_{n\to\infty} S_n = \frac{1}{b-a}\int_a^b f(t)\phi(t)dt.$$

*Proof.* Rewrite $S_n$ as

$$S_n = \frac{1}{n}\left[\sum_{j=1}^{k} f(x_j) + \sum_{j=n-l}^{n} f(x_j) + \sum_{j=k+1}^{n-l-1} f(x_j)\right] \equiv \Sigma' + \Sigma'' + \Sigma'''.$$

Denoting by $M$ the upper bound of $|f(x)|$ in $(a, b)$,

$$|\Sigma'| \leqq M\frac{k}{n} \to 0, \qquad |\Sigma''| \leqq M\frac{l+1}{n} \to 0, \qquad (n\to\infty).$$

Rewrite $\Sigma'''$, by means of (1), as

$$\begin{aligned}\Sigma''' = \frac{1}{b-a}\Bigg\{ &\sum_{j=k+1}^{n-l-1} f(x_j)\phi(x_j)(x_{j+1} - x_j) \\ &+ \sum_{j=1}^{k}\frac{b-a}{r} f\left[a + \frac{j(b-a)}{n}\right]\phi\left[a + \frac{j(b-a)}{n}\right] + \sum_{j=n-l}^{n}(\text{id.})\Bigg\} \\ &- \frac{1}{n}\left\{\sum_{j=1}^{k} f\left[a + \frac{j(b-a)}{n}\right]\phi\left[a + \frac{j(b-a)}{n}\right] + \sum_{j=n-l}^{n}(\text{id.})\right\} \\ &- \frac{\epsilon_{n,s}}{n(b-a)}\sum_{j=k+1}^{n-l-1} f(x_j),\end{aligned}$$

and let $n\to\infty$. The first sum on the right evidently tends to $\int_a^b f(t)\phi(t)dt/(b-a)$. The second and the third sums do not exceed in absolute value, respectively,

$$\frac{1}{n}MM'(k+l) \to 0, \qquad \frac{|\epsilon_n|}{n(b-a)}Mn = \frac{|\epsilon_n|}{b-a}M \to 0$$

$[n\to\infty;\ |\phi(x)| \leqq M'$ in $(a, b)]$. The Lemma is proved. Denote by $N(n; c, d)$ the number of points $x_s$ contained in $(c, d) \subset (a, b)$. Taking above $f(x) \equiv 1$ in $(a, b)$, or $f(x) \equiv 1$ in $(c, d)$ and $\equiv 0$ elsewhere, we get:

$$(4) \qquad \frac{1}{b-a}\int_a^b \phi(t)dt = 1; \qquad \lim_{n\to\infty}\frac{N(n; c, d)}{n} = \frac{1}{b-a}\int_c^d \phi(t)dt,$$

which justifies the terminology adopted. An immediate generalization is

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* $f(x)$ may be real-valued or of the form $f_1(x) + if_2(x)$, $f_{1,2}(x)$ real-valued.

LEMMA II. *Let*

$$x_s = F(\theta_s), \ (s = 1, 2, \cdots, n); \quad a = F(\alpha), \ b = F(\beta), \tag{5}$$

*where* $F(\theta)$ *is monotone in* $(\alpha, \beta)$ *and the* $\theta_s$ *are asymptotically* $\phi$*-equidistributed in* $(\alpha, \beta)$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} [f(x_1) + f(x_2) + \cdots + f(x_n)] = \frac{1}{\beta - \alpha}\int_\alpha^\beta f[F(\theta)]\phi(\theta)d\theta.*$$

The most interesting applications are derived by taking $f(t) = \log(x-t)$ (principal value of log), and $f(t) = 1/(x-t)$, with a certain fixed $x$, real or complex, not on $(a, b)$ (eventually coinciding with $a$ or $b$). These yield the following:

THEOREM I. *Let* $P_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$. *If the* $x_s$ *are asymptotically* $\phi$*-equidistributed on* $(a, b)$, *then both limits*: $\lim_{n\to\infty} \log \sqrt[n]{P_n(x)}$ *and* $\lim_{n\to\infty} (d/dx) \log \sqrt[n]{P_n(x)}$ *exist, with* $x$ *fixed as above. Moreover,*

$$\lim_{n\to\infty} \log \sqrt[n]{P_n(x)} = \frac{1}{b-a}\int_a^b \phi(t) \log (x-t)dt, \tag{6}$$

$$\lim_{n\to\infty} \frac{d}{dx} \log \sqrt[n]{P_n(x)} = \frac{1}{b-a}\int_a^b \frac{\phi(t)dt}{x-t}. \tag{7}$$

THEOREM II. *If the* $x_s$ *satisfy the conditions of Lemma* II, *then*

$$\lim_{n\to\infty} \log \sqrt[n]{P_n(x)} = \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(\theta) \log [x - F(\theta)]d\theta, \tag{8}$$

$$\lim_{n\to\infty} \frac{d}{dx} \log \sqrt[n]{P_n(x)} = \frac{1}{\beta-\alpha}\int_\alpha^\beta \frac{\phi(\theta)}{x-F(\theta)}d\theta. \tag{9}$$

We see that under the conditions stated, the limiting relations (6), (8) may be differentiated with respect to $x$.

The import of Theorems I and II is twofold. (a) They assert the existence of certain limits related to the polynomial $P_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$, and give the explicit expressions of these limits.† This is of importance in the Theory of Interpolation. It yields, for example, the following result.‡,§ Let $F(x)$ be analytic in a certain region in the complex $x$-plane, enclosing the segment $(a, b)$. Construct the Lagrange interpolation polynomial $\Pi_n(x)$, of degree $\leqq n-1$, coin-

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* If $F'(\theta)$, $F''(\theta)$ exist, we may write $x_{s+1} - x_s = (\theta_{s+1} - \theta_s)F'(\theta_s) + \epsilon_{n,s}(\theta_{s+1} - \theta_s)$, which reduces the present case to (1), if $F'(\theta) \neq 0$ in $(\alpha, \beta)$.

† The existence of $\lim_{n\to\infty} \log\sqrt[n]{|P_n(x)|}$ follows in the same manner.

‡ *Cf.* C. Runge, Theorie und Praxis der Reihen, Leipzig, 1904, pp. 134–142, where (7) is first derived, then (8) from it. Runge's analysis lacks rigor, due to the general character of his book.

§ P. Montel, Leçons sur les séries de polynomes à une variable complexe, Paris, 1910, pp. 51–52, where $\phi(t) \equiv 1$ and the $x_s$ are equidistant on $(a, b)$.

ciding in value with $F(x)$ at the points $x_i$ assumed to be asymptotically equidistributed on $(a, b)$. Then, as $n\rightarrow\infty$, $\Pi_n(x)$ converges to $F(x)$ in a certain subregion containing part or the whole of $(a, b)$, as the case may be [we omit the proof which is a textual repetition of that by Montel (*loc. cit.*)].

(b) The above theorems give at once the values of a great variety of definite integrals, as we proceed to show in a few examples.

**3. Evaluation of certain definite integrals.** We shall evaluate two definite integrals:

(i) $\int_0^\pi \log (x-\cos\theta)d\theta$.

Here $F(\theta)\equiv\cos\theta$. Take $\theta_s=(2s-1)\pi/2n$, $(s=1, 2, \cdots, n)$. Then

$$P_n(x) = \prod_{s=1}^{n}(x-\cos\theta) = \frac{\cos n\theta}{2^{n-1}} = \frac{(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n}{2}, \qquad (x = \cos\theta),$$

$$\lim_{n\rightarrow\infty}\sqrt[n]{P_n(x)} = \lim_{n\rightarrow\infty}\frac{P_{n+1}(x)}{P_n(x)} = \frac{x+\sqrt{x^2-1}}{2}, \quad (x, \text{ real or complex, not on } (-1, 1);\ x\sqrt{x^2-1}>0 \text{ for real } x),$$

and formulas (8), (9) give (with $\phi(\theta)\equiv 1$):

$$\frac{1}{\pi}\int_0^\pi \log(x-\cos\theta) = \log\frac{x+\sqrt{x^2-1}}{2}, \qquad (x \text{ as above}), \tag{10}$$

$$\frac{1}{\pi}\int_0^\pi\frac{d\theta}{x-\cos\theta} = \frac{1}{\sqrt{x^2-1}}. \tag{11}$$

Break up (10) into $\int_0^{\pi/2}\cdots+\int_{\pi/2}^{\pi}\cdots$, replace in the first integral $\theta$ by $\pi-\theta$ and add:

$$\frac{1}{\pi}\int_0^{\pi/2}\log(x^2-\cos^2\theta)d\theta = \log\frac{x+\sqrt{x^2-1}}{2},$$

whence, for $x=1$,

$$\int_0^{\pi/2}\log\sin\theta\, d\theta = -\frac{\pi}{2}\log 2 \quad \text{(Euler)}.$$

(ii) $\int_0^\pi\log(x-e^{i\theta})\,d\theta$.

Here $F(\theta)\equiv e^{i\theta}$. We get, with $\theta_s=2s\pi/n$, $(s=1, 2, \cdots, n)$:

$$P_n(x) = \prod_{s=1}^{n}(x-e^{2s\pi i/n}) = x^n-1;$$

$$\lim_{n\rightarrow\infty}\sqrt[n]{x^n-1} = \lim_{n\rightarrow\infty}\frac{x^{n+1}-1}{x^n-1} = \begin{cases} x, & x\geqq 1;\\ 1, & 0<x\leqq 1;\end{cases}$$

whence, by (7), with $\phi(\theta)\equiv 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}\log(x-e^{i\theta})d\theta=\begin{cases}\log x, & x\geqq 1;\\ 0, & 0<x\leqq 1;\end{cases}$$

(both sides being analytic for $x\neq 0$). Separate real and imaginaries:

$$\frac{1}{4\pi}\int_0^{2\pi}\log(1-2x\cos\theta+x^2)=\begin{cases}\log x, & x\geqq 1;\\ 0, & 0<x\leqq 1;\end{cases}$$

and breaking up $\int_0^{2\pi}\cdots=\int_0^{\pi}\cdots+\int_{\pi}^{2\pi}\cdots$:

$$\int_0^{2\pi}\log(1-2x\cos\theta+x^2)d\theta=\begin{cases}2\pi\log x, & x\geqq 1;\\ 0, & 0<x\leqq 1;\end{cases}\qquad\text{(Poisson)}.\tag{12}$$

*Remarks.* (i) Even without using a wider choice of $f(t)$, $\phi(t)$, we may obtain many new integrals from (10–12), by differentiating under the integral sign, by power series expansion, *etc.* Thus (11), with $x=a/b$, yields

$$\int_0^{\pi}\frac{d\theta}{a-b\cos\theta}=\frac{\pi}{\sqrt{a^2-b^2}},\qquad(|a|>|b|),\tag{13}$$

of importance in the theory of Legendre Polynomials.*

The same formula (11) yields, expanding both sides in powers of $1/x$ and comparing coefficients:

$$\int_0^{\pi}\cos^{2n}\theta\,d\theta=(-1)^n\pi\binom{-\frac{1}{2}}{n};$$

also, differentiating with respect to $x$:

$$\frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{(x-\cos\theta)^2}=\frac{x}{(x^2-1)^{3/2}},\quad etc.$$

(ii) For a very general class of orthogonal Tchebychef polynomials $\phi_n(x)=x^n+\cdots$, the following relation holds:†

$$\lim_{n\to\infty}\sqrt[n]{\phi_n(x)}=\lim_{n\to\infty}\left[\frac{\phi_{n+1}(x)}{\phi_n(x)}\right]=\frac{x+\sqrt{x^2-1}}{2},\qquad(x\text{ as above}).\tag{14}$$

The question naturally arises: let $x_s=\cos\theta_s$ be the $n$ zeros of $\phi_n(x)$ known to lie in $(-1,1)$; can we conclude from (14), in view of (10), that the $\theta_s$ are equidistributed in $(0,\pi)$? The answer in many cases is in the affirmative. This important result I learned recently from Dr. P. Erdös.

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* Heine, Handbuch der Kugelfunktionen, 2d ed., Berlin, 1878, pp. 24, 35.

† J. Shohat, Théorie générale des polynomes orthogonaux de Tchebychef, Mémorial des sciences mathématiques, No. 66, 1934, pp. 55–56.

**4. Approximation of a definite integral by Riemann Sums.*** (i) *Euler-Maclaurin Summation Formula and the Trapezoidal Rule.* Assume $f(x)$ to be such that its Euler-Maclaurin Summation Formula may be continued indefinitely. Write this formula as follows:

$$\begin{aligned}\int_a^b f(x)dx - h[f(a) + f(a+h) + \cdots + f(a+\overline{n-1}\,h)]\\ = \frac{h}{2}[f(b) - f(a)] - \frac{B_1}{2!}h^2[f'(b) - f'(a)]\\ + \frac{B_3}{4!}h^4[f'''(b) - f'''(a)] \mp \cdots .\end{aligned} \tag{15.1}$$

$$\begin{aligned}S_n \equiv \int_a^b f(x)dx - \frac{h}{2}\{f(a) + f(b)\\ + 2[f(a+h) + f(a+2h) + \cdots + f(a+\overline{n-1}\,h)]\}\\ = -\frac{B_1}{2!}h^2[f'(b) - f'(a)] + \frac{B_3}{4!}h^4[f'''(b) - f'''(a)] \mp \cdots ,\end{aligned} \tag{15.2}$$

$$\left(h = \frac{b-a}{n};\ B_{2i-1} = \text{Bernoulli Numbers } \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \cdots\right).$$

It is seen that (15.1) *represents the expansion, in powers of h, of the difference between* $\int_a^b f(x)dx$ *and its Riemann sum corresponding to equidistant ordinates, and* (15.2) *represents a similar expansion for the difference between* $\int_a^b f(x)dx$ *and its Trapezoidal Formula.*

(ii) *Simpson's Formula.* Denote for brevity

$$f_s = f\left[a + \frac{s(b-a)}{n}\right], \qquad (s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots).$$

We proceed to derive an expansion similar to (15.2) for the difference

$$\begin{aligned}\Delta_n = \int_a^b f(x)dx - \frac{h}{6}\{f_0 + f_n + 4[f_{1/2} + f_{3/2} + \cdots + f_{n-1/2}]\\ + 2[f_1 + f_2 + \cdots + f_{n-1}]\}, \qquad \left(h = \frac{b-a}{n}\right),\end{aligned} \tag{16}$$

between a definite integral and its Simpson's Formula. [The existence of such an expansion is evident *a priori*, for Simpson's Formula is a linear combination of two trapezoidal formulas corresponding to $h = (b-a)/n$ and $h = (b-a)/2n$.] Thus, we may write, following (15.2),

* *Cf.* J. L. Walsh and W. E. Sewell, Note on degree of approximation of an integral by Riemann sums, this MONTHLY, vol. 44, 1937, pp. 155–160.

$$\Delta_n = h^k \sum_{i=1}^{\infty} C_i h^i [f^{(i)}(b) - f^{(i)}(a)], \tag{17}$$

where $k(\geqq 0)$ and $C_i$ are constants independent of $h$ and $f(x)$. In order to determine $k$ and the $C_i$, apply (17) to

$$f(x) \equiv e^x, \quad \text{with} \quad a = 0, \qquad b = x = nh.$$

We get successively

$$\begin{gathered} e^x - 1 - \frac{h}{6}\left\{1 + e^x + 4\frac{e^{x+h/2} - e^{h/2}}{e^h - 1} + 2\frac{e^x - e^h}{e^h - 1}\right\} = h^k \sum_{i=1}^{\infty} h^i (e^x - 1), \\ \frac{h}{e^h - 1} + \frac{h}{6} + \frac{4}{3}\,\frac{h/2}{e^{h/2} + 1} = 1 - h^k \sum_{i=1}^{\infty} C_i h^i. \end{gathered} \tag{18}$$

We know that

$$\frac{h}{e^h - 1} - \frac{h}{2} = 1 + \sum_{i=1}^{\infty} (-1)^{i-1} \frac{B_{2i-1}}{(2i)!} h^{2i}, \qquad (|h| < 2\pi). \tag{19}$$

Furthermore, we readily verify the following identity

$$\frac{t}{e^t + 1} = \frac{t}{2} - \frac{t^2/4}{\dfrac{t}{e^t - 1} + \dfrac{t}{2}}. \tag{20}$$

Combining (18), (19), (20) we have

$$\begin{aligned} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{B_{2i-1}}{(2i)!} h^{2i} - \frac{h^2}{12}\, & \frac{1}{1 + \displaystyle\sum_{i=1}^{\infty} (-1)^{i-1} \frac{B_{2i-1}}{(2i)!}\left(\frac{h}{2}\right)^{2i}} \\ & \equiv - h^k \sum_{i=1}^{\infty} C_i h^i. \end{aligned} \tag{21}$$

Hence,

$$\begin{gathered} k = 1; \quad C_1 = 0; \quad C_2 = C_4 = C_6 = \cdots = 0; \\ C_3 = -\frac{1}{32 \cdot 90}, \qquad C_5 = +\frac{1}{7 \cdot 243}, \cdots. \end{gathered} \tag{22}$$

All $C_{2i-1}$ may be computed by means of (21). [They could be expressed in determinantal form in terms of the $B_{2i-1}$.] Thus, for the difference $\Delta_n$ in (16),

$$\Delta_n = \sum_{i=1}^{\infty} C_{2i+1} h^{2i+2} [f^{(2i+1)}(b) - f^{(2i+1)}(a)], \tag{23}$$

where

$$\sum_{i=1}^{\infty} C_{2i+1}h^{2i+2} \equiv -\sum_{i=1}^{\infty}(-1)^{i-1}\frac{B_{2i-1}}{(2i)!}h^{2i} + \frac{h^2/12}{1+\sum_{i=1}^{\infty}(-1)^{i-1}\frac{B_{2i-1}}{(2i)!}\left(\frac{h}{2}\right)^{2i}},$$

$$\Delta_n = -\frac{h^4}{32.90}[f'''(b) - f'''(a)] + \frac{h^6}{7.243}[f^{\mathrm{V}}(b) - f^{\mathrm{V}}(a)] + \cdots. \tag{24}$$

Formulas (15.2), (24) may be of use in practical computations. They yield, for example,

$$S_n = -\frac{h^2}{12}[f'(b) - f'(a)] + O(h^4), \qquad \left(h = \frac{b-a}{n}\right), \tag{25}$$

$$\Delta_n = -\frac{h^4}{32.90}[f'''(b) - f'''(a)] + O(h^6), \tag{26}$$

in full agreement with the known remainders

$$S_n = -\frac{(b-a)^3}{12n^2}f''(\xi), \qquad \Delta_n = -\frac{(b-a)^5}{32.90n^4}f^{\mathrm{IV}}(\eta),$$

[$\xi$, $\eta$ in $(a, b)$].

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## ON A CERTAIN DIOPHANTINE PROBLEM

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**1. Introduction.** Consider the expression

$$C_n(\alpha, \beta) \equiv \cos n\alpha + \cos n\beta \tag{1}$$

where $n$ is an integer $\geqq 1$. The first problem to be solved in this paper is to determine all real solutions $\alpha$, $\beta$ of the equation

$$C_n(\alpha, \beta) = 0 \tag{2}$$

for which $\cos\alpha$ and $\cos\beta$ are rational, and all integers $n$ for which such solutions exist.

We should observe that $\cos n\alpha$ is a polynomial of the $n$th degree in $\cos\alpha$ (Tchebyscheff polynomial). We shall apply the results thus obtained for the first problem to the discussion of the more general equation

$$\cos sn\alpha + \cos n\beta = 0,$$

where $s$ is an arbitrary integer. Finally, we shall consider certain systems of equations of the type (2) in more than two variables.

**2. The Diophantine equation.** Writing (1) in the form

$$C_n(\alpha, \beta) \equiv 2\cos\frac{n}{2}(\beta + \alpha)\cos\frac{n}{2}(\beta - \alpha),$$

we find that every solution of (2) must satisfy either

$$\frac{n}{2}(\beta + \alpha) = (2\nu - 1)\frac{\pi}{2}, \quad (\nu = 0, \pm 1, \pm 2, \cdots),$$

or

$$\frac{n}{2}(\beta - \alpha) = (2\lambda - 1)\frac{\pi}{2}, \quad (\lambda = 0, \pm 1, \pm 2, \cdots),$$

so that in either case we find

$$\left(xy - \cos\frac{2k-1}{n}\pi\right)^2 = (1 - x^2)(1 - y^2),$$

where $x = \cos\alpha$, $y = \cos\beta$, $(k = 0, \pm 1, \pm 2, \cdots)$.

We shall distinguish two cases:

1° $$\cos\frac{2k-1}{n}\pi = r \qquad \text{(rational)},$$

and

2° $$\cos\frac{2k-1}{n}\pi = \rho \qquad \text{(irrational)}.$$

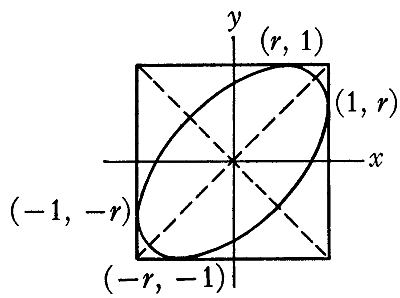
Thus our problem is reduced to the consideration of the quadratic equations in rational $x$ and $y$,

(3) $$x^2 + y^2 - 2xyr + r^2 - 1 = 0,$$

(4) $$x^2 + y^2 - 2xy\rho + \rho^2 - 1 = 0,$$

where $r$ is any rational number of the form $\cos(2k-1)\pi/n$ and $\rho$ is any irrational number of the form $\cos(2k-1)\pi/n$.

It is readily seen that for each admissible $r$ or $\rho$ equation (3) or (4) represents an ellipse. The figure below is drawn for $r = \frac{1}{2}$; for $r = 1$, we have $y = x$; for $r = -1$, we have $y = -x$.




**3. The rational values of** $\cos(2k-1)\pi/n = r$. Suppose that $\cos(2k-1)\pi/n$ is a rational number $r$. Write $r' = 1 - r^2$ and $\rho_1 = r + i\sqrt{r'}$, $\rho_2 = r - i\sqrt{r'}$. Hence $\rho_1$

and $\rho_2$ are roots of unity, so that $(x-\rho_1)(x-\rho_2)$ is a factor of the polynomial $x^n-1$, and we may write

$$(5) \qquad x^n - 1 = (x^2 - 2rx + 1)(x^{n-2} + a_1x^{n-3} + \cdots - 1), \text{ if } r^2 \neq 1.$$

From the rationality of $r$ it follows that the second factor in the right member of (5) has rational coefficients. Moreover, the coefficient of the highest power of $x$ in each member of (5) is equal to unity, so that by Gauss's Lemma* it follows that the coefficients of the polynomials occurring in (5) are integers or zero. Thus $2r$ is an integer or zero and we have the possibilities

$$r = 0, \pm \tfrac{1}{2}, \pm 1.$$

**4. The $n$'s corresponding to rational $r$'s and the solutions of equation (3).** We now consider the various possibilities for $r$. First, let $r = \cos (2k-1)\pi/n = 0$; then

$$(6_1) \qquad \frac{2k-1}{n}\pi = \frac{\pi}{2}(2\nu - 1), \qquad (\nu = 0, \pm 1, \cdots).$$

Thus $n$ is even, and writing $n=2m$, we have

$$(6_2) \qquad \frac{2k-1}{m} = 2\nu - 1, \quad \text{an odd integer}.$$

Noting that $C_n(\alpha, \beta)$ is a polynomial in $\cos\alpha$ and $\cos\beta$, and recalling that we are asking for all solutions of (2) for which $\cos\alpha$ and $\cos\beta$ are the unknowns, we may restrict ourselves to the intervals $0 \leqq \alpha \leqq \pi$, $0 \leqq \beta \leqq \pi$. Similarly, in $(6_1)$ we may confine ourselves to the $k$'s for which

$$(7) \qquad 0 \leqq \frac{2k-1}{n} \leqq 1.$$

In the case under consideration this becomes $0 \leqq (2k-1)/m \leqq 2$, which in view of $(6_2)$ reduces to $(2k-1)/m = 1$. Thus $2k-1=m$, which shows that $m$ must be an odd integer, $m = 2l-1$, so that $n = 2(2l-1)$, $k=l$.

The equation corresponding to (3) becomes in this case ($r=0$)

$$(8) \qquad x^2 + y^2 = 1;$$

and the solutions are given by

$$(9) \qquad x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}, \qquad (t \text{ rational}).$$

Consider next $r = 1/2$. In this case we find in a similar way $(2k-1)/n = 1/3$; so that $n$ must be a multiple of 3, say $n=3m$; thus $m=2k-1$, so that $m$ must be an odd integer and $n=3(2l-1)$, $k=l$.

* Dickson, Modern Algebraic Theories, p. 154.

Equation (3) becomes, for this value of $r$,

(10) $$x^2+y^2-xy-\tfrac{3}{4}=0.$$

By the usual methods of solving a quadratic equation in rationals we readily find that the general solution of (10) is given by

(11) $$x=\frac{1}{2}\,\frac{2t^2-2t-1}{t^2-t+1},\qquad y=\frac{1}{2}\,\frac{t^2-4t+1}{t^2-t+1},\qquad (t \text{ rational}).$$

The case $r=-1/2$ yields $(2k-1)/n=2/3$ which is obviously impossible.

If $r=\cos(2k-1)\pi/n=1$ we find $2k-1=0$, which is again impossible.

Finally, if $r=-1$, then $2k-1=n$, so that $n$ is an odd number. To this case corresponds the solution $(x,-x)$, where $-1\leqq x\leqq 1$, and $x$ is rational.

**5. The irrational values of $\rho=\cos(2k-1)\pi/n$ and the corresponding solutions of equation (4).** We now determine all the possible values of an irrational number $\rho$ of the form $\cos(2k-1)\pi/n$ for which the equation

($4_1$) $$x^2+y^2-2xy\rho+\rho^2-1=0$$

shall have rational solutions $x$ and $y$. Rewriting (4) in the form

($4_2$) $$(\rho-xy)^2=(1-x^2)(1-y^2),$$

we see that $\rho$ must be a quadratic irrationality, so that

$$\rho=a\pm\sqrt{c},\qquad (a,\,c \text{ rational},\ c>0).$$

Hence by ($4_2$), $xy=a$, $(1-x^2)(1-y^2)=c$, so that

$$(x+y)^2=(1+a)^2-c,\qquad (x-y)^2=(1-a)^2-c.$$

We see then that in order that ($4_1$) have a rational solution, we must have

(12) $$(1+a)^2-c=u^2,\qquad (1-a)^2-c=v^2,$$

where $u$ and $v$ are rational. We then have exactly four solutions, one of which is

(13) $$x_0=\frac{u+v}{2},\qquad y_0=\frac{u-v}{2},$$

and the others are

(14) $$(-x_0,-y_0),\ (y_0,x_0),\ (-y_0,-x_0).$$

We are now ready to determine all values of $k$ and $n$ for which $\rho=\cos(2k-1)\pi/n$ satisfies the conditions (12).

Let us write

$$e^{i(2k-1)\pi/n}=\tau\equiv\rho+i\sigma.$$

It is clear that $|\rho|<1$, and we shall prove that $|\rho'|<1$, where $\rho'=a\mp\sqrt{c}$. For this purpose we observe that $\rho$ is a zero of the polynomial

$$\frac{\sin n\theta}{\sin\theta} \equiv U_{n-1}(\cos\theta) \equiv U_{n-1}(x)$$

with *rational* coefficients; from which follows that $\rho'$ is also a zero of the same polynomial. But all these zero's are real and between $-1$ and $+1$; in fact, they are given by $\cos \nu\pi/n$, $(\nu=1, 2, \cdots, n-1)$. Thus $|\rho'|<1$, $\sigma\neq 0$, $\sigma'\neq 0$.

Writing

$$\rho' = \cos\frac{\mu\pi}{n}, \qquad \tau' = e^{i\mu\pi/n} \equiv \rho' + i\sigma',$$

we see that $\tau$ and $\tau'$ are roots of unity, as are also

$$\bar\tau = e^{-i(2k-1)\pi/n}, \qquad \bar\tau' = e^{-i\mu\pi/n}.$$

Thus the biquadratic polynomial

$$\begin{aligned}(x-\tau)(x-\bar\tau)(x-\tau')(x-\bar\tau') &= (x^2-2\rho x+1)(x^2-2\rho' x+1)\\ &= x^4-2x^3(\rho+\rho')+2x^2(1+2\rho\rho')-2x(\rho+\rho')+1\\ &= x^4-4ax^3+2x^2(1+2a^2-2c)-4ax+1\end{aligned}$$

is a factor of the polynomial $x^n-1$. Applying Gauss's Lemma we see that $4a$ and $2(1+2a^2-2c)$ are integers or zero, and it follows that

$$a = \frac{a'}{4}, \qquad c = \frac{a'^2-4b'}{16},$$

where $a'$ and $b'$ are integers or zero. Thus we have

$$\rho = \frac{a' \pm \sqrt{a'^2-4b'}}{4}, \qquad a'^2 > 4b'.$$

But $|\rho|<1$ and $|\rho'|<1$, so that $|a|+\sqrt{c}<1$ and $|a'|\leqq 3$. We have also $|a'|+\sqrt{a'^2-4b'}<4$, from which we obtain $2|a'|<4+b'$.

Summarizing, we have

$$|a'|\leqq 3, \quad a'^2 \geqq 4b'+1, \quad 2|a'|\leqq 3+b, \quad 2|a'|-3\leqq b'\leqq\frac{a'^2-1}{4},$$

from which follows readily $|a'|<4-\sqrt{5}<2$, $b'\leqq 0$.

Thus the following cases may arise

1) $\qquad a'=0; \quad b'=-1,-2,-3; \quad c=-\dfrac{b'}{4},$

2) $\qquad |a'|=1; \quad b'=0,-1; \quad c=\dfrac{1-4b'}{16}.$

Since $c$ cannot be a perfect square, the above cases reduce to

1) $a' = 0$; $b' = -2$, or $-3$; $\rho = a \pm \sqrt{c} = \pm \frac{1}{2}\sqrt{2}$, or $\pm \frac{1}{2}\sqrt{3}$;

2) $|a'| = 1$; $b' = -1$; $\rho = \dfrac{1 \pm \sqrt{5}}{4}$, or $\dfrac{-1 \pm \sqrt{5}}{4}$.

In addition, if condition (12) is to be satisfied, the only cases that remain are

$$\rho = \pm \tfrac{1}{2}\sqrt{3}.$$

From this it is seen that $u^2 = 1/4$, $v^2 = 1/4$.

The case $\rho = \cos (2k-1)\pi/n = \sqrt{3}/2$ yields $(2k-1)/n = 1/6$, by (7), so that $n = 6m$ and $m = 2k-1$, and hence $n = 6(2l-1)$.

Finally, if $\rho = -\sqrt{3}/2$ we find $(2k-1)/n = 5/6$, so that $n = 6m$ and $2k-1 = 5m$, from which it is seen that $m$ is an odd integer $2l-1$ or $n = 6(2l-1)$, and $k = 5l-2$.

The corresponding solutions are found by (13) and (14) to be

$$(\tfrac{1}{2}, 0),\ (-\tfrac{1}{2}, 0),\ (0, \tfrac{1}{2}),\ (0, -\tfrac{1}{2}). \tag{15}$$

**6. Summary of the results.** A substitution of (9), (11), (15) in equation (3) for the corresponding values of $n$ shows that this equation is actually satisfied. Hence, we have proved the following:

THEOREM. *The equation*

$$\cos n\alpha + \cos n\beta = 0 \tag{2}$$

*has rational solutions in* $\cos \alpha$ *and* $\cos \beta$ *if and only if* $n$ *has one of the two forms*

$$n = 2(2l-1), \qquad n = 3(2l-1), \tag{16}$$

*in which cases the solutions are given respectively by* (9) *and* (11). *Furthermore, for every odd* $n$ *there are the trivial solutions* $\cos \alpha = -\cos \beta$. *Finally, if* $n$ *is of the particular form* $6(2l-1)$ *we obtain the additional solutions given by* (15).

It should be noted that the solutions in $\cos \alpha$ and $\cos \beta$ are the same for all $n$'s belonging to the same arithmetical sequence so that it is sufficient to consider $l=1$ in each of the cases (16).

Finally, we observe that in case $n = 2(2l-1)$ corresponding to $r=0$, equation (3) reduces to the Pythagorean equation $x^2+y^2=1$.

**7. The equation** $\cos sn\alpha + \cos n\beta = 0$. We shall now apply the results of the previous section to find all the solutions in rational $\cos \alpha$ and $\cos \beta$ of the equation

$$\cos sn\alpha + \cos n\beta = 0, \tag{17}$$

where $s$ is an arbitrary integer.

Writing (17) in the form

$$2\cos\frac{n(s\alpha+\beta)}{2}\cos\frac{n(s\alpha-\beta)}{2}=0,$$

we find, as in §2,

$$\left[yT_s(x)-\cos\frac{2k-1}{n}\pi\right]^2=[1-T_s^2(x)][1-y^2], \tag{18}$$

where $x=\cos\alpha$, $y=\cos\beta$, $T_s(x)=\cos s\alpha$. Finally, (18) may be written

$$T_s^2(x)+y^2-2T_s(x)yr+r^2-1=0, \tag{19}$$

where as before $r=\cos(2k-1)\pi/n$.

Since $x$ is rational, so is $T_s(x)$ and hence by the results of §4, §5 there may be rational solutions of (19) in rational $x$ and $y$ only if

$$r=0,\ \tfrac{1}{2},\ -1,\ \pm\tfrac{1}{2}\sqrt{3}.$$

For $r=0$ equation (19) reduces to

$$T_s^2(x)+y^2=1. \tag{20}$$

Now it is readily seen that by differentiating $T_s(x)=\cos s\alpha$ with respect to $x=\cos\alpha$ we find

$$\frac{d}{dx}(\cos s\alpha)=\frac{d}{d\alpha}(\cos s\alpha)\frac{d\alpha}{dx}=\frac{s\sin s\alpha}{\sin\alpha},$$

or

$$1-T_s^2(x)=(1-x^2)\left[\frac{T_s'(x)}{s}\right]^2, \tag{21}$$

where

$$T_s'(x)=\frac{d}{dx}T_s(x).$$

By means of (21) we may write (20) in the form

$$x^2+\left[\frac{sy}{T_s'(x)}\right]^2=1,$$

so that

$$x=\frac{2\lambda}{1+\lambda^2},\qquad y=\frac{1}{s}\,\frac{1-\lambda^2}{1+\lambda^2}\,T_s'\left(\frac{2\lambda}{1+\lambda^2}\right), \tag{22}$$

where $\lambda$ is an arbitrary rational parameter. Thus we have found the general solution of (20).

Consider next $r=1/2$; equation (19) becomes

$$y^2-yT_s(x)+T_s^2(x)=\tfrac{3}{4}. \tag{23}$$

In view of (21) this may be written as

$$3(1 - x^2) = u^2, \tag{24}$$

where

$$u = s\,\frac{2y - T_s(x)}{T_s'(x)}\,.$$

But the general solution of (24) is

$$x = \frac{\sigma^2 - 3}{\sigma^2 + 3}, \qquad u = \frac{6\sigma}{\sigma^2 + 3},$$

where $\sigma$ is a rational parameter. Thus the general solution of (23) is given by

$$x = \frac{\sigma^2 - 3}{\sigma^2 + 3}, \qquad y = \frac{1}{2}\left\{T_s\left(\frac{\sigma^2 - 3}{\sigma^2 + 3}\right) + \frac{1}{s}\,\frac{6\sigma}{\sigma^2 + 3}\,T_s'\left(\frac{\sigma^2 - 3}{\sigma^2 + 3}\right)\right\}. \tag{25}$$

When $r = -1$ equation (19) reduces to

$$T_s^2(x) + y^2 + 2yT_s(x) = 0, \tag{26}$$

and $y = -T_s(x)$, $x$ rational, gives the general solution of (26).

Finally, if $r = \pm\sqrt{3}/2$, equation (19) becomes

$$T_s^2(x) + y^2 \pm yT_s(x)\cdot\sqrt{3} - \tfrac{1}{4} = 0,$$

so that $yT_s(x) = 0$. If $T_s(x) = 0$ we have $x = \cos\,(2k-1)\pi/2s$ which, by §3, is rational if and only if $s$ is of the form $2l-1$ and $k = l$; and we find $x = 0$, $y = \pm 1/2$.

If $y = 0$ then $T_s(x) = \pm 1/2$, and $x = \cos\,(3k \pm 1)\pi/3s$. We must now determine the values of $s$ and $k$ in order that $x$ be rational. The argument of §3, §4 may be extended to show that the only possible rational values of $x$ are $0$, $\pm 1/2$, $\pm 1$. If $x = 0$, then $(3k \pm 1)/3s = 1/2$, which is impossible. If $x = 1/2$, then $(3k \pm 1)/3s = 1/3$ so that $s$ is of the form $3l \pm 1$, with $k = l$. If $x = -1/2$, then $(3k \pm 1)/3s = 2/3$ so that $s$ is either $3l-1$, with $k = 2l-1$ or $s = 3l-2$, with $k = 2l-1$. Finally, $x = \pm 1$ gives $(3k \pm 1)/3s = 0$ or $1$, both of which are impossible. Thus the only solutions are $x = \pm 1/2$, $y = 0$ with the corresponding values of $s$ and $k$.

**8. Another generalization of equation** (2). As an extension in a different direction, let us consider the simultaneous equations

$$\cos n\alpha + \cos n\beta + \cos n\gamma + \cos n\delta = 0, \tag{27}$$

$$\alpha + \delta = \beta + \gamma, \tag{28}$$

to be solved in rational $\cos\alpha$, $\cos\beta$, $\cos\gamma$, $\cos\delta$. This system reduces to (2) for $\alpha = \gamma$, $\beta = \delta$.

Introducing the three angles $\theta$, $\phi$, $\psi$ defined by

$$\theta = \tfrac{1}{4}(\alpha + \beta + \gamma + \delta), \qquad \phi = \tfrac{1}{4}(\alpha + \beta - \gamma - \delta), \qquad \psi = \tfrac{1}{4}(\alpha - \beta + \gamma - \delta),$$

and making use of (28), we find that

$$\alpha = \theta + \phi + \psi, \qquad \beta = \theta + \phi - \psi,$$
$$\gamma = \theta - \phi + \psi, \qquad \delta = \theta - \phi - \psi.$$

By the well known product theorem for the sum of cosines, we find, in view of (28),

$$\cos n\alpha + \cos n\beta + \cos n\gamma + \cos n\delta = 4 \cos n\theta \cos n\phi \cos n\psi.$$

Hence all solutions of the system (27), (28) are given by one or more of the angles

$$n\theta = (2\kappa - 1)\frac{\pi}{2}, \qquad n\phi = (2\lambda - 1)\frac{\pi}{2}, \qquad n\psi = (2\mu - 1)\frac{\pi}{2},$$

where $\kappa, \lambda, \mu = 0, \pm 1, \pm 2, \cdots$. From these we find

$$\beta + \gamma = \frac{\pi}{n}(2\kappa - 1), \qquad \alpha - \gamma = \frac{\pi}{n}(2\lambda - 1),$$

$$\alpha - \beta = \frac{\pi}{n}(2\mu - 1), \qquad \delta = \beta + \gamma - \alpha.$$

Setting

$$r = \cos\frac{\pi}{n}(2\kappa - 1), \qquad s = \cos\frac{\pi}{n}(2\lambda - 1), \qquad t = \cos\frac{\pi}{n}(2\mu - 1),$$

$$x = \cos\alpha, \qquad y = \cos\beta, \qquad z = \cos\gamma,$$

our problem is reduced to that of finding $r$, $s$ and $t$ for which one or more of the equations

$$(29a) \qquad (yz - r)^2 = (1 - y^2)(1 - z^2),$$

$$(29b) \qquad (xz - s)^2 = (1 - x^2)(1 - z^2),$$

$$(29c) \qquad (xy - t)^2 = (1 - x^2)(1 - y^2),$$

has rational solutions $x$, $y$, $z$, and to determine all such solutions. We may use our previous results and consider only equation (29$c$), the two others being entirely analogous.

We find in this way that the system (27) and (28) has rational solutions if and only if $n = 2(2l-1)$, $3(2l-1)$ or $n$ is odd.

In the first case $t = 0$, and the solution is given by (9) for $x$ and $y$, while $z$ is any rational number between $-1$ and 1. Moreover in the subclass $l = 3l' - 1$ we have the particular solutions given by (15) and $z$ as before, any rational number between $-1$ and 1.

In the second case $t = \frac{1}{2}$, and the solutions for $x$ and $y$ are given by (11) with $z$ as before. Finally, for every odd $n$ ($t = -1$) we find the solution $x = -y$ and $z$ as before.

**9. On certain systems of equations.** Consider first the system

$$\cos n\alpha + \cos n\beta = 0, \quad \cos n\beta + \cos n\gamma = 0, \tag{30}$$

to be solved in rational $\cos\alpha$, $\cos\beta$, $\cos\gamma$. This system may be readily discussed by using the results of §6.

Setting $x=\cos\alpha$, $y=\cos\beta$, $z=\cos\gamma$ and considering first the case $n=2(2l-1)$, we have the simultaneous system (8)

$$x^2+y^2=1, \qquad y^2+z^2=1,$$

so that

$$x=\pm z=\frac{2\lambda}{1+\lambda^2}, \qquad y=\frac{1-\lambda^2}{1+\lambda^2}.$$

If $n=3(2l-1)$ we have, by (10),

$$x^2+y^2-xy-\tfrac{3}{4}=0,$$
$$y^2+z^2-yz-\tfrac{3}{4}=0,$$

from which follows that either $z=x$ or $z=y-x$, where $x$ and $y$ are given by (11). A similar discussion can be made for the odd $n$'s and for the particular solutions.

Finally, consider the system

$$\cos n_i\alpha_i + \cos n_i\alpha_{i+1} = 0, \qquad (i=1,2,\cdots,k),$$

to be solved in rational $\cos\alpha_i=x_i$. We shall confine ourselves to the case $k=2$, since the general case is exactly analogous. We have

$$\cos m\alpha + \cos m\beta = 0,$$
$$\cos n\beta + \cos n\gamma = 0,$$

where $m$ and $n$ are integers $\geqq 1$.

The case $m=n$ has just been considered. If $m\neq n$ then $m$ and $n$ may belong either to the same or to the two different arithmetic sequences $2(2l-1)$, $3(2l-1)$. In the first case the situation is as before. In the second case we have the possibility, for example, of

$$x^2+y^2=1, \qquad y^2+z^2-yz-\tfrac{3}{4}=0,$$

from which follows $(z-\frac{1}{2}y)^2=\frac{3}{4}x^2$, so that $z-\frac{1}{2}y=0$, $x=0$; and the solutions are given by $x=0$, $y=1$, $z=\frac{1}{2}$; or $x=0$, $y=-1$, $z=-\frac{1}{2}$.

A similar discussion may be applied to the other cases.

# GENERAL FORMULAS FOR THE NUMBER OF MAGIC SQUARES BELONGING TO CERTAIN CLASSES

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## A. Introduction

**1. Preliminary remarks.** By a *magic square* of order $n$ is meant a square arrangement of $n^2$ consecutive integers so that every row, every column and each of the two diagonals have the same sum—called the *magic sum.*

Figure 1, for example, shows a magic square of $5^2$ numbers with the magic sum 60.

| | | | | |
|---|---|---|---|---|
| 0 | 7 | 13 | 16 | 24 |
| 11 | 19 | 20 | 2 | 8 |
| 22 | 3 | 6 | 14 | 15 |
| 9 | 10 | 17 | 23 | 1 |
| 18 | 21 | 4 | 5 | 12 |

Fig. 1

The condition that the numbers be consecutive and hence *all different*—unlike the requirement of equal sums—cannot be completely expressed by means of equations. Therefore the problem of constructing magic squares is not the algebraic one of finding unknown quantities from known, but rather the combinatorial problem of giving to known quantities a definite order [**1**].*

Since methods for the construction of magic squares for any value of $n$ ($n>2$) have long been known, the problem in this sense may be regarded as solved. The general question, however, of finding all possible magic squares or of calculating their number seems to be resolvable only within certain limits, and then only by the introduction of special conditions which make it possible to use combinatorial, analytical, number-theoretical or group-theoretical methods.

One such procedure of particular importance, and of a predominantly number-theoretical character, will be treated in the following paragraphs.

* These references in bold type are to *Notes* at the end of the paper.

**2. Certain general conditions. 2.1.** The essential properties of a magic square remain unchanged if all its numbers are increased or decreased by the same amount. Therefore the smallest number may be arbitrarily chosen. Magic squares have been traditionally made up of the numbers from 1 to $n^2$. However, it seems desirable to take zero as the first element in order to be able to express any element in the square uniquely as a binomial of the form $k_0+k_1n$, where $k_0$ and $k_1$ take the values from 0 to $n-1$, each $n$ times. Then all these elements can be expressed by digits of the $n$-adic number scale and, if a zero be written to the left of the elements 0 to $n-1$, by two-digit numbers of this scale.

In these squares the magic sum ($S$) is equal to the sum of the integers from 0 to $n^2-1$, divided by $n$; hence

$$S = \frac{n(n^2-1)}{2}. \tag{1}$$

**2.2.** It has long been known that the formation of magic squares can be referred to the construction of two or more suitable *auxiliary squares*. The magic square written in the $n$-adic number scale may be broken up into two such auxiliary squares by putting all the first digits in the corresponding cells of a first auxiliary square ($U$), and the second digits in the corresponding cells of a second auxiliary square ($V$).

If, for instance, every number of Fig. 1 be divided by 5, the quotients form the auxiliary square $U$ (Fig. 2) and the remainders the auxiliary square $V$ (Fig. 3).

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 0 | 1 |
| 4 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 0 |
| 3 | 4 | 0 | 1 | 2 |

$U$

FIG. 2

| | | | | |
|---|---|---|---|---|
| 0 | 2 | 3 | 1 | 4 |
| 1 | 4 | 0 | 2 | 3 |
| 2 | 3 | 1 | 4 | 0 |
| 4 | 0 | 2 | 3 | 1 |
| 3 | 1 | 4 | 0 | 2 |

$V$

FIG. 3

Conversely, a magic square can be obtained from two suitable auxiliary squares by the use of the formula $Un+V$, that is to say, by adding to the numbers of $V$ the corresponding numbers of $U$ multiplied by $n$.

The following condition is necessary and sufficient that the square so formed shall contain each of the numbers 0 to $n^2-1$ exactly once:

(2) *The two auxiliary squares must match, i.e. if they are superposed, each of the numbers* 0 *to* $n-1$ *of the one square must coincide with every number* 0 *to* $n-1$ *of the other square exactly once.*

Hence:

(3) *Each of the numbers* 0 *to* $n-1$ *must appear exactly* $n$ *times in each of the auxiliary squares.*

Now let $S_u$ denote the sum of an arbitrary row, column or diagonal of $U$ and $S_v$ the sum of the corresponding rank of $V$. In order that the combined square may have equal sums in all four magic directions, it is necessary and

sufficient that for every row, every column, and each of the two diagonals the diophantine equation

$$nS_u + S_v = S = \frac{n(n^2 - 1)}{2} \tag{4}$$

be satisfied.

This equation becomes an identity in the particular case

$$S_u = S_v = \frac{n(n - 1)}{2}. \tag{5}$$

If this condition is fulfilled for all rows, columns, and diagonals, the two auxiliary squares also will have equal sums in all magic directions, and we accordingly call them "magic" and $n(n-1)/2$ their magic sum.

Now $n(n-)1/2$ is the sum of the $n$ numbers, 0 to $n-1$. The conditions (3), (4), and (5) are therefore fulfilled if each of the numbers 0 to $n-1$ appear exactly once in each row, column, and diagonal of both auxiliary squares. In this case $U$ and $V$ are so-called diagonal *latin squares*; the magic square itself is then a diagonal *Euler square* [**2**].

**2.3.** This special form of auxiliary square is particularly important for the formation of so-called *pandiagonal* magic squares. By this expression [**3**] are meant magic squares in which the *broken diagonals* also give the magic sum. A broken diagonal is traced out by starting from any cell and moving parallel to a principal diagonal $(n-1)$-times, one cell at a time, *cyclically*; that is as though each edge of the square were joined to the opposite edge [**4**]. A magic square is then pandiagonal if equation (4) is satisfied also for the broken diagonals. In the sequel the term *diagonals* will be used to include both the principal and the broken diagonals.

Correspondingly, latin squares, which also contain each element exactly once in each broken diagonal, are called pandiagonal. From two matching pandiagonal latin squares $U$ and $V$, a pandiagonal magic square can be derived by the formula $Un+V$ (or, equally well, $Vn+U$).

**3. Outline of the paper.** In this paper, which forms only the first chapter of a complete and systematic investigation of the whole subject [**5**], we shall restrict ourselves to *prime orders*.

For these it will be shown that all previously known methods for the formation of pandiagonal magic squares can be reduced to the construction of suitable pandiagonal latin squares.

These pandiagonal latin squares have certain special properties, on account of which we classify them in a group, which we call *cyclic squares*.

We then calculate the total number of existing cyclic pandiagonal latin squares (**B.**) and the number of pandiagonal magic squares derivable from them (**C.**).

It will be shown that, on the one hand, the numerical formula thus found contains all previously known pandiagonal magic squares (**D.**), including the

symmetric squares of this kind (**E.**), but that, on the other hand, there exist still other pandiagonal magic squares which are new.

These latter can be formed from pandiagonal latin squares of a new type, which may be obtained from cyclic pandiagonal latin squares, and which we call *semi-cyclic* (**F.**).

Finally the number of *non-pandiagonal* cyclic squares will be calculated as well as the number of non-pandiagonal magic squares which are obtainable directly from cyclic squares (**G.**).

## B. Cyclic Squares

**1. Cyclic arrangements.** We shall designate by *cyclic arrangement* a sequence of $n$ elements (not necessarily all different) in which the last element is followed by the first, so that a closed cycle is formed.

Any arbitrary arrangement, in which the elements are all different, will be denoted by $A_1$. We designate by $a_0, a_1, \cdots, a_{n-1}$ the elements of the sequence determined by this arrangement. We consider all arrangements which can be obtained from $A_1$ by a cyclic shifting of the initial member as identical with $A_1$.

We choose from $A_1$ an arbitrary element $a_t$, and add to its index the numbers $0, p, 2p, 3p, \cdots$ one after another, or, in other words, starting with $a_t$ we jump ahead $p$ elements at a time, coming back at last, with the $n$th jump, to $a_t$.

The cyclic arrangement

$$a_t, a_{t+p}, a_{t+2p}, \cdots, a_{t+(n-1)p}$$

is thus obtained.

We call this arrangement $A_p$ and $p$ its *cycle number.*

All arrangements obtained from $A_1$ with the same cycle number will be considered identical without regard to the choice of $t$.

We can, in $A_1$ and the cyclic arrangements obtained from it, drop the letter $a$ and treat the indices as natural numbers with the stipulation that all numbers greater than $n-1$ or less than zero must be brought into the range 0 to $n-1$ by replacing them by their remainders modulo $n$.

We shall reduce the cycle number $p$ (and likewise all other cycle numbers encountered) to its smallest positive remainder modulo $n$, since all arrangements of the form $A_{p \pm rn}$ are identical with $A_p$.

Hence $p$ can take on, in all, the $n$ different values $0, 1, 2, \cdots, n-1$.

Throughout the rest of this paper we shall assume that *$n$ is an odd prime.* Therefore all values of $p$, excepting 0, are relatively prime to $n$.

If $p=0$, $A_p$ consists of $n$ equal elements; we shall call such an arrangement *equiform.*

If $p \neq 0$, $A_p$ consists of $n$ different elements; such an arrangement we call *latin,* for in a latin square all rows and columns must be arrangements of this form.

There is, therefore, *one* equiform cyclic arrangement, and $n-1$ latin cyclic arrangements.

**2. Construction of cyclic squares.** By a *cyclic square*, we mean a square in which we think of each edge as joined to the opposite edge, and which possesses the following properties:

(a) $n$ different elements appear in the square $n$ times each,
(b) all rows have the same cyclic arrangement,
(c) all columns have the same cyclic arrangement.

*Construction*: Choose two cycle numbers $p$ and $q$ so that at least one of them is different from zero. We fill the first row of the square to be formed with $A_p$, and then complete the columns according to $A_q$.

The square so formed is cyclic.

*Proof:* Any section of the square is of the form shown in Figure 4, in which $B$

$$\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdots & B-2p-2q & B-p-2q & B-2q & B+p-2q & B+2p-2q & \cdots \\
\cdots & B-2p-q & B-p-q & B-q & B+p-q & B+2p-q & \cdots \\
\cdots & B-2p & B-p & B & B+p & B+2p & \cdots \\
\cdots & B-2p+q & B-p+q & B+q & B+p+q & B+2p+q & \cdots \\
\cdots & B-2p+2q & B-p+2q & B+2q & B+p+2q & B+2p+2q & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

FIG. 4

is the index of an arbitrary element $a_B$. Suppose that any arbitrary cell, lying $x$ columns to the right and $y$ rows down from the cell filled by $a_B$, contains the element $a_Z$. Then

$$Z = B + px + qy.$$

Likewise for another arbitrary cell

$$Z' = B + px' + qy',$$

so that

$$Z' = Z + p(x' - x) + q(y' - y).$$

If we set

$$x' - x = \alpha, \qquad y' - y = \beta$$

we obtain

$$Z' = Z + p\alpha + q\beta. \tag{6}$$

For all cells of the same column $\alpha = 0$, so that

$$Z' = Z + q\beta \tag{6a}$$

and for all cells of the same row $\beta = 0$, so that

(6b) $$Z' = Z + p\alpha.$$

Therefore not only does every column form an arrangement $A_q$ (as follows immediately from the construction), but also every row forms an arrangement $A_p$.

At least one of these two is a latin arrangement since the numbers $p$ and $q$ are not both zero; therefore, in each row or in each column there will appear $n$ different elements; hence each element will appear $n$ times in all.

Thus we see that the square is cyclic.

Of the many properties of cyclic squares, we note here only the following:

In (6), for all cells of the same left diagonal, [6] $\alpha=\beta$, so that

(6c) $$Z' = Z + (p + q)\alpha;$$

and for all cells of the same right diagonal, [7] $\alpha = -\beta$, so that

(6d) $$Z' = Z + (q - p)\beta.$$

Hence every left diagonal has the arrangement $A_{p+q}$ and every right diagonal the arrangement $A_{q-p}$.

Two cyclic squares which are obtained, with the same choice of $p$ and $q$, either from the same initial arrangement $A_1$ with different first numbers, or from different initial arrangements $A_1$, will have their groups of $n$ like elements in the same relative positions and will differ only in that the indices which designate each group will be permuted, for certain groups. We shall therefore, consider two such squares as identical.

There are, then, in all *two* different cyclic squares for which either $p$ or $q$ is zero, namely if $p=0$ and $q$ is arbitrary or vice versa.

These two cyclic squares will be excluded in the sequel and will not be taken account of in finding the numerical formulas. Since they have either every row or every column made up of $n$ equal elements, they cannot be used as auxiliary squares in the formation of magic squares.

We therefore may assume that $A_p$ consists of $n$ different elements.

Since we have chosen $A_1$ arbitrarily we can, without loss of generality, take $A_1=A_p$ and thus have

(7) $$p = 1$$

always. Hence we obtain *all* the different cyclic squares when $q$ runs through the values 1 to $n-1$. The total number of cyclic squares, which we call $C$ [8], is then:

(8) $$C = n - 1.$$

**3. Pandiagonal latin squares.** Since we have excluded the cases where $p=0$ or $q=0$, $A_p$ and $A_q$ are always latin arrangements; thus all cyclic squares considered here are also latin squares.

In order that a cyclic latin square be *pandiagonal*, the cycle numbers of the two diagonal directions must also be different from zero. By (6c) and (6d) these

cycle numbers are $p+q$ and $q-p$, which, by (7), become $q+1$ and $q-1$. Therefore necessary and sufficient conditions for pandiagonality are:

(9) $$q \neq n - 1$$

and

(10) $$q \neq 1.$$

Hence only the $n-3$ values from 2 to $n-2$ are permissible for $q$, so that the total number of pandiagonal cyclic squares [9] is:

(11) $$C(P) = n - 3.$$

**4. Matching of two cyclic squares.** The following condition is necessary and sufficient that two cyclic squares *match* to form an Euler square:

(12) *If two cells are filled with the same element in one cyclic square, they must be filled with different elements in the other.*

It follows from (6) that two cells of a cyclic square can be filled with the same element if and only if

(13) $$p\alpha + q\beta \equiv 0 \pmod{n}.$$

By (7) this becomes

(14) $$\alpha + q\beta \equiv 0 \pmod{n}.$$

If, now, in the first cyclic square

(14a) $$\alpha + q_1\beta \equiv 0 \pmod{n}$$

and if at the same time in the second cyclic square

(14b) $$\alpha + q_2\beta \equiv 0 \pmod{n},$$

it follows that

$$\beta(q_2 - q_1) \equiv 0 \pmod{n};$$

and also we readily find that

$$\alpha(q_2 - q_1) \equiv 0 \pmod{n}.$$

Since, however, for two different cells $\alpha$ and $\beta$ cannot both be zero, $q_2$ would have to equal $q_1$. Hence the squares would be identical, since a cyclic square is uniquely determined by the choice of $q$.

Consequently all cyclic squares which are different from one another match in pairs to form an Euler square, since the condition for this is merely:

(15) $$q_2 \neq q_1.$$

If $q_1$ be chosen, $q_2$ has then one less value it can take on; so that for each of the $n-1$ cyclic squares counted in (8) there are $n-2$ matching pairs, and for

each of the $n-3$ pandiagonal cyclic squares counted in (11) there are $n-4$ matching pandiagonal pairs. Hence the total number of Euler squares formed in this way is

$$CC = (n - 1)(n - 2), \tag{16}$$

and the number of pandiagonal Euler squares so found is

$$C(PP) = (n - 3)(n - 4). \tag{17}$$

In these formulas all Euler squares are considered as identical which are obtainable from each other by interchanging the indices of two or more groups of $n$ like elements within one or both of the two formative latin squares.

## C. Derivation and Enumeration of Pandiagonal Magic Squares

If we choose as elements of the arrangement $A_1$ the numbers $0, 1, \cdots, n-1$ in any order, we obtain from every pandiagonal cyclic square a latin square which contains each of the $n$ numbers $n$ times. Since all $n$ numbers, which together give the magic sum $n(n-1)/2$, appear in every row, column, and diagonal, the square is pandiagonal.

We can associate the numbers $0, 1, \cdots, n-1$ with the elements $a_0, a_1, \cdots, a_{n-1}$ arbitrarily, hence the number of possible ways of making this association is:

$$W(P) = n!. \tag{18}$$

If we form two such squares $U$ and $V$ with cycle numbers $q_1$ and $q_2$, the association of numerical values in $V$ is independent of that in $U$ (and *vice versa*). The total number of ways of association for each combination is then:

$$W(PP) = n!^2. \tag{19}$$

If $q_2$ is different from $q_1$ and hence condition (12) satisfied, condition (2) is also fulfilled. *The matching of two auxiliary squares is, then, independent of the association of numerical values.*

Since $U$ and $V$ are pandiagonal latin squares, equation (5), and therefore also equation (4), is satisfied for all rows, columns and diagonals.

We can consequently combine $U$ and $V$ to form a pandiagonal magic square, using either the formula $Un+V$ or $Vn+U$. We shall, however, count such a pair as only *one* square, for these two forms can be obtained from one another by interchanging $q_1$ and $q_2$, and if we let $q_1$ and $q_2$ run through all possible values we obtain them both using only one method of combination.

Since for each of the $(n-3)(n-4)$ combinations counted in (17) we can, according to (19), associate the numerical values in $n!^2$ ways, we obtain the total number of pandiagonal magic squares so formed by multiplication of formulas (17) and (19). These squares fall into sets of eight each, such that the squares of each set can be obtained from each other by rotation or reflection. We shall, as is customary, regard these as identical and count them as only one square.

The number of pandiagonal magic squares formed from cyclic squares is therefore:

$$M(PP) = \frac{(n-3)(n-4)n!^2}{8}. \tag{20}$$

## D. Relations to Known Methods

**1. Particular properties of cyclic squares.** In order to show that formula (20) really includes all previously known pandiagonal magic squares of prime order, we next call attention to the following property of cyclic squares:

Two arbitrary elements may have as indices the numbers

$$Z_0 = Z - B \qquad \text{and} \qquad Z_1 = Z' - B,$$

so that (6) becomes:

$$Z_1 = Z_0 + p\alpha + q\beta. \tag{21}$$

Now suppose that $Z_0$ lies in the cell $F_{0,0}$ and $Z_1$ in the cell $F_{\alpha,\beta}$, so that we can pass from $F_{0,0}$ to $F_{\alpha,\beta}$ by a *step* of $\alpha$ columns to the right and $\beta$ rows downward, or briefly, by the step $(\alpha, \beta)$.

If this step be repeated, we arrive at, one after another,

$$\begin{array}{lll} F_{2\alpha,2\beta} & \text{with} & Z_2 = Z_1 + p\alpha + q\beta = Z_0 + 2(p\alpha + q\beta), \\ F_{3\alpha,3\beta} & \text{with} & Z_3 = Z_0 + 3(p\alpha + q\beta), \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots . \end{array}$$

With the $w$th step we arrive at

$$F_{w\alpha,w\beta} \qquad \text{with} \qquad Z_w = Z_0 + w(p\alpha + q\beta), \tag{22}$$

and with the $n$th step at

$$F_{n\alpha,n\beta} = F_{0,0} \qquad \text{with} \qquad Z_n = Z_0 + n(p\alpha + q\beta) \equiv Z_0 \pmod{n},$$

that is, back to the cell with which we started.

Hence we can always replace $\alpha$ and $\beta$ by their smallest remainders modulo $n$. If at least one of them is not zero the cells $F_{0,0}$, $F_{\alpha,\beta}$, $F_{2\alpha,2\beta}$, $\cdots$, $F_{(n-1)\alpha,(n-1)\beta}$ are necessarily different. We call $n$ cells connected in this way—independently of their order—the *path* $(\alpha, \beta)$.

Two paths $(\alpha, \beta)$ and $(\alpha', \beta')$, starting from the same initial cell, will be considered identical if

$$\alpha\beta' \equiv \alpha'\beta \pmod{n}$$

since both paths then contain the same cells and differ only in their order.

We obtain, therefore, starting from an arbitrarily chosen cell:

(a) $n-1$ different paths, if we set, for example, $\beta=1$ and let $\alpha$ run through the values 1 to $n-1$;

(b) two more paths, different from these and from each other, namely, the columns with $\alpha=0$, $\beta$ arbitrary and the rows with $\beta=0$, $\alpha$ arbitrary.

So there are, in all, $n+1$ paths each of which contains the same initial cell and $n-1$ other cells. Hence the $n+1$ paths contain all $1+(n+1)(n-1)=n^2$ cells of the square.

We obtain the $n$ indices lying in the cells of the path $(\alpha, \beta)$ if, in the expression

$$Z_w = Z_0 + w(p\alpha + q\beta) \tag{22}$$

$w$ takes on the values 0 to $n-1$. The path $(\alpha, \beta)$ contains therefore the elements with indices $Z_0, Z_1, \cdots, Z_{n-1}$ in the arrangement $A_{p\alpha+q\beta}$.

This arrangement is equiform if

$$p\alpha + q\beta \equiv 0 \pmod{n}. \tag{13}$$

In this case the path consists of $n$ equal elements; accordingly we call it an *equiform path.*

If (13) is not satisfied, $A_{p\alpha+q\beta}$ is a latin arrangement; the path then consists of $n$ different elements. We call it, therefore, a *latin path.*

If $p$ and $q$ are different from zero, by use of (7), formula (13) becomes

$$\alpha + q\beta \equiv 0 \pmod{n}. \tag{14}$$

Then the arrangements of the $n$ paths which are obtained from $\beta=1$, $\alpha=0, 1, \cdots, n-1$, are all different. For only one of them is (14) satisfied, namely when

$$\alpha = -q. \tag{23}$$

The $n-1$ other paths are latin.

For the one remaining path $\beta=0$ while $\alpha$ may be taken equal to any of the numbers $1, 2, \cdots, n-1$; and (14) is not satisfied for any of these values.

Hence the theorem:

(24) *Of the $n+1$ paths, one is equiform; the others are latin.*

This theorem remains true if one of the two numbers $p$ and $q$ be set equal to zero.

If $p=0$, (13) becomes $q\beta\equiv 0 \pmod{n}$, and is therefore satisfied only for the *one* path for which $\beta=0$; in this case all rows are equiform.

If $q=0$, (13) becomes $p\alpha\equiv 0 \pmod{n}$ and is therefore satisfied only for the *one* path for which $\alpha=0$; in this case all columns are equiform.

The two cyclic squares so formed (excluded in the enumeration in Part B, §2,

above) are, it is true, not suitable as auxiliary squares in the formation of magic squares; they nevertheless serve to make clear the relation to a group of other, very widely known, methods for the formation of magic squares, as we now wish to show.

**2. Transformation of a "position fondamentale."** We first form two non-latin cyclic squares, $U$ and $V$ with $p_1=0$, $q_1=1$ and $p_2=1$, $q_2=0$. With the $n$ elements of the arrangement $A_1$ we associate in both cyclic squares the numbers 0 to $n-1$ in natural order (beginning with 0). We combine the two squares so obtained by the formula $Un+V$.

Since in $U$ all paths excepting the rows, and in $V$ all paths excepting the columns are latin, the combined square is not magic in the rows and columns (except for the middle ones), but it is magic in all other step-directions.

The square so obtained contains the numbers 0 to $n^2-1$ in their natural order. It can therefore be written down immediately and, under the name of *carré naturel* or *position fondamentale*, it has served as a foundation for a large group of known methods for the construction of magic squares (for pandiagonal squares, in particular). These methods make use—in the most varied disguises—of the fact that all step-directions of the "carré naturel" (excepting the rows and columns) give the magic sum. They rest on the possibility of bringing such magic paths (with the help of the middle row and middle column, which are magic in any case) by proper transformations into the rows, columns, and diagonals of the square to be formed. It really makes no difference whether the $n$ numbers belonging to a magic path be obtained by cyclic steps within the square, or whether, for this purpose, the "carré naturel" (or another "position fondamentale" obtained from it by suitable permutations) be thought of as written infinitely often in a coördinate system (whose axes are parallel to two edges of the square) and the different step-directions be characterized by their coördinates (corresponding to $\alpha$ and $\beta$) [10]. Since, as has been shown, the "carré naturel" can always be broken up into two cyclic squares, and since these can be transformed by a corresponding change of the values of $p$ and $q$ into pandiagonal cyclic squares which are among those counted in (11), no pandiagonal magic squares can be formed in this way other than those already included in formula (20).

**3. Uniform step squares. 3.1.** On another aspect of this same principle rests the other group of previously known methods, which depend on the possibility of filling the magic square to be formed with the numbers 1 to $n^2$ (or, as we should say, 0 to $n^2-1$) by traversing the square, proceeding uniformly. To this end we divide the numbers 0 to $n^2-1$ into $n$ groups of $n$ numbers each, the first group comprising the numbers 0 to $n-1$, the second $n$ to $2n-1$, *etc.* We call such a group an *n-ad*. To fill the square the numbers of an $n$-ad in natural order are placed in a path, and the numbers of every other $n$-ad are placed in the same order in paths parallel to the first. Furthermore, the initial cell of each $n$-ad is reached from the end cell of the previous $n$-ad by a step which is the

same for all initial cells. In other words, the initial cells of each $n$-ad form a path and so also do any $n$ cells which are filled with numbers congruent modulo $n$. In order to be able to write down the magic square by proceeding uniformly, it is then sufficient to know the step with which to pass from one number to the next of the same $n$-ad (*principal step*), and the step with which to pass from the end number of one $n$-ad to the first number of the next $n$-ad (*cross step*) [11].

If such a uniform step square be broken up into two auxiliary squares by the formula $Un+V$, any $n$ numbers belonging to the same $n$-ad form a path which is equiform in $U$ and latin in $V$; every $n$ congruent numbers form a path which is latin in $U$ and equiform in $V$. Since all paths of the first group are parallel to one another, and likewise for the second group, both auxiliary squares must be cyclic.

**3.2.** All cyclic squares are, as has been shown, uniquely determined by the choice of $q$. They can be equally well characterized, however, by their equiform step-direction, since, according to (14), $\alpha/\beta$ is determined by $q$. Let $G$ denote the equiform step-direction and set, by use of (14),

$$G_1 \equiv \frac{\alpha_1}{\beta_1} \equiv -q_1 \pmod{n}, \qquad G_2 \equiv \frac{\alpha_2}{\beta_2} \equiv -q_2 \pmod{n}.$$

Then in a uniform step square every path $G_1$ is filled with numbers of the same $n$-ad and every path $G_2$ with numbers which give equal remainders on division by $n$. So we pass from any number to the next number of the same $n$-ad by a step $(\alpha_1, \beta_1) \equiv G_1 \pmod{n}$ and from the last number of the $k$th $n$-ad (namely $(k-1)n+n-1=kn-1$) to the first number of the following $n$-ad (namely $kn$) by a step $(\alpha_1, \beta_1)+(\alpha_2, \beta_2) \equiv G_1+G_2 \pmod{n}$.

The uniform step method, using the principal step and the cross step, depends then on the fact that, instead of $q_1$ and $q_2$, the two quantities

$$G_1 \equiv -q_1 \pmod{n}, \qquad G_1 + G_2 \equiv -(q_1+q_2) \pmod{n}$$

are taken as fixed. The method further requires that all paths $G_1$ in $V$ and all paths $G_2$ in $U$ consist of the numbers 0 to $n-1$ in natural order. These conditions may also be expressed in the following manner:

(25) *Uniform step squares are made up of cyclic squares, in which the elements of the first row are the numbers* 0 *to* $n-1$ *either in natural order with the first number arbitrary, or in a cyclic arrangement derived from the natural order by use of an arbitrary cycle number.*

*Proof*: Denote by $A_g$ the arrangement of the $n$ elements obtained by traversing the $n$ cells of the path $G$ with the step $(\alpha, \beta)$, and by $A_{g'}$ the arrangement obtained by replacing $(\alpha, \beta)$ by an equivalent step $(\alpha', \beta')$, so that $\alpha\beta' \equiv \alpha'\beta \pmod{n}$. Then $A_g$ and all arrangements $A_{g'}$ obtained from it are cyclic arrangements derivable from a common initial arrangement $A_1$. If, now, $G$ be latin in one auxiliary square and equiform in the other so that, for instance, $g_1 \neq 0$, $g_2=0$,

then all values of $g_2'$ are zero while $g_1'$ can take on $n-1$ different values (including $g_1$ itself). It suffices, therefore, that *one* of these arrangements contains the numbers 0 to $n-1$ in natural order, and for this it is necessary and sufficient that the numbers in this order be associated with the elements in the arrangement $A_1$, where by $A_1$ we may mean the arrangement of the elements in the first row (or, equally well, of those in any other latin path).

**3.3.** In order to determine the total number of pandiagonal uniform step squares we have thus only to find in how many ways the numerical values can be associated in each of the $(n-3)(n-4)$ combinations counted in (17).

This association is determined for each of the two auxiliary squares

(a) by a cycle number $c$, by means of which, out of the natural sequence $0, 1, 2, \cdots, n-1$, the sequence is derived in which the numbers are allotted to the elements of the arrangement $A_1$;

(b) by the number with which $A_c$ begins.

Since $c$ can be chosen in $n-1$ ways, and the initial number of any arrangement in $n$ ways, there are for each of the two auxiliary squares, independently of the other,

$$ {}_uW(P) = n(n-1) \tag{26} $$

possibilities of association, hence for each combination

$$ {}_uW(PP) = n^2(n-1)^2 \tag{27} $$

possibilities.

Multiplication of formulas (17) and (27) gives therefore $(n-3)(n-4)(n-1)^2n^2$ for the total number of pandiagonal uniform step squares [**12**]; hence excluding the forms obtainable from one another by rotation or reflection:

$$ {}_uM(PP) = \frac{(n-3)(n-4)(n-1)^2n^2}{8}. \tag{28} $$

From (18) and (26) there results:

$$ \frac{W(P)}{{}_uW(P)} = \frac{n!}{n(n-1)} = (n-2)!. $$

Hence

$$ W(P) = (n-2)!\,{}_uW(P), \tag{29} $$

$$ W(PP) = (n-2)!^2\,{}_uW(PP), \tag{30} $$

$$ M(PP) = (n-2)!^2\,{}_uM(PP). \tag{31} $$

Thus in singling out the uniform step squares we obtain only the $(n-2)!^2$th part of the total number. Hence, conversely, the total number can be obtained from (28) by multiplication by $(n-2)!^2$ [**13**].

**4. Other methods.** All other previously known procedures for the formation of pandiagonal magic squares of prime order—insofar as they are not experimental—may be referred to one of the methods already discussed. In particular, it makes really no difference whether the magic squares be broken up into two auxiliary squares in the manner described or whether the numbers 0 to $n^2-1$ (or 1 to $n^2$) be expressed as binomials or trinomials [**14**]. Also, the conditions for the formation and combination of auxiliary squares remain essentially the same if, instead of the quantities $p$ and $q$, other quantities dependent on them be used to characterize a cyclic square. The two procedures,

(a) the formation of a basic square filled with arbitrary elements or letters (*carré litéral*) to which the numerical values will later be associated,

(b) the replacement of the two cycle numbers by *one* of them, the other being set equal to one,

have previously been used, independently of one another, in a great many different ways [**15**]. The combination, however, made above of these two procedures makes it possible to bring a variety of known methods under a particularly simple general form. Thereby the cycle numbers $p$ and $q$ will no longer be looked upon as absolute quantities, but their *ratio* alone will be regarded as characteristic of the structure of a cyclic square. For composite orders large groups of new methods can be obtained in this way. For prime orders the derivation of general numerical formulas is made possible (or at least facilitated), and likewise for certain classes of magic squares which we now consider.

## E. Symmetry

**1. General conditions for symmetry.** Several authors have given great attention to the question of forming *symmetric* magic squares, *i.e.* squares in which any two numbers lying symmetrically with respect to the middle point have the same sum. This sum will be called the *complementary sum* of the magic square.

Generally we define the *complementary sum* as the sum of the largest and the smallest number in a square or (equivalently) as the $n$th part of twice the magic sum. Thus the complementary sum in a square formed from the numbers 0 to $n^2-1$ is $n^2-1$, and in each of the two auxiliary squares the complementary sum is $n-1$. Then we have the theorem:

(32) *A square formed from $n^2$ consecutive integers is symmetric when, and only when, both its auxiliary squares are symmetric* [**16**].

*Proof*: We denote by $K_u$ the sum of two arbitrary numbers which lie symmetrically in the auxiliary square $U$ and by $K_v$ the sum of the corresponding pair in the auxiliary square $V$. In order that the combined square, obtained by the formula $Un+V$, be symmetric the diophantine equation

$$nK_u + K_v = n^2 - 1 \tag{33}$$

must hold throughout. The only suitable positive integral solution of this equation is

$$K_u = K_v = n - 1, \tag{34}$$

which is equivalent to (32). For if $K_u$ were smaller than $n-1$, $K_v$ would have to be greater than $2(n-1)$ which is impossible since no number in $V$ is greater than $n-1$; on the other hand, if $K_u$ were greater than $n-1$, $K_v$ would have to be negative.

In a cyclic square (which may or may not be pandiagonal) let us denote by $F_{0,0}$ the middle cell and by $F_{i,j}$ an arbitrary cell which lies $i$ columns to the right and $j$ rows down from the middle cell. The cell lying symmetrically is then $F_{n-i,n-j} = F_{-i,-j}$.

Suppose now the cell $F_{i,j}$ filled with the element $a$ and let $(\alpha, \beta)$ be the equiform step-direction. Then the $n$ elements $a$ lie in cells of the form $F_{i+w\alpha,j+w\beta}$ where $w$ runs through the values 0 to $n-1$.

Furthermore, suppose $F_{-i,-j}$ filled with the element $b$, then the $n$ elements $b$ lie in cells of the form $F_{-i+w\alpha,-j+w\beta}$. We should obtain the same cells if, starting with $F_{-i,-j}$, we traced out the path in the opposite direction, so that we may just as well write $F_{-i-w\alpha,-j-w\beta}$. Since for any value of $w$ the two cells $F_{i+w\alpha,j+w\beta}$ and $F_{-i-w\alpha,-j-w\beta}$ lie symmetrically, we have the following theorem:

(35) *If, in a cyclic square one element a lies symmetrically to an element b, then every element a lies symmetrically to an element b, and vice versa.*

If $i=j=0$, then the cells $F_{i+w\alpha,j+w\beta} = F_{w\alpha,w\beta}$ and $F_{-i-w\alpha,-j-w\beta} = F_{-w\alpha,-w\beta}$ become cells of the same path passing through $F_{0,0}$ so that $a=b$. If we set $a=b=m$, then $m$ is the element lying in the middle cell. Thus

(36) *Any cell which is filled with the same element as the middle cell lies symmetrically to a cell filled with the same element.*

**2. Enumeration.** In order that an auxiliary square be symmetric, the numerical values must be associated in such a way that, in all pairs of equiform paths lying symmetrically,

$$a + b = n - 1; \tag{37}$$

hence in the path passing through the middle cell

$$a = b = m = \frac{n-1}{2} \cdot \tag{38}$$

From (35) we see that in order to do this it is sufficient to associate the numbers properly in any *one* latin path which passes through the middle cell. Thereupon we obtain the following rule for the formation of symmetric auxiliary squares:

(39) *First the middle cell is filled with the number* $m=(n-1)/2$. *Then the cells of the middle row to the left of the middle cell are filled arbitrarily with numbers different from one another and from the middle number, but no two of which are complementary* (*i.e. have the sum* $n-1$). *Finally the cells of the middle row to the right of the middle cell are filled with the numbers complementary to the corresponding numbers on the left.*

By (38) we have

(40) $$n = 2m + 1.$$

Not counting the middle number $m$, there remain $n-1=2m$ numbers which form $m$ complementary pairs each with the sum $2m$. These $m$ pairs can be assigned in $m!$ ways to the $m$ elements lying in the middle row to the left of the middle cell. Also, from each of these pairs, independently of the others, we may choose either the smaller or the larger number, which gives us $2^m$ possibilities. Once this choice has been made the assignment of numerical values is, according to (39), completely fixed. There are in all, then, in each cyclic square, and hence also in each pandiagonal cyclic square,

(41) $${}_sW(P) = m!\cdot 2^m$$

possible ways of assigning numerical values symmetrically, therefore, in each combination of two pandiagonal cyclic squares,

(42) $${}_sW(PP) = m!^2\cdot 2^{2m}$$

possibilities. From this number we can, referring to (17) and (40), obtain the total number of symmetric pandiagonal magic squares by multiplying by $(n-3)(n-4)=2(m-1)(2m-3)$. For this total number we have (if, as before, we count the eight related squares as one):

(43) $${}_sM(PP) = 2^{2m-2}(m-1)(2m-3)m!^2 = 2^{n-4}(n-3)(n-4)\left(\frac{n-1}{2}\right)!^2.$$

**3. Symmetric uniform step squares.** In order that a symmetric pandiagonal magic square be also a uniform step square, we must have (25) satisfied as well as (39). Therefore we can no longer assign the $m$ complementary pairs of numbers with the $m$ elements, lying in the middle row on the left of the middle cell, in $m!$ ways as before, but only in $m$ ways, for if *one* of these numbers is chosen, the cycle number $c$ is determined by the difference between this number and the middle number $m$ and hence the position of the other numbers is fixed. In each of the $m$ pairs then the position of the greater number or of the smaller number is fixed by the first number chosen. Thus the $m!\cdot 2^m$ ways of association, counted in (41), are reduced to

(44) $${}_{su}W(P) = 2m = n-1.$$

The same result may be reached by starting with uniform step squares. By

the choice of the cycle number $c$, which may be made in $n-1$ ways, and by the middle number $m$, the initial cell of the arrangement $A_c$ (which we here take for convenience as the arrangement of the middle row) is determined, so that (26) reduces to (44).

For each combination of two pandiagonal cyclic squares the number of ways in which the numerical values can be associated so as to satisfy the conditions of symmetry and the uniform step-property are, then

$$(45) \qquad {}_{su}W(PP) = (n-1)^2.$$

From this number we obtain the total number of pandiagonal symmetric uniform step squares by multiplying by (17) and there results (again dividing by 8):

$$(46) \qquad {}_{su}M(PP) = \frac{(n-3)(n-4)(n-1)^2}{8}.$$

From (26) and (44) it follows that

$$\frac{{}_{u}W(P)}{{}_{su}W(P)} = \frac{n(n-1)}{n-1} = n.$$

Hence

$$(47) \qquad {}_{u}W(P) = n \cdot {}_{su}W(P),$$

$$(48) \qquad {}_{u}W(PP) = n^2 \cdot {}_{su}W(PP),$$

$$(49) \qquad {}_{u}M(PP) = n^2 \cdot {}_{su}M(PP).$$

So we see that the number of pandiagonal symmetric uniform step squares may be obtained from the total number of pandiagonal uniform step squares by division by $n^2$ [17]. That indeed this must be true is seen also from the following consideration:

Every pandiagonal magic square remains pandiagonal and magic if its rows or its columns be shifted cyclically. This can be done in $n$ ways with the rows and in $n$ ways with the columns, thus in $n^2$ ways in all. In each of the $n^2$ squares related in this way the middle cell is filled with a different number, in *one* of them, therefore, with the middle number $\frac{1}{2}(n^2-1)=2m(m+1)$. If *one* of these $n^2$ squares is a uniform step square so are all the others and therefore so is the one with the middle number $\frac{1}{2}(n^2-1)$ in the middle cell. However, a uniform step square which has this middle number in the middle cell is then symmetric, for every arrangement $A_c$ derived cyclically from $A_1$ becomes symmetric if the initial number be so chosen that $m=\frac{1}{2}(n-1)$ becomes the middle number. Among the $n^2$ pandiagonal uniform step squares there is, therefore, always *one* symmetric square.

## F. Semi-cyclic Pandiagonal Squares and Derivation of New Kinds of Magic Squares

As we have seen, the numerical formula (20) includes all previously known pandiagonal magic squares of prime order. The question which next arises is whether other such squares exist. This question—so far as I have been able to ascertain—has, in the previous literature on the subject, either been passed over or answered in the negative without proof. That this question requires investigation is indicated, however, by the fact that formula (5), which served as our point of departure, gave only a special solution of equation (4), and since by the requirement that the two auxiliary squares be pandiagonal latin squares, we have introduced a further special condition which is not essential.

However, even if these limitations be maintained, we can infer the completeness of formula (20) only insofar as we deal only with pandiagonal magic squares which are reducible to *cyclic* pandiagonal latin squares. It is therefore of particular interest to see if there exist pandiagonal latin squares which are *not* cyclic. For composite orders this is immediately evident; for prime orders, however, the question has never been considered.

A latin square may be

(a) either cyclic in *two* magic directions, whereupon it follows from (6a, b, c, d) that the two other magic directions are also cyclic,

(b) or cyclic in only *one* magic direction,

(c) or cyclic in *no* magic direction.

Since we have called the squares of group (a) *cyclic*, we accordingly call those of group (b) *semi-cyclic* and those of group (c) *non-cyclic*. In order to show that there are pandiagonal latin squares which do not belong to (a) and hence that there are possibly pandiagonal magic squares not included in (20), it is then sufficient to demonstrate the existence of semi-cyclic or non-cyclic pandiagonal latin squares.

This investigation would carry us beyond the limits of this paper. I have already discussed a special kind of semi-cyclic pandiagonal latin squares and the pandiagonal magic squares obtained from it in the paper "*Ueber irreguläre pandiagonale lateinische Quadrate mit Primzahlseitenlänge und ihre Bedeutung für das n-Königinnenproblem sowie für die Bildung magischer Quadrate*" [18]. Therein I have shown that semi-cyclic pandiagonal latin squares can be derived from pandiagonal cyclic squares by interchanging, in pairs, 4 parallel rows, columns, or diagonals, if

$$q^2 \equiv -1 \pmod{n}, \tag{50}$$

and therefore $n$ (which is supposed to be a prime number greater than 5) has the form $4k+1$. I have generalized this result in an article soon to appear, in the course of which the following theorem is proved:

*If the prime order $n$ has the form*

(51), (52) $$n = 2sk + 1, \qquad s > 1, \qquad k > 1,$$

*and if*

(53) $$(q - 1)^{2s} \equiv (q + 1)^{2s} \pmod{n},$$

(54), (55) $$(q - 1)^{2s'} \not\equiv (q + 1)^{2s'} \pmod{n}, \qquad \text{whenever } 0 < s' < s,$$

*then from each pandiagonal cyclic square formed with $q$, semi-cyclic pandiagonal latin squares can be derived by interchanging, in pairs, $2s$ rows or $2s$ columns. If the pandiagonal cyclic square be formed with*

(56) $$q' \equiv \frac{q - 1}{q + 1} \pmod{n},$$

*then semi-cyclic pandiagonal latin squares can be derived from it by interchanging, in pairs, $2s$ parallel diagonals.*

Since the problem is thus referred to that of finding suitable solutions of (53), (54), we can in this way also derive numerical formulas, and determine the conditions of symmetry of the pandiagonal latin and magic squares (as well as of the solutions of the problem of $n$ queens) thus obtained.

## G. Derivation of Non-pandiagonal Magic Squares from Cyclic Squares

**1. Non-pandiagonal cyclic squares.** If we abandon the condition of pandiagonality we may in the formation of cyclic squares use for $q$ also the two values, $n-1$ and 1, hitherto excluded by (9) and (10). The number of non-pandiagonal cyclic squares is, therefore:

(57) $$C(N) = 2.$$

Addition of formulas (11) and (57) gives us formula (8) again, *viz.*,

(8a) $$C(N) + C(P) = C = n - 1.$$

If $q=1$, we have $q-1\equiv 0 \pmod{n}$ so that, by (6d), (7), (13), and (14), the equiform step-direction coincides with the right diagonals, as shown, for example, by Figure 5. If $q=n-1$, the left diagonals become equiform, since then $q+1\equiv 0 \pmod{n}$.

| | | | | |
|---|---|---|---|---|
| $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_0$ |
| $a_2$ | $a_3$ | $a_4$ | $a_0$ | $a_1$ |
| $a_3$ | $a_4$ | $a_0$ | $a_1$ | $a_2$ |
| $a_4$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ |

Fig. 5

If we associate the numbers $0, 1, \cdots, n-1$ with the elements arbitrarily, only one of the $n$ equiform diagonals will give the magic sum $n(n-1)/2$, namely, the one filled with the middle number $m=(n-1)/2$.

In order that such an auxiliary square be magic we must therefore assign the middle number $m$ to the element which fills the equiform principal diagonal (and hence the middle cell). For example, in Figure 5 we must set $a_4=2$.

The three other magic directions will immediately give the required equal sums, since they contain latin arrangements. Hence the $n-1$ other numbers can be associated with the $n-1$ remaining elements arbitrarily.

The number of ways in which the numerical values may be associated in non-pandiagonal cyclic squares is then

$$W(N) = (n - 1)!. \tag{58}$$

Non-pandiagonal magic squares can be formed from cyclic auxiliary squares in two ways:

(a) by combination of two non-pandiagonal cyclic squares,
(b) by combination of a non-pandiagonal with a pandiagonal cyclic square.

Combinations of the first group will be denoted by $(NN)$, those of the second by $(NP)$ or $(PN)$ according to the order of the two auxiliary squares and, if this order is indifferent, by $(\overline{NP})$.

**2. Combination of two non-pandiagonal cyclic squares** $(NN)$. From (15) we see that each of the two non-pandiagonal cyclic squares matches with the other so that the number of matching combinations of two such squares is:

$$C(NN) = 2. \tag{59}$$

The number of ways in which the numerical values can be associated for each of these combinations is, according to (58),

$$W(NN) = (n - 1)!^2. \tag{60}$$

Hence the number of non-pandiagonal magic squares formed in this way is $2(n-1)!^2$, and hence, by the reasoning immediately preceding (20), we have, on dividing by 8,

$$M(NN) = \frac{(n - 1)!^2}{4} \cdot \tag{61}$$

It may be remarked that this formula is applicable not only for prime orders but for all odd orders; for from $q = \pm 1$ it follows that $q \pm 1 = \pm 2$ and therefore is relatively prime to any odd $n$ [**19**].

**3. Combination of a non-pandiagonal with a pandiagonal cyclic square $(\overline{NP})$.** Formula (15) shows that every non-pandiagonal cyclic square matches with every pandiagonal one; thus the total number of combinations of cyclic squares one of which is pandiagonal but the other not, is obtained by combining each of the $n-3$ pandiagonal cyclic squares, counted in (11), with each of the 2 non-pandiagonal cyclic squares, counted in (57). Hence:

$$C(NP) = C(PN) = 2(n - 3). \tag{62}$$

Each of these combinations consists of two cyclic squares which, in contrast with the cases $(PP)$ and $(NN)$, are formed in *different* ways; therefore no Euler square of the group $(NP)$ can be identical with a square of the group $(PN)$. Hence the total number of matching combinations of a pandiagonal cyclic

square and a non-pandiagonal cyclic square is:

$$(63)\qquad C(NP) + C(PN) = C(\overline{NP}) = 4(n - 3).$$

Addition of formulas (17), (59) and (63) gives formula (16) again, *viz.*:

$$(16a)\qquad C(PP) + C(NN) + C(\overline{NP}) = CC = (n - 1)(n - 2).$$

The number of ways in which the numerical values can be associated is, by (18) and (58), for every combination $(\overline{NP})$:

$$(64)\qquad W(\overline{NP}) = n!(n - 1)! = n(n - 1)!^2.$$

From this number, we obtain the total number of non-pandiagonal magic squares so formed [20] by multiplication by (63); so that (again dividing by 8):

$$(65)\qquad M(\overline{NP}) = \frac{n(n - 3)(n - 1)!^2}{2}.$$

In these squares either all left or all right broken diagonals are magic, according as $q$ is equal to 1 or to $n-1$ in the non-pandiagonal auxiliary square.

**4. Uniform step squares.** In order to construct uniform step squares condition (25) must be fulfilled for the non-pandiagonal cyclic squares just as for the pandiagonal cyclic squares. Since the middle number is fixed in a non-pandiagonal cyclic square, the association of numerical values can no longer be made in $(n-1)!$ ways but in only

$$(66)\qquad {}_uW(N) = n - 1$$

ways. Hence, for every combination of two non-pandiagonal cyclic squares, we have as the number of possible ways of assigning numerical values:

$$(67)\qquad {}_uW(NN) = (n - 1)^2.$$

Multiplying by (59) and dividing by 8, we obtain as the total number of uniform step squares so formed:

$$(68)\qquad {}_uM(NN) = \frac{(n - 1)^2}{4}.$$

For combinations of a pandiagonal cyclic square and a non-pandiagonal cyclic square we obtain, by multiplication of (26) and (66), as the number of ways of numerical association:

$$(69)\qquad {}_uW(\overline{NP}) = n(n - 1)^2,$$

and from this, multiplying by (63) and dividing by 8, we find for the number of non-pandiagonal uniform step squares in this group:

$$(70)\qquad {}_uM(\overline{NP}) = \frac{n(n - 3)(n - 1)^2}{2}.$$

From (58) and (66) it follows that

$$\frac{W(N)}{{}_uW(N)} = \frac{(n-1)!}{n-1} = (n-2)!.$$

Hence

$$W(N) = (n-2)! \cdot {}_uW(N). \tag{71}$$

In the same way, as above in (30) and (31), we obtain from this:

$$M(NN) = (n-2)!^2 \cdot {}_uM(NN), \tag{72}$$

and

$$M(\overline{NP}) = (n-2)!^2 \cdot {}_uM(\overline{NP}). \tag{73}$$

Therefore here, too, if the number of non-pandiagonal uniform step squares is known, the total number of non-pandiagonal magic squares formed from cyclic squares can be obtained from it by multiplication by $(n-2)!^2$ [**21**].

**5. Symmetry. 5.1.** In deriving the conditions for assigning numerical values so that the complementary pairs would lie symmetrically, in paragraphs E1, E2 above, we made no use of the supposition of pandiagonality. Hence the number of ways in which this can be done is the same for non-pandiagonal as for pandiagonal cyclic squares, so that we have, as in (41):

$${}_sW(N) = {}_sW(P) = m! \cdot 2^m. \tag{74}$$

Therefore, for each arbitrary combination

$${}_sW(NN) = {}_sW(\overline{NP}) = {}_sW(PP) = m!^2 \cdot 2^{2m} \tag{75}$$

in accordance with (42).

From this number we obtain the number of non-pandiagonal symmetric squares contained in (61) and the number in (65) by multiplying by (59) and (63) respectively and dividing by 8, *viz.*:

$${}_sM(NN) = 2^{2m-2} \cdot m!^2 = 2^{n-3}\left(\frac{n-1}{2}\right)!^2; \tag{76}$$

$${}_sM(\overline{NP}) = 2^{2m}(m-1)m!^2 = 2^{n-2}(n-3)\left(\frac{n-1}{2}\right)!^2. \tag{77}$$

**5.2. Symmetric uniform step squares.** Every non-pandiagonal cyclic square which satisfies the uniform step condition (25) is also symmetric, since the middle number already lies in the middle cell. For symmetric uniform step squares, then:

$${}_{su}W(N) = {}_uW(N) = n - 1, \tag{78}$$

$${}_{su}W(NN) = {}_uW(NN) = (n-1)^2, \tag{79}$$

(80) $$_{su}M(NN) = {_u}M(NN) = \frac{(n-1)^2}{4},$$

in accordance with (66), (67) and (68).

From (44) and (78) follows:

(81) $$_{su}W(N) = {_{su}}W(P) = n - 1,$$

(82) $$_{su}W(\overline{NP}) = {_{su}}W(NN) = {_{su}}W(PP) = (n-1)^2,$$

in accordance with (45) and (79).

Therefore, multiplying by (63) and dividing by 8 we obtain:

(83) $$_{su}M(\overline{NP}) = \frac{(n-3)(n-1)^2}{2}.$$

**Concluding remark.** The derivation of formulas for the different classes of non-pandiagonal magic squares treated here shows, even more clearly than in the case of pandiagonal magic squares, the great advantage, for the purpose of enumeration, in *separating* the conditions for the association of numerical values from the conditions for the formation of cyclic squares and for their matching. This procedure is even more fruitful in dealing with *composite orders* as well as in the formation and enumeration of *magic cubes*. This will be shown in a later paper. We conclude the present one with a Table which summarizes the formulas found and gives the numerical results of these formulas for the four smallest prime orders [22].

## Summary of Symbols

a). The principal symbols mean:

- $C$ the number of Cyclic squares;
- $W$ the number of Ways of associating numerical values with the elements;
- $M$ the number of Magic squares not obtainable from one another by rotation or reflection.

b). The symbols used in parentheses mean that the principal symbol refers:

- $(P)$ to a Pandiagonal cyclic square;
- $(N)$ to a Non-pandiagonal cyclic square;
- $(PP)$ to a combination of two pandiagonal cyclic squares;
- $(NN)$ to a combination of two non-pandiagonal cyclic squares;
- $(NP)$, $(PN)$ to a combination of a non-pandiagonal cyclic square with a pandiagonal cyclic square with regard to order;
- $(\overline{NP})$ to a combination of a non-pandiagonal cyclic square with a pandiagonal cyclic square independently of order.

c). The subscripts written to the left of the symbols $W$ and $M$ mean:

- $u$ uniform step squares;
- $s$ symmetric squares;
- $su$ symmetric uniform step squares.

NUMERICAL TABLE

| $PP$ = pandiagonal<br>$NN$ and $\overline{NP}$ = nonpandiagonal | Formula | $n=$ 3 | 5 | 7 | 11 |
|---|---|---|---|---|---|
| $M(PP)=\dfrac{(n-3)(n-4)n!^2}{8}$ | (20) | 0 | 3600 | 38 102 400 | 11 153 456 455 680 000 |
| $M(NN)=\dfrac{(n-1)!^2}{4}$ | (61) | 1 | 144 | 129 600 | 3 292 047 360 000 |
| $M(\overline{NP})=\dfrac{n(n-3)(n-1)!^2}{2}$ | (65) | 0 | 2880 | 7 257 600 | 579 400 335 360 000 |
| Total: | | 1 | 6624 | 45 489 600 | 11 736 148 838 400 000 |
| Of these, there are: | | | | | |
| a) uniform step squares: | | | | | |
| ${}_uM(PP)=\dfrac{(n-3)(n-4)(n-1)^2n^2}{8}$ | (28) | 0 | 100 | 2 646 | 84 700 |
| ${}_uM(NN)=\dfrac{(n-1)^2}{4}$ | (68) | 1 | 4 | 9 | 25 |
| ${}_uM(\overline{NP})=\dfrac{n(n-3)(n-1)^2}{2}$ | (70) | 0 | 80 | 504 | 4 400 |
| Total: | | 1 | 184 | 3 159 | 89 125 |
| b) symmetric: | | | | | |
| ${}_sM(PP)=2^{n-4}(n-3)(n-4)\left(\dfrac{n-1}{2}\right)!^2$ | (43) | 0 | 16 | 3 456 | 103 219 200 |
| ${}_sM(NN)=2^{n-3}\left(\dfrac{n-1}{2}\right)!^2$ | (76) | 1 | 16 | 576 | 3 686 400 |
| ${}_sM(\overline{NP})=2^{n-2}(n-3)\left(\dfrac{n-1}{2}\right)!^2$ | (77) | 0 | 64 | 4 608 | 58 982 400 |
| Total: | | 1 | 96 | 8 640 | 165 888 000 |
| c) symm. unif. step squ. | | | | | |
| ${}_{su}M(PP)=\dfrac{(n-3)(n-4)(n-1)^2}{8}$ | (46) | 0 | 4 | 54 | 700 |
| ${}_{su}M(NN)=\dfrac{(n-1)^2}{4}$ | (80) | 1 | 4 | 9 | 25 |
| ${}_{su}M(\overline{NP})=\dfrac{(n-3)(n-1)^2}{2}$ | (83) | 0 | 16 | 72 | 400 |
| Total: | | 1 | 24 | 135 | 1 125 |

## *Notes*

**1.** If we restrict ourselves to the condition of equal sums, there results a far simpler (only apparently more general) problem, whose solution gives squares which satisfy the condition for equal sums but which are made up of arbitrary integers, not necessarily all different. Such a square J. Chernick (this MONTHLY, 45, 1938, pp. 172–175) calls a *general magic square* and shows that it can always be obtained in a purely algebraic way from $n^2-2n$ arbitrarily given numbers.

**2.** A *latin square* is a square array of $n$ different elements in $n^2$ cells, each element being contained exactly once in each row and in each column (in *diagonal* latin squares, also in each of the

two diagonals), therefore, $n$ times in all. This designation dates from Euler, who in his *Recherches sur une nouvelle espèce de quarrés magiques*—cf. Verhandelingen uitgegeven door het Genootschap der Wetenschappen te Vlissingen vol. 9, 1782, pp. 85–239 = Leonh. Euleri Opera omnia sub ausp. soc. scient. nat. Helveticae, ser. I, vol. 7 (Lipsiae et Berolini 1923) pp. 291–392—by superposition of two such latin squares (the first of which he wrote in Latin characters) constructed the so-called *Euler squares*, in which every element of the first latin square coincides with every element of the second once and only once.

**3.** Synonymous terms in use are *panmagic, diabolic*, and others.

**4.** Such broken diagonals are illustrated in Fig. 1 by the numbers 0, 8, 14, 17, 21 or 7, 11, 15, 23, 4 or 7, 20, 14, 1, 18 *etc.*

**5.** The most important results of these—for the most part still unpublished—investigations, are indicated in the *Résumé de contributions à une théorie générale mathématique des carrés magiques* (Brussels, 1937), which also appeared under the title *Bericht über Studien zu einer allgemeinen mathematischen Theorie der magischen Quadrate* in the *Comptes-Rendus du deuxième Congrès international de récréation mathématique* (Paris, 1937), pp. 88–94 and in the journal "Sphinx" VIII (Brussels, 1938), pp. 62–69; there I have tried to unite the principal features of the most important known methods under the general viewpoints given in §1 above, and to suggest new possibilities of generalization.

**6.** That is, in all diagonals which slope downward from left to right.

**7.** That is, in all diagonals which slope downward from right to left.

**8.** A summary of the symbols used in the numerical formulas is adjoined to the Table at the end of the paper.

**9.** In each of these squares any $n$ cells filled with the same element give a solution of the *problem of n queens, i.e.*, the problem of placing $n$ queens on a board of $n^2$ cells so that no queen can be captured by any other.

**10.** Various French authors, in particular, proceed in this way. Of these we mention here only G. Arnoux, Les espaces arithmétiques hypermagiques, Paris, 1894. The same idea was presented earlier by H. Scheffler, Die magischen Figuren, Leipzig, 1882, who showed, that as "magic coördinate table," certain other arrangements besides the natural order from 1 to $n^2$ could be used as a base.—The procedure of G. Arnoux was further developed, in particular by G. Tarry in *La magie arithmétique dévoilée* (Bulletin de la société philomathematiques de Paris IX, 1907, pp. 182–195) and in other papers by the introduction of the idea of *séries numérales.* The method was perfected by E. Cazalas in *Carrés magiques au degré n*, Paris, 1934.—A summary of methods of this kind is given by M. Kraitchik, Traité des Carrés Magiques, Paris, 1930, p. 22.

**11.** One of the best known methods of this kind is the so-called *knight's-move method* of Moschopoulos (14th century), in which the principal step is a knight's move to the right and downward and the cross step is four cells downward. The same pandiagonal magic squares are formed from two cyclic squares if we set $p_1 = p_2 = 1$, $q_1 = (n-1)/2$, $q_2 = (n+1)/2$, and, in each of the two cyclic squares, associate the numbers in the arrangement $A_q$ with the elements.

**12.** One obtains the same result from the formula derived in a different way for arbitrary $n$ by D. N. Lehmer, *On the congruences connected with certain magic squares*, Transactions of the American Mathematical Society, vol. 31, 1929, pp. 529–551. But his formula on p. 540 should be multiplied by the factor $n^2$ and consequently also the numerical values given in the table on the same page. This has been pointed out by N. G. W. H. Beeger, On certain magic squares, Nieuw Archief voor Wiskunde, vol. 17, 1932, pp. 151–162 (*cf.* p. 158).

**13.** This is the basis of the procedure of N. G. W. H. Beeger (*loc. cit.* p. 161) who next calculates the number of pandiagonal magic squares which are *not* uniform step squares. For this purpose he derives from each uniform step square $n!^2$ pandiagonal magic squares by transformations, which are really equivalent to permutations of the first digits and of the second digits of the numbers written in the $n$-adic scale. Then he excludes by division by $n^2(n-1)^2$ the uniform step squares. The same result is therefore arrived at by correspondingly varying the way of associating numbers within the two auxiliary squares, and it follows that both methods of enumeration deal with the same pandiagonal magic squares.

**14.** *Cf.*, for example, H. Scheffler (*cf.* note **10**) who writes the numbers 1 to $n^2$ in the form $1+r+sn$ and indeed was the first to set forth the method from a general point of view. His procedure is also given by W. Ahrens, Mathematische Unterhaltungen und Spiele, vol. II, Leipzig, 1918, p. 43 *sqq.*, who also includes a summary of the best known other methods.

**15.** a) was used by Euler (*cf.* note **2**), b) by G. Tarry (*cf.* note **10**).

**16.** This theorem is true not only for magic squares, but in general; *e.g.* the "carré naturel" is symmetric and can be used for the formation of symmetric magic squares.

**17.** A corresponding rule has been applied by D. N. Lehmer (*cf.* note **12**) to symmetric semi-magic uniform step squares of arbitrary order (*cf. loc. cit.* p. 539 *sq.*).

**18.** *Cf.* Nieuw Archief voor Wiskunde, vol. 19, 1938, pp. 257–271. Instead of the word *cyclic* I there used *regular* and grouped together all latin squares which are not regular (therefore semi-cyclic and non-cyclic) under the designation *irregular*. The names "cyclic," "semi-cyclic," and "non-cyclic," however, emphasize more sharply the essential distinctions and are not limited to latin squares. Moreover, the division of latin squares into "regular" and "irregular" squares has already been made from a different viewpoint by other writers (Margossian, Kraitchik).

**19.** If we set $p_1=p_2=q_2=1$, $q_1=n-1$, and start with the numerical arrangement $A_m$ (beginning with $m$) as a basis for the first auxiliary square, and with the arrangement $A_{m+1}$ (beginning with zero) as the basis of the second auxiliary square, we obtain the uniform step squares formed by the so-called *Bachet method* (also called the *terrace method*). These squares can be written down immediately according to the following rule: Starting cell: the cell directly under the middle cell; principal step: one cell diagonally down and to the right; cross step: two cells downward.

**20.** If we set $p_1=p_2=q_1=1$, $q_2=m+1$ and start with the numerical arrangement $A_1$ (beginning with $m+1$) as a basis for the first auxiliary square and with the arrangement $A_2$ (beginning with 1) as a basis for the second auxiliary square, we obtain the uniform step squares formed by the so-called *Indian method*. These squares can be written down immediately according to the following rule: Starting cell: middle cell of the first row; principal step: one cell diagonally to the right and upward; cross step: one cell downward.

**21.** N. G. W. H. Beeger applies this principle here in the same way as for pandiagonal uniform step squares (*cf.* note **13**). In order to do this he first divides the non-pandiagonal uniform step squares into six classes, of which four contain $n(n-1)^2(n-3)$ squares each and two contain $(n-1)^2$ squares each (*loc. cit.* p. 158). This result corresponds to formulas (70) and (68) if they be multiplied again by 8.

**22.** For the case $n=5$, the first four numbers of the table have been obtained several times in other ways, *e.g.* by M. J. Van Driel, Magic squares of $(2n+1)^2$ cells, London, 1936, who also counts as uniform step squares those obtained from proper uniform step squares by permutations. He finds a total of 6624 such "magic squares of order 5 made by the method of uniform steps," which may be divided into 3600 pandiagonal and 144+1440+1440 non-pandiagonal squares (*loc. cit.* p. 24, and also in earlier writings of the same author). The 144 squares correspond to the group ($NN$), and the two classes of 1440 each to the groups ($NP$) and ($PN$); correspondingly, he gives (p. 42), as the number of symmetric squares included therein, 16 pandiagonal and 16+32+32 non-pandiagonal squares.

The number 3600 for 25-celled pandiagonal magic squares was also found by J. C. Burnett, Easy methods for the construction of magic squares, London, 1936, by use of formula (20) and he remarks, without proof: "This applies to all odd prime orders" (p. 68). He treats, in addition, a group of 25-celled non-pandiagonal magic squares whose number he gives (without proof) as 3024. As may be seen from his method of construction, these are the 144+2880 magic squares of the groups ($NN$) and ($\overline{NP}$).

W. W. Rouse Ball, Mathematical Recreations and Essays, 7th edition, London, 1917, in Chapter VII ("Magic Squares"), from a 25-celled magic square formed by the "Indian method," derives by multiplication by $4!\times 5!$ (hence in accordance with formula (64)), 2880 magic squares, "though only 720 of them are really distinct" (p. 141). Since here, in the group ($\overline{NP}$), only the case ($NP$) is taken account of, and of these squares only the half for which $q_1=1$ (*cf.* note **20**), and since in the case $n=5$, the factor $(n-3)/2$ is equal to 1, this is the fourth part of the squares

counted in formula (65). Then (p. 142), the number of magic squares derivable from a 25-celled square formed by the "Bachet method" is likewise given as 720. This number should be divided by $\frac{1}{2}n(n-3)=5$ since $\frac{1}{4}M(\overline{NP})=\frac{1}{2}n(n-3)\cdot M(NN)$ (*cf.* formulas (61) and (65)). The correct number, 144, is obtained (p. 144) in another way by the combination of two non-pandiagonal auxiliary squares.

M. Kraitchik (*cf.* note **10**) has calculated (p. 55) the number of 25-celled magic squares which are derivable from an Euler square to be $2\times 5!^2=28{,}800$ (therefore $8\times 3600$). Since the basic Euler square is pandiagonal, and is used in the two forms which are obtainable from one another by interchanging the two latin squares and since, in case $n=5$ there is, by formula (17), only a single such pair of squares, the number given really includes all pandiagonal magic squares of order 5 obtainable from cyclic squares.

A. L. Candy, Construction, classification and census of magic squares of order 5, Lincoln, Nebraska, 1938, p. 59 explicitly leaves open the question as to the number of 25-celled pandiagonal magic squares.

ADDENDUM—In the second edition of his book (1939), A. L. Candy calculates, on p. 65, the number 3600 for 25-celled pandiagonal magic squares, by use of formula (20).

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# MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This department of the* MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

## MATHEMATICS IN THE JUNIOR COLLEGE

R. J. HANNELLY, Phoenix Junior College

**1. Introduction.** *The junior college.* From a modest beginning at Joliet in 1902 the junior college has struggled for over three decades to find its place in the systems of education in the United States. Today there are more than 550 junior colleges distributed in forty-four states and the District of Columbia. The public junior college has thrived in the far west and in the middle west whereas the private variety has taken root in the south. California has fifty-seven; Texas, thirty-eight; Iowa, thirty-seven; and Oklahoma, thirty-two. junior colleges are of one-year, two-year, and four-year types. In January, 1939, 408 were two-year units.* Only thirty-four were four-year units. Educators carry on lively arguments as to whether the 6-4-4 or the 8-4-2 (or 6-3-3-2) plan is better.

The functions of the junior college are to offer: (1) the first two years of pre-professional education in a manner acceptable to the four-year colleges and universities; (2) the first two years of general and liberal arts education; and (3) semi-professional education for which there is a community need. A semi-profession is here defined as a vocation for which two years of post-high school education is necessary and sufficient.

* Junior College Directory, 1939. Washington: American Association of Junior Colleges, 1939.

Uncritical observers have considered junior college education as secondary. Secondary education may be defined upon at least seven different bases.* Junior college education may be properly considered as partially collegiate. It has been stated that education in grades thirteen and fourteen is general. Definitions of general education have been unconvincing. In fine, there is confusion as to the classification and nature of junior college education.

*Mathematics in the junior college.* Mathematics courses in the junior college have generally been patterned after the lower division courses in the universities.† An examination of 352 junior college bulletins and mathematics texts in use in junior colleges in 1939 revealed the typical offering to be: intermediate algebra (offered by 146 junior colleges); college algebra (248); trigonometry (295); analytic geometry (229); and calculus (176). There were 143 combined courses for freshmen. Elementary, algebra, plane geometry, solid geometry, solid analytical geometry, differential equations, functions of a complex variable, and combinations of these with other subjects were offered, each by a few junior colleges. Mathematics of finance was offered by 69.‡ Few semi-professional courses were found. This corroborates previous studies.§

**2. Problems and issues.** Correspondence with fifty teachers of junior college mathematics served further to set forth the following issues with their attendant problems:||

*Articulation with the high school.* The high school graduate enters the junior college with from none to four years of mathematics. If two years are required, the chances are that the student who has taken the minimum has forgotten most of it. This situation appeared to be an important problem. There is no great difference in calibre between freshman mathematics, and intermediate algebra and solid geometry. Since the freshman usually begins with college algebra or trigonometry or both, there is overlapping. A review does some good and for that reason some duplication is desirable, but too much is a waste of educational effort. Some mature persons who avoided mathematics in high school wish to take it on the college level.

*Differentiation of mathematical offerings in the junior college.* There are three bases for differentiation of offerings: (1) high school preparation; (2) purposes of students; and (3) community resources.

Attempts have been made to differentiate on the basis of high school courses taken, *e.g.*, if a student has had only two years of high school mathematics, he

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* Cf. B. F. Pittenger, "Use of the Term 'Secondary' in American Education," School Review, vol. 24, 1916, pp. 130–141.

† Cf. Justin E. Hills, "Junior College Mathematics," School Science and Mathematics, vol. 29, 1929, pp. 880–885.

‡ Cf. Robert J. Hannelly. *The Mathematics Program in the Junior College.* Unpublished doctor's dissertation. Boulder, Colorado: University of Colorado, 1939.

§ Cf. J. Calvin Funk et al. *Report of the Mathematics Committee of the California Junior College Association.* Mimeographed materials. Berkeley: California State Department of Education, 1935.

|| Hannelly, *loc. cit.*

has been required to attend the college algebra class five times a week instead of three. Teacher-opinion indicated that this method is not satisfactory. Another method is for the junior college to offer high school courses with or without credit. A third is to offer a rapid survey of high school mathematics. On the whole, this basis of differentiation is inadequate because it does not sufficiently take into consideration the purpose of the student.

Differentiation on the basis of the direction in which the student is moving is sound. If he is going on to the university for professional education he can be advised accordingly, but if he purposes to study liberal arts or general education from one to four years, suitable courses are not generally available. Only a beginning has been made in vocational mathematics.

The third basis of differentiation, *i.e.*, community resources, connotes limitation of offerings. Population and financial resources are reflected in the number and kinds of courses and in the number and preparation of the staff. A junior college situated in a rich agricultural region may be obliged to offer a course in mathematics with agricultural applications.

*Articulation with the university*. During the infancy of the junior college, the university was hypercritical of it, but, with experience, a better understanding has been engendered. In pre-professional mathematics, the junior college accepts, with reasonable reservations, the requirements imposed by the university. However, the university is not in general convinced that it likes general mathematics. With semi-professional courses the university is not concerned.

**3. Proposed solutions.** The ultimate solutions will depend upon the insight, industry, research, and coöperation of the teachers and educators involved. However, it is axiomatic that permanent change for the better is usually gradual, not saltatory.

Articulation of high school and junior college mathematics is part of the larger problem of the articulation of the high school and the junior college. Junior college mathematics teachers will do well to study this broad problem. For the present the junior college can, if its resources permit, offer solid geometry and intermediate algebra for credit. Furthermore, if resources permit and if there is sufficient demand, it can offer elementary algebra and plane geometry, without credit, to those persons who must have these courses to satisfy professional requirements. To prevent overlapping for those students who have taken trigonometry, solid geometry, and three or four semesters of algebra in high school, the freshman course may well begin with analytic geometry. In fact this plan is now in use in some junior colleges. Under such a plan, it may be decided locally whether or not to give credit in college algebra and trigonometry to these students. If so, it should certainly be by examination.

Other problems of differentiation and articulation are discussed under the following headings:

*General mathematics*. Under this title the purpose is to indicate the solution of the problem of what kind of mathematics to offer the junior college freshman

who (1) intends to take only one year of mathematics in a liberal arts or in a general curriculum, and (2) has taken only two years of high school mathematics, now well forgotten. There is ample testimony to the fact that college algebra, trigonometry, and analytic geometry constitute an unsatisfactory offering from the points of view of the student and the teacher.

Textbooks in the field illustrate varying degrees of unity and various principles for unification. The lowest degree is the "tandem arrangement" of the three subjects mentioned in one book. The next lowest degree is the "eclectic chapter arrangement." Some authors use the idea of functionality in the first chapters of the book and from then on a rather capricious order prevails. Others use the idea of functionality throughout. Properly done, this is defensible. Still others sprinkle a bit of calculus into the "eclectic chapter arrangement" and dignify it by the name of mathematical analysis. One of the more defensible principles is the arrangement on the basis of cultural importance of topics. Another arrangement is made on the basis of job-analysis,—the practical course.

Tenable ordering principles for general mathematics are functionality, cultural importance, and reality of problems. Many confuse reality with utility. A person's sphere of reality may include immaterial as well as material things, *e.g.*, it may include escalators, new-mown hay, the organic theory of evolution, and the Pythagorean theorem. Many of the real problems in life have nothing to do with utility. Real problems are not always practical problems.

General mathematics may be characterized by at least two notions:

(1) When the psychological and logical organizations of subject-matter are different, emphasis is placed upon the former;

(2) General mathematics implies range of materials and is not confined to a single well-knit entity of materials.

From the foregoing considerations it is suggested that junior college teachers experiment with a general course covering the function concept, probability, mathematics in the physical sciences, mathematics in the life sciences, mathematics in the social sciences, mathematics in art, mathematics and philosophy, and the like. Such a course may be carried out by the students and teachers by the unit plan and gradually refined. A unit outline may be divided into overview, suggested activities, non-mathematical outcomes, mathematical outcomes, instruments of evaluation, and bibliography.

*Semi-professional mathematics.* A semi-professional curriculum includes a course in mathematics when and if those who build the curriculum are convinced that sufficient mathematics is used in the semi-profession to warrant a course in it. Semi-professional curricula are feasible only in large municipal junior colleges or in those small junior colleges situated in areas in which one or more common semi-professions are practised, *e.g.*, in rich agricultural or mining regions. Semi-professional curricula in actual operation include: general business; secretarial; medical secretary; merchandising; banking and finance; technician in engineering; technician in chemistry; government; mining; business

management; agriculture; aviation-technology; and family relationships.* It is clear that these are confined chiefly to the fields of business and engineering. If mathematics is required in a business semi-profession, it is usually chosen from business arithmetic, machine courses, commercial algebra, mathematics of finance, or statistics.

The number and types of semi-professional curricula which can be offered by a junior college depend directly upon the size and character of the community and can be determined only from a survey of the same. Semi-professional curricula instituted in small junior colleges without suitable background may have a precarious existence. In general, however, junior colleges should be urged to survey their communities for semi-professions.

*Pre-professional mathematics.* Pre-professional is the designation applied to the lower division mathematics for engineering students, actuarial students, prospective mathematics teachers, and majors in mathematics or the physical sciences. The university requirements are clear: college algebra, trigonometry, analytic geometry, and calculus. The heavy scholastic burden carried by the engineering student, even in the junior college, is plain. He does not have time for enough of the humanities. This is recognized by members of the National Society for the Promotion of Engineering Education.† The solution of this problem as far as the junior college is concerned is dependent upon the change of emphasis, and an extension of the period of education to be initiated, if at all, by the university. In a word, the engineer needs to study more social subjects or else the subjects he now takes should be studied also in their social context. A few junior college mathematics teachers are in favor of general mathematics for engineering freshman, to be followed by the more highly distilled courses in mathematics.

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* Junior College Journal, vol. 9, 1939.

† Cf. W. H. Rayner, "The Cultural Element in Engineering Education," Society for the Promotion of Engineering Education, Proceedings, 1922. Lancaster, Pennsylvania: New Era Printing Company, 1923, pp. 153–160.

## QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

*The department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

### COWS AND COSINES

L. R. FORD, Armour Institute of Technology

The problem solved here was brought in by a resident of a Chicago suburb. The formula embodying the solution caused him no little astonishment. It happens rather often, I think, that connections which are all a part of the day's work in some field seem far-fetched and unexpected to the uninitiated.

One is reminded of DeMorgan's actuarial friend.* While explaining certain probabilities of survival, DeMorgan used a formula involving $\pi$, explaining that it was the ratio of the circumference of a circle to its diameter. At this the friend exclaimed, "That must be a delusion. What can the circle have to do with the numbers alive at a given time?" Our suburban friend might have exclaimed in like vein, "What have *angles* to do with the question, either?"

*The Problem.*—A boy is given a young heifer calf. On its third birthday and each year thereafter, it gives birth to a heifer calf. Each of its descendants likewise produces a heifer calf on each birthday, beginning with the third. Assuming no deaths, find a formula for the number of animals in existence $n$ years hence.

Let $u_n$ be the required number. We have

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \cdots,$$
$$u_n = 1,\ 1,\ 1,\ 2,\ 3,\ 4,\ 6,\ \cdots.$$

The number for any year is equal to the number the year before plus the number of births which have just occurred, and the latter is equal to the number of animals three years earlier. This gives the difference equation

$$u_{n+3} = u_{n+2} + u_n. \tag{1}$$

To solve this, set $u_n = r^n$. We have $r^{n+3} = r^{n+2} + r^n$, and cancelling $r^n$, we have a solution provided

$$r^3 - r^2 - 1 = 0. \tag{2}$$

The general solution of (1) is then

$$u_n = ar_1^n + br_2^n + cr_3^n, \tag{3}$$

where $r_1$, $r_2$, $r_3$, are the roots of (2) and $a$, $b$, $c$ are constants.

The constants in (3) are to be determined so that $u_n$ shall have proper values for $n = 0, 1, 2$, whence

* A. DeMorgan, A Budget of Paradoxes, London, 1872, p. 172.

$$a + b + c = 1,$$
$$ar_1 + br_2 + cr_3 = 1,$$
$$ar_1^2 + br_2^2 + cr_3^2 = 1.$$

Solving these and simplifying by the use of the relations between the roots of (2),

$$r_1 + r_2 + r_3 = 1, \qquad r_1r_2 + r_1r_3 + r_2r_3 = 0, \qquad r_1r_2r_3 = 1,$$

we have the solution in the form

$$u_n = \frac{r_1^2 + 1}{r_1^2 + 3} r_1^n + \frac{r_2^2 + 1}{r_2^2 + 3} r_2^n + \frac{r_3^2 + 1}{r_3^2 + 3} r_3^n. \tag{4}$$

Now (2) has one real positive root and two imaginary roots. We find

$$r_1 = 1.465576, \qquad r_2 = -0.232788 + 0.792557i$$

and $r_3$ is the conjugate imaginary of $r_2$. We write $r_2$ in the polar form

$$r_2 = 0.826030(\cos 106.369° + i \sin 106.369°).$$

Using these values to determine the coefficients in the solution, we find, after some calculation,

$$a = 0.61149, \qquad b = 0.229679(\cos 32.248° - i \sin 32.248°),$$

$c$ being the conjugate of the latter.

We can now write the solution. Noting that the second and third terms of (4) are conjugates, their sum is twice the real part of $br_2^n$. We have

$$\begin{aligned} u_n &= 0.61149(1.465576)^n \\ &\quad + 0.45936(0.826030)^n \cos(n \cdot 106.369° - 32.248°). \end{aligned} \tag{5}$$

This is the required formula. As a matter of fact, the second term of the formula damps out rather rapidly. Thus for $n=10$, the two terms are as follows:

$$u_{10} = 27.956 + 0.045 = 28.001.$$

From the very first $(n=0)$ the second term is numerically less than one-half; so $u_n$ is the nearest integer to the first term in all cases.

## A METHOD FOR OBTAINING A CONJUGATE FUNCTION

JOHN BEEK, JR., National Bureau of Standards

The text-books give an expression for a conjugate function based on the Cauchy-Riemann partial differential equations. The purpose of this paper is to give an expression which involves only substitution in the form of the given function; the function must, however, be defined for complex values of the variables.

Let $U(x, y)$ be a real function, defined for complex values of $x$ and $y$, and satisfying Laplace's equation. Then the conjugate function is the imaginary part

of $2U[(z+c)/2, (z-c)/2i]$, where $z=x+iy$, and $c$ is a constant restricted only by the condition that the function be finite.

The proof follows immediately from a consideration of the expression for $U$ as a real solution of Laplace's equation. We have $U(x, y)=f(x+iy)+f^*(x-iy)$, where the form of $f$ is not known. Then

$$2U\left(\frac{z+c}{2}, \frac{z-c}{2i}\right) = 2f\left(\frac{z+c}{2}+\frac{z-c}{2}\right) + 2f^*\left(\frac{z+c}{2}-\frac{z-c}{2}\right)$$
$$= 2f(z) + 2f^*(c).$$

The real part of this function is $f(z)+f^*(z^*)+f^*(c)+f(c^*)$, which is $U$ plus a constant. Therefore, the imaginary part is the conjugate of $U$.

## AN APPLICATION OF INEQUALITIES BETWEEN SYMMETRIC FUNCTIONS TO THE THEORY OF EQUATIONS

L. TORALBALLA

The fundamental theorem on symmetric functions is to the effect that every polynomial symmetric in $x_1, x_2, \cdots, x_n$ is equal to a polynomial, with integral coefficients, in the elementary symmetric functions,

$$\Sigma x_1, \Sigma x_1 x_2, \cdots, x_1 x_2 \cdots x_n,$$

and the coefficients of the given polynomial. Because of the relationships between these elementary symmetric functions and the coefficients of a polynomial equation, every inequality between symmetric polynomials gives rise to a theorem in the theory of equations. We shall consider some cases.

(1) Suppose $A_1, A_2, \cdots, A_n$ are all real. Then

$$(A_i - A_j)^2 \geqq 0, \qquad i<j, \qquad (i, j = 1, 2, \cdots, n);$$
$$A_i^2 + A_j^2 \geqq 2A_iA_j;$$
$$(n-1)\Sigma A_1^2 \geqq 2\Sigma A_1A_2;$$

or in terms of the elementary symmetric functions,

$$(n-1)(\Sigma A_1)^2 \geqq 2n\Sigma A_1A_2.$$

Hence, in order that the equation

$$(A) \qquad X^n + b_1X^{n-1} + b_2X^{n-2} + \cdots + b_n = 0,$$

have all its roots real, it is necessary (though not sufficient) that

$$(n-1)b_1^2 \geqq 2nb_2.$$

A necessary condition that all the roots be real and distinct from each other is

$$(n-1)b_1^2 > 2nb_2.$$

For the quadratic $X^2+b_1X+b_2=0$, this reduces to $b_1^2>4b_2$.

## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*College Algebra.* By P. K. Rees and F. W. Sparks. New York and London, McGraw-Hill Book Company, 1939. 11+312 pages. $2.25.

*Differential and Integral Calculus.* Second edition. By J. H. Neelley and J. I. Tracey. New York, The Macmillan Company, 1939. 9+495 pages. $3.25.

*La Ecuación de Segundo Grado a Dos y Tres Variables.* (Método para su estudio y reducción.) By J. B. Kervor. Buenos Aires, Librería y Editorial "El Ateneo," 1938. 135 pages.

*New First Course in the Theory of Equations.* By Leonard Eugene Dickson. New York, John Wiley and Sons; London, Chapman and Hall, 1939. 9+185 pages. $1.75.

*Tafeln und Aufgaben zur Harmonischen Analyse und Periodogrammrechnung.* By Karl Stumpff. Berlin, Julius Springer, 1939. 7+174 pages.

*Orthogonal Polynomials.* By Gabor Szegö. (American Mathematical Society Colloquium Publications, vol. XXIII.) New York, American Mathematical Society, 1939. 9+401 pages. $6.00.

### REVIEWS

*Coordinate Solid Geometry.* By Robert J. T. Bell. London, The Macmillan Co., 1938. 12+175+39 pages. $2.25.

Metrical solid analytic geometry is a sort of step-child in the curriculum of our American colleges. Most of the introductory courses in analytic geometry given in our schools include the rudiments of solid analytics, barely sufficient to make intelligible to the student the geometric applications of the calculus to functions of more than one variable. When the student is ready to take up the subject of analytic geometry in space more extensively, the writers of textbooks are so eager to introduce him into the realm of modern geometry that metrical geometry is again treated in a perfunctory fashion.

The book under review is a decided departure from this attitude. About half of the material is devoted to the point, the line, and the plane, followed by a chapter on the sphere, and another on the cone. The rest of the book deals with the conicoids. The exercises are numerous and constitute an essential part of the book. The student who will work a considerable part of them will not only gain a mastery of the analytical method, but will learn a good deal of geometry not discussed in the text. The algebraic prerequisites are modest. The material is more than sufficient for a semester's work even for students of more than average ability.

The book is a part of a more extensive treatise on coördinate geometry by

the same author. Some of the pages give the impression of being somewhat crowded, but the book as a whole is quite satisfactory in appearance and seems to be free of typographical errors.

N. A. COURT

*Modern-School Solid Geometry*. By Rolland R. Smith and John R. Clark. Yonkers, World Book Company, 1939. 8+248 pages. $1.28.

This book is one in the Schorling-Clark-Smith mathematics series. In several respects it departs from the traditional course in solid geometry. There are nine chapters, rather well unified in content. The theorems are numbered continuously throughout the book and end with theorem 61. This seems an unusually small number until it is observed that some of the propositions conventionally stated as theorems have been given as corollaries. There is a real reduction, however, effected by assuming some 6 or 7 theorems without proof. In addition to cylindrical and conical surfaces the authors define prismatic and pyramidal surfaces, deriving the prism and pyramid from these. They also postulate that: *Any property of prisms* (*pyramids*) *with regular bases which does not depend upon the number of lateral faces is also a property of circular cylinders* (*cones*). They fail to make full use of this postulate, however, and fall back on limits to justify the formulas for the surface area and volume of the cylinder and cone.

A plane is not defined. Instead a postulate gives the usual "two-points of a line" property of the plane. Just why this is done is not clear.

The book is well written, has adequate simple exercises, is pleasingly illustrated, and shows evidence of the skill in teaching which would be expected of its competent authorship.

V. S. MALLORY

*Construction, Classification and Census of Magic Squares of Order Five*. Second edition, revised and enlarged. By A. L. Candy. Published by the author, 1003 H Street, Lincoln, Nebraska, 1939. 7+221 pages. $1.00.

The first edition of the book was reviewed in this MONTHLY, vol. 45, 1938, p. 381. The principal general result of the new edition is that the estimated total number of $5\times 5$ magic squares, independent of rotations and reflections, has been increased from 12,860,440 to 13,288,952. This has been accomplished by a further intensive study of the various classes and types treated in the first edition.

Chapter V on pandiagonal squares has been considerably augmented. In particular, it is shown how the complete set of 3600 pandiagonal squares of order 5 can be obtained from 36 basic squares. A method is also described by which pandiagonal squares of higher prime order may be constructed.

Chapter XI is new and is devoted to a special class of squares from which bordered squares of order 5, and of higher orders, may be obtained.

Chapter XII is also new and, although not strictly within the scope of the

book, is interesting in that it shows how the material developed on $5\times 5$ squares can be used in the construction of $10\times 10$ squares.

Professor Candy has done an amazing amount of work on magic squares of order 5. It is to be hoped that he will continue until he has found the exact number of such squares, and has given a complete classification of them.

G. E. RAYNOR

*Mathematics in Action.* By Walter W. Hart and Lora D. Jahn. New York, D. C. Heath and Company, 1939. 9+344 pp. $0.88.

This book has been designed to meet the first year's work of a "practical, socialized course in mathematics" for the junior high school. As set forth in the preface, the title exemplifies the theme of the book: "mathematics used to solve typical significant problems of real life."

Topics covered are fractions, measurement, percentage, graphs, straight lines and angles, circles, triangles, parallelograms, trapezoids, and especially instructive chapters on arithmetic in the home, thrift, banking practices, and transportation. Emphasis is placed upon percentage and its applications, and the various topics are distributed over the year rather than concentrated in units. Furthermore, the inductive method of reasoning has been employed frequently as a means of unifying the course.

The student's interest is aroused by many devices such as pictures, cleverly chosen problem material, and an informal style which makes the book read like fiction in many places. Diagnostic tests, remedial instruction, progress and mastery tests, and an abundance of drill material are other features of this textbook.

The reviewer feels, however, that the typography leaves much to be desired. In the effort to apply many diversified techniques, designed to hold the student's attention, many pages present a confused and crowded appearance, with the point of instruction poorly set forth. The desire of the authors to utilize all available page space may have a tendency to discourage the average student confronted with such a textbook to read.

R. A. HARRISON

*Elementary Mathematical Statistics.* By William D. Baten. New York, John Wiley & Sons, 1938. 10+338 pages. $3.00.

The aim of this book is to serve as an introduction to the practical use of statistical methods by students whose mathematical training does not include calculus.

Topics covered include graphs, averages, measures of dispersion, the normal curve, skewness and kurtosis, probability and sampling, index numbers, observational equations, correlation and regression, time series, analysis of variance and standard errors of certain statistics. Topics which are more carefully developed than might be expected from the generally elementary nature of the book include the non-linear correlation coefficient, the analysis of variance and,

under sampling, the full algebraic derivation of certain fundamental formulas. On the other hand, certain topics of elementary nature but of fundamental importance are relatively neglected, as for instance, non-normal frequency distributions and frequency curves other than the normal curve, specific birth rates and mortality tables, and fitting empirical equations. The treatment of index numbers is too sketchy to constitute an adequate introduction for students of business or economics.

The book accomplishes its aim with a fair degree of success for students whose primary interest is agriculture and related sciences but is not satisfactory for students interested in business, economics, or vital statistics. The book is noteworthy for careful exposition of technique of such things as fitting a normal curve to a given frequency distribution, computing the correlation coefficient, or testing the significance of the difference between the means of two samples. On the other hand, it does not seem to fulfill very well the natural function of the first book in statistics of bringing out the real meaning of the statistical point of view and of beginning to develop the student's statistical maturity.

In certain cases the exposition does not bring out the real points clearly. For example the statement as to the usefulness of charts is not broad enough in that it is limited to their use as picturing figures and does not suggest their significance as a method for making and communicating complicated quantitative analyses. On page 47, the examples under geometric mean are all concerned with the process of inserting $N$ means between two extremes and do not cover the process of finding the geometric mean of $N$ quantities. On page 85, the explanation of the use of the normal curve in connection with academic grades does not state the important assumptions that have been made in the proposed treatment. On page 184, the discussion of the value of the correlation coefficient in predicting, for the case of $r=.5$, is badly jumbled, with several non-sequiturs and an incorrect conclusion.

The worst mistake in the book, in the opinion of the reviewer, is the failure to make the essential preliminary analysis before starting on page 233 to fit an equation to an historical series. It is recognized by time-series analysts that if it is desired to get a trend line for such a series which will really represent the growth or decay tendency as contrasted with seasonal and cyclical variability, it is essential to select the period covered so as to avoid a spurious trend which merely reflects cyclical or seasonal changes. Specifically, the period should cover at least two cycles and the period chosen should not begin or end at the high or the low point of a cycle or of a marked seasonal swing. In the example chosen to illustrate secular trend, the period is in fact coincident for the most part with the downward phase of the 1929–1937 business cycle and the straight line equation which is determined does not show the real trend but merely a version of the cyclical decline.

Attention is called to this mistake in order to emphasize one of the principal distinguishing characteristics of statistical work, namely, that good non-mathematical reasoning must be mixed with sound mathematics if one is to get

significant statistical results. This point of view is repugnant to dyed-in-the-wool mathematicians, who must be born again, in a sense, before they can become good statisticians. It would seem to be one of the obligations of the writer of an elementary book on mathematical statistics to start the process of re-education of mathematical students, and also, of course, to show by his example that the non-mathematical reasoning incorporated in a statistical argument must be tested carefully and made as objective as is practicable under the circumstances.

On the whole the book can be recommended as a usable elementary textbook for students who wish to become routine statistical computers in fields related to agriculture, and should be of some value for reference purposes in other lines for students with more ambitious aims. For general use, however, it seems inadequate from the point of view of imparting a broad grasp of statistical principles and developing statistical judgment and does not contribute much to the major task of developing a true appreciation of statistics as a major essential method of the biological and social sciences.

R. W. BURGESS

*Engineering Descriptive Geometry.* By Charles E. Rowe. New York, D. Van Nostrand Co., 1939. 8+299 pages. $2.50.

Following a brief historical sketch, the author states fourteen fundamental principles of descriptive geometry. These principles are carefully developed through the next four chapters under the general headings: principal views, auxiliary views, oblique views, line and plane problems. Numerous problems are analyzed and discussed as applications of these fundamental principles. This material covers about fifty percent of the textual matter of the book. Then follow chapters on surfaces and developments, surfaces and intersections, and warped surfaces. There is a chapter on geology and mining problems, and one on engineering problems. The treatment of shades and shadows, and of perspective drawing is thorough.

The "direct method" is used because the author indicates that years of teaching experience have convinced him that the direct method has distinct advantages. However, a chapter is devoted to a discussion of the older or "Mongean method."

Approximately one thousand problems for solution are offered. These are completely classified and indexed for convenience in assignment or study.

The book has been carefully written; the drawings are accurate and follow standard practices; the printing is excellent—it is a splendid book.

B. R. BEISEL

# MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

*All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.*

## SUGGESTIONS FOR DECEMBER CLUB PROGRAM

### CHRISTMAS TOPIC

Connecticut State College Mathematics Club has, as the title on last December's program, "What is the Star of Bethlehem?" This topic has appeared from time to time on other club programs and its popular appeal is illustrated by the annual December lecture given at the Hayden Planetarium in New York City. Short discussions of these lectures have appeared in the publications of the planetarium "The Sky" December, 1937, page 4–8, and December, 1938, page 4–5. The *Astronomical Society of the Pacific* has also published a pamphlet entitled:—"Is that Star the Star of Bethlehem?" Leaflet 106, December, 1937. Perhaps the questions of greatest importance are: Was the star of Bethlehem a nova which appeared for just a few days, or a comet; or is there some foundation for the theory that the planets Saturn, Jupiter, and Mars are calculated as having been in conjunction around 6 or 4 B.C.?

### CHRISTMAS STUNTS

In our stunt library there are many contributions which might be useful to you in planning your December programs. The best and most appreciated stunts are those which you create to meet the needs of your own club. This is well illustrated by some questions quoted from a Christmas questionnaire devised by Mary Catherine Donahoe of the *Euclid's Circle* of Mount Scholastica College, Atchinson, Kansas.

## SPEAKING OF MATHEMATICS

Did you ever think that perhaps you are a budding Einstein? Do you believe that possibly you possess hidden powers of genius?

Take a pencil and check the particular item in each of the following which will make the statement correct. If you are above the average you will score around 95. To determine your MQ (Mathematics Quotient) subtract 5 points for every mistaken choice, then total your missed points and deduct these from a perfect score of 100. (Three mistakes mean 15 points off or a correct score of 85.) When finished turn to page 596 to determine your score.

1. The awe-inspiring but unsolvable problem which baffled the greatest mathematicians of antiquity is: a. the computation of compound interest; b. the solution of indeterminate equations; c. the rate of decrease of the national debt; d. the trisection of the angle with ruler and compasses.

2. A well known quantity that is correctly termed irrational is: a. the betting pandemonium of the Kentucky Derby; b. the square root of seven; c. sine squared plus cosine squared equals one; d. the social hysteria of the Mardi Gras.

3. You are the young wife of a rich old man. On your tenth wedding anniversary, he is 75 years old. The chances that he will live for the Silver Jubilee are: a. none at all; b. one out of a hundred; c. thirty-two out of a thousand; d. twenty-five out of a thousand.

4. "Let no one who is unacquainted with geometry enter here" is: a. the slogan of the Mathematical Association of America; b. a rule of *Euclid's Circle*; c. the inscription on Plato's porch; d. Einstein's motto.

5. In only one of the following, mathematics did not play a major part: a. Ancient Egyptian Public Works; b. the Golden Gate Bridge; c. the Grand Coulee Dam; d. U. S. Rehabilitation.

6. The abacus was used by the Oriental peoples: a. as a device for counting; b. as an instrument for astronomical observation; c. as a method of determining the speed of ships at sea; d. as a formula for the area of the circle.

7. *Mathematics for the Million*, which recently outsold fiction in the author's native land, was written by one of the following: a. Lancelot Hogben; b. Alexis Carrel; c. Compton McKenzie; d. Hilaire Belloc.

8. The invention of the so-called musical proportion is attributed to one of these: a. Mendelssohn; b. the Babylonians; c. Bach; d. the French.

9. The name Briggs is most frequently associated with which of the following? a. Euthanasia; b. Algorithm; c. Euphuism; d. Common logarithms.

10. If you were writing a book entitled *World Famous Mathematicians*, which of the names listed below would you exclude? a. Demosthenes; b. Euclid; c. Einstein; d. Plato.

11. The unknown algebraic quantity among the Egyptians was: a. $x$ or $y$; b. "gee" or "haw"; c. "mu" or "nu"; d. "heap" or "hau."

12. The Romans boast of the fact that Victorian of Aquitanis is best known for: a. the origination of the polka dance step; b. the discovery of the law of gravity; c. the rule for determining the date of Easter; d. the construction of the Coliseum.

13. The man who raised mathematics to the ranks of a science and placed geometry among the liberal arts was: a. Aristotle; b. Menaechmus; c. Euclid; d. Pythagoras.

14. Historians are of universal agreement that seventh century Greek mathematics should make mention of: a. Ahmes; b. Achilles; c. Al-Biruni; d. Boethius.

15. Magic squares and circles owe their origin to: a. the American Indian; b. the development of spiritualism; c. primitive magicians; d. the Chinese.

16. Your 1939 calendar would not bear its present form except for the work of: a. Benjamin Franklin; b. Pope Gregory XIII; c. Julius Caesar; d. Napoleon.

17. "Depart for thou hast not the grip of philosophy" was expressed by one of the following: a. St. Paul; b. William Shakespeare; c. a follower of Plato; d. a follower of Euclid.

18. Which series would most delight the modern mathematician? a. World Series; b. Taylor's Series; c. Science and Culture Series; d. Aromatic Series.

19. Pythagoras called ——————— the most beautiful of all solid figures: a. the cube; b. the cone; c. the sphere; d. the statute of Aphrodite.

20. If you have chosen an answer for each of the preceding 19, don't feel too optimistic for you have missed at least one. Can you find the question for which no correct answer was given?

## ANSWERS TO "SPEAKING OF MATHEMATICS"

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. d | 2. b | 3. c | 4. c | 5. d | 6. a | 7. a | 8. b | 9. d | 10. a |
| 11. d | 12. c | 13. d | 14. no answer | 15. d | 16. b | 17. c | 18. b | 19. c | 20. 14 |

This department wishes that it could send individual acknowledgements to the various clubs for the stunts and suggestions they have contributed. Such material is always welcome. The Mathematics and Physics Club sponsored by Sister Leontius of the College of St. Teresa, Winona, Minnesota, has been very generous in its contributions to the Stunt Library.

### Christmas Tree

Has your club ever tried constructing polyhedra from suitable colored cardboard and using these for ornaments on a mathematics Christmas tree? Some members may prefer to make only the simpler ones, such as pyramids, prisms, the platonic solids; others may venture to construct star polyhedra by adding pyramids to each of the faces of the regular solids. Archimedean solids may be treated similarly. Those with greater knowledge of or interest in model-making may find ways of constructing these solids out of single templets. With such a beginning there may be demands for a program of mathematical models later in the year. Mr. A. Harry Wheeler of Wooster, Massachusetts, gave a talk to the Mathematics Clubs of Greater Boston on model-making. The lecture illustrated as well as discussed the construction of many interesting complex polyhedra. This is one of many lectures that Mr. Wheeler has given before many eastern groups in the last several years. He has been responsible for cabinet displays of various types of polyhedra being placed at Brown University, Massachusetts Institute of Technology, and Wellesley College.

# PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND H. S. M. COXETER

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 391. *Proposed by J. Travers, Harrow, England.*

If $P$ is a point inside a square $ABCD$, so situated that $PA:PB:PC=1:2:3$, calculate the angle $APB$. Use only the methods of Euclid, Book I.

E 392. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Determine the locus of a point from which the four normals to a given ellipse form a harmonic pencil.

E 393. *Proposed by V. Thébault, Le Mans, France.*

Find two perfect squares, of five digits each, which together contain all the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. How many solutions are there?

E 394. *Proposed by N. A. Court, University of Oklahoma.*

If the lines $AM$, $BM$, $CM$ joining any point $M$ to the vertices $A$, $B$, $C$ of a tetrahedron $ABCD$ meet the respective opposite faces in the points $P$, $Q$, $R$, and if the lines $DM$, $DP$, $DQ$, $DR$ meet the face $ABC$ in the points $S$, $X$, $Y$, $Z$, prove that (both in magnitude and in sign)

$$\frac{DM}{MS}=\frac{1}{2}\left(\frac{DP}{PX}+\frac{DQ}{QY}+\frac{DR}{RZ}\right).$$

E 395. *Proposed by E. P. Starke, Rutgers University.*

In high school geometry texts and elsewhere one frequently meets the statement that the reason for the straightness of the crease in a folded piece of paper is that the intersection of two planes is a straight line. This is fallacious. What is the correct reason?

### SOLUTIONS

E 353 [1938, 691]. *Proposed by J. E. Trevor, Cornell University.*

Mrs. Black and Mrs. Brown bought cloth, each paying as many cents per yard as she bought yards. Mrs. Green and Mrs. White bought groceries. Mrs. Green spent one cent more than the excess of twice Mrs. Black's payment over seven-ninths of Mrs. Brown's. Mrs. White spent as much as the sum of two-thirds of Mrs. Brown's payment and one-half of Mrs. Green's. Mrs. Black's expenditure was equal to five-sixths of Mrs. Green's payment, plus one-third of Mrs. White's. The total expenditure was more than one dollar and less than

$$PX^2 = PX \cdot XC = AX \cdot XB = QX^2.$$

This construction fails if $P$ lies on $AB$; in that case the chord $QPQ'$ of the given circle such that angle $QPA = \pi/4$ locates $X$ as the projection of $Q$ or $Q'$ on $AB$.

Also solved by V. W. Graham, E. S. Smith, E. P. Starke, Herbert Tate, and the proposer.

*Editorial Note.* The proposer determined $X$ as lying on the circle whose diameter is the radius to $P$ of the circle $ABP$. All the other solutions were quite different. Smith made use of the following theorem: If from any point $X$ in the conjugate axis of a rectangular hyperbola, $XP$ be drawn to the vertex, and $XQ$ parallel to the transverse axis to meet the curve, then $PX = XQ$. (See Asquith's *Pure Geometry*, Cambridge, 1921, p. 200.) Tate remarked that the problem proposed is equivalent to finding the locus of the midpoints of all chords of a circle which subtend a right angle at a fixed point $P$ within the circle.

E 355 [1938, 691]. *Proposed by K. W. Miller, Utilities Research Commission, Chicago.*

If $a_1, a_2, a_3, \cdots, a_n$ are any set of $n$ arbitrary numbers, real or complex, rational or not, with or without repetitions or gaps, zero and unity excluded, which are assigned serial numbers $i$ from 1 to $n$, respectively, in any arbitrarily designated order (which may bear no relation to natural sequence or to order of magnitude), prove the identity:

$$\frac{1}{a_1} + \sum_{i=2}^{i=n} \frac{1}{a_i} \prod_{j=1}^{j=i-1} \left( \frac{a_j - 1}{a_j} \right) = 1 - \prod_{i=1}^{i=n} \left( \frac{a_i - 1}{a_i} \right).$$

*Solution by E. P. Starke, Rutgers University.*

For brevity, write $A_i = 1/a_i$. After verifying the proposed relation

$$A_1 + \sum_{i=2}^{n} A_i \prod_{j=1}^{i-1} (1 - A_j) = 1 - \prod_{i=1}^{n} (1 - A_i)$$

for $n = 1$, put it in the form

$$A_1 + \sum_{i=2}^{n-1} A_i \prod_{j=1}^{i-1} (1 - A_j) + A_n \prod_{j=1}^{n-1} (1 - A_j) = 1 - \left[ \prod_{i=1}^{n-1} (1 - A_i) \right] (1 - A_n)$$

$$= 1 - \prod_{i=1}^{n-1} (1 - A_i) + A_n \prod_{i=1}^{n-1} (1 - A_i).$$

Then its truth for every integer $n$ is evident by induction.

Also solved by E. F. Allen, and V. W. Graham.

E 356 [1939, 48]. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of enumeration there exists a two-place number with equal digits, whose square is a four-place number with equal digits. If each digit of the four-place number is itself a four-place number in the decimal system,

determine the base of the unknown system of enumeration, and show that the solution is unique.

*Solution by W. R. Talbot, Lincoln University, Jefferson City, Mo.*

Let $k$ be the base of the system, $aa$ the two-digit number, and $bbbb$ its square. Then

$$\{a(k+1)\}^2 = b(k^3 + k^2 + k + 1).$$

Dividing by $k+1$,

$$a^2(k+1) = b(k^2+1) = b(k^2-1) + 2b.$$

Thus $k+1$ divides $2b$. Since $b<k$, we must have $k+1=2b$. Substituting $2b-1$ for $k$, we find that $2a^2=(2b-1)^2+1$, whence

$$a^2 = (b-1)^2 + b^2.$$

Thus $a$ is the hypotenuse of a right triangle whose other sides are two consecutive integers. By the well known solution of that problem (see, for instance, Rouse Ball's *Mathematical Recreations and Essays*, London, 1939, p. 58), $a=c_n^2+c_{n-1}^2$, where $c_1=1$, $c_2=2$, and $c_n=2c_{n-1}+c_{n-2}$; $b-1$ and $b$ (not respectively) are $c_n^2-c_{n-1}^2$ and $2c_nc_{n-1}$. The first few values are as follows:

$$\begin{array}{lllllll}
c_n = 1, & 2, & 5, & 12, & 29, & 70, & 169, \cdots, \\
a = & 5, & 29, & 169, & 985, & 5741, & 33461, \cdots, \\
b = & 4, & 21, & 120, & 697, & 4060, & 23661, \cdots.
\end{array}$$

Since $b$ is to be a four-digit number in the decimal system, the only admissible solution is $a=5741$, $b=4060$, whence

$$k = 2b - 1 = 8119.$$

Also solved by Wm. Forman, E. P. Starke, G. W. Wishard, and the proposer.

*Editorial Note.* Starke solved the equation $2a^2=k^2+1$, and likewise the equation (1) of E 353 above, by the method of Problem 3677, this MONTHLY, November, 1935, pp. 572–577.

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

*Proposed by M. M. Flood, Princeton University.*

3927. If

$$\sum_{k=1}^{n} S_k = I_t, \qquad \sum_{k=1}^{n} p_k = t,$$

where $S_k$ is a real square symmetric matrix of order $t$ and rank $p_k$; then a real orthogonal matrix $T$ exists such that $TS_kT^{-1}=E_k^t$, where $E_k^t$ is a matrix with $p_k$ consecutive units on its principal diagonal and zeros elsewhere and such that $\sum_{k=1}^{n} E_k^t = I_t$.

3928. *Proposed by J. R. Musselman, Western Reserve University.*

If $O$ is the circumcenter of triangle $A_1A_2A_3$, and $B_i$ is the image of $A_i$ in the side $A_jA_k$; show that the circles $A_1OB_1$, $A_2OB_2$, $A_3OB_3$ meet in that point which is the inverse in the circumcircle of the isogonal conjugate point of the nine-point center.

3929. *Proposed by J. R. Musselman, Western Reserve University.*

The perpendiculars to the sides of triangle $A_1A_2A_3$ from any point $T$ on the circumcircle of the triangle cut the circle again in the points $B_1$, $B_2$, $B_3$. Show that the image lines of $B_i$ cut the sides $A_jA_k$ in three collinear points. The line of these points is $\Delta_2$ in the problem 3758 [1937, 668].

*Editorial Note.* For the definition of image line, see the article by the proposer in this MONTHLY [1938, p. 421] entitled *On the line of images.*

3930. *Proposed by V. Thébault, Le Mans, France.*

Three forces are applied at any point $P$ of the circumcircle of the triangle $ABC$ directed toward the vertices $A$, $B$, $C$. Show that: (1) if the three forces are equal, their resultant passes through a fixed point; (2) if the forces are proportional to the lengths of the sides $BC$, $CA$, $AB$, they are in equilibrium.

3931. *Proposed by V. Thébault, Le Mans, France.*

What must be the base of a number system such that numbers of the form *abcabc*004004 and 4004*abcabc* are the squares of numbers of the form *defdef* and *ghighi*?

3932. *Proposed by V. Thébault, Le Mans, France.*

What must be the base of a number system such that a number of the form *aabb* is the square of a number of the form *cc*, where $c$ is a multiple of $b$? Show that $b$ is always a perfect square, and that there exists no number system possessing squares of the form *aabb*, where $c$ is a multiple of $a$.

## SOLUTIONS

3838 [1937, 395]. *Proposed by J. R. Musselman, Western Reserve University.*

If $I$ be the incenter of triangle $A_1A_2A_3$ with $I_i$ the point of contact on the side $A_jA_k$, and $I_i'$ the image of $I$ in the side $I_jI_k$, then the circles $I_i'I_j'I_k$ pass through $\phi$, the point of Feuerbach of $A_1A_2A_3$; also the circles $I_iI_jI_k'$ meet on $I_1'I_2'I_3'$ at a point $\psi$ such that $\phi$ and $\psi$ are symmetric as to the center of the common nine-point circle of $I_1I_2I_3$ and $I_1'I_2'I_3'$.

*Note.* The point of Feuerbach for triangle $A_1A_2A_3$ is the point of tangency of its inscribed circle with its nine-point circle. See Johnson's *Modern Geometry* for a discussion of this point.

*Solution by R. Goormaghtigh, Bruges, Belgium.*

Let $r$ be the radius of the circle $I_1I_2I_3$, $\alpha_i$ the angles and $H$ the orthocenter of the triangle $I_1I_2I_3$. The angles $I_3'I_1H$ and $HI_1I_2'$ being equal to $\alpha_1$, the perpendiculars on $I_1I_2'$ and $I_1I_3'$ at $I_2'$ and $I_3'$ meet on $I_1H$ at a point $J_1$ such that $I_1J_1=r/\cos\alpha_1$. Let $Q_1$ be the image of $I_1$ through $I$; then, as

$$I_1J_1:I_1Q_1 = 1:2\cos\alpha_1 = I_1I:I_1H,$$

$IH$ and $J_1Q_1$ are antiparallel with respect to $I_1H$ and $I_1I$.

The intersection $\phi$ of the circles $I_1I_2I_3$ and $I_1I_2'J_1I_3'$ is the projection of $I_1$ on $J_1Q_1$, and $I_1\phi$ and the perpendicular from $I_1$ on the Euler line $IH$ of the triangle $I_1I_2I_3$ are isogonal in the triangle $I_1I_2I_3$. Hence $\phi$ is the focus of a parabola inscribed in $I_1I_2I_3$ and having the Euler line $IH$ as directrix, *i.e.*, by a well known property, the Feuerbach point of $A_1A_2A_3$; from analogy, the circles $I_1'I_2I_3'$ and $I_1'I_2'I_3$ also meet the circle $I_1I_2I_3$ at $\phi$.

Let, further, $\omega$ be the midpoint of $IH$; $\omega$ is the nine-point center of $I_1I_2I_3$ and also of $I_1'I_2'I_3'$ as this triangle is homothetic to the medial triangle of $I_1I_2I_3$ for the pole $I$ and the ratio 2. As $II_1'$ is equipollent to $I_1H$, $I_1'$, $I_2'$, $I_3'$ and $I_1$, $I_2$, $I_3$ are symmetric as to $\omega$, and this proves the second part of the proposed question.

*Editorial Note.* The theorem of the problem may be proved by the use of only the definition of the Feuerbach point $\phi$ in the note to the problem. Then, using the reasoning of the above solution, we shall have a proof that $\phi$ is the focus of the parabola tangent to the sides of $I_1I_2I_3$ and having the Euler line of this triangle as its directrix. It will be shown also that $\phi$ and $\psi$ lie on the conic through $I_1I_2I_3I_1'I_2'I_3'$ with its center at the center of the common nine-point circle of $I_1I_2I_3$ and $I_1'I_2'I_3'$. Also $\phi$ is the center of the rectangular hyperbola through $\overline{H}$, the orthocenter, and the exmedian points of triangle $I_1I_2I_3$.

The isosceles triangle $I_2' I_1 I_3'$ has its sides equal and parallel to the corresponding sides of $I_3 I_1 I_2$. Hence the diameter $I_1 J_1$ of the circumcircle $(I_2' I_1 I_3')$ is parallel and equal to the diameter $IA_1$ of $(I_3 I_1 I_2)$; and it follows first that $A_1 J_1$ is equal and parallel to $II_1$, and then that $A_1 J_1$ is perpendicular to $A_2 A_3$. Thus $J_1$ is on the altitude $A_1 H_1$ of $A_1 A_2 A_3$. Let the internal bisector $A_1 I$ cut side $A_2 A_3$ in $V_1$, and take $I_{11}$ on this side so that $I_{11} A_3 = A_2 I_1$: it is well known that $I_{11}$ is the point of contact of this side with the escribed circle in the angle $A_1$. Then $H_1$, $I_1$, $V_1$, $I_{11}$ form an harmonic set, since they are the orthogonal projections of the centers of similitude and the centers of the inscribed and escribed circles for $A_1$. We shall use inversion with respect to the circle $(M_1)$ with center $M_1$ and diameter $I_1 I_{11}$. In this inversion the inscribed circle $(I)$ and the escribed circle are self-corresponding; $H_1$, $V_1$ form an inverse pair; and the nine-point circle $(N)$, passing through $H_1$ and $M_1$, inverts into a straight line through $V_1$ which with the tangent to $(N)$ at $H_1$ form with $A_2 A_3$ an isosceles triangle. The tangents to $(N)$ at $H_1$ and $M_1$ form with the same side a similar isosceles triangle, and the tangent at $M_1$ is antiparallel to $A_2 A_3$ with respect to angle $A_2 A_1 A_3$. Hence the straight line through $V_1$ is also antiparallel to $A_2 A_3$ with respect to the same angle; and it now follows that this straight line is the other tangent to $(I)$ from $V_1$, with the point of contact $U_1$. It is also tangent to the escribed circle. Hence $(I)$ and $(N)$ are tangent at $\phi$, the inverse of $U_1$; and $U_1$ is the symmetric of $I_1$ with respect to $A_1 V_1$. The circles $(II_1 V_1 U_1)$ and $(I_2' I_1 I_3')$ have the parallel diameters $V_1 I$ and $I_1 J_1$, and hence the tangents to the first at $I_1$ and $V_1$, and those to the second at $H_1$ and $I_1$ form similar isosceles triangles with $A_2 A_3$. Hence these two circles are inverses, and it now follows that $(I_2' I_1 I_3')$ must pass through $\phi$. Similarly, $(I_3' I_2 I_1')$ and $(I_1' I_3 I_2')$ pass through $\phi$.

What follows is a varied form of Goormaghtigh's proof of the focal properties of $\phi$ with more detail. The triangle $I_1 I_2 I_3$ has the circumcenter $I$ and the orthocenter $\overline{H}$, and hence $I_1 \overline{H}$ is equal and parallel to $II_1'$; and it also follows that $\overline{H}$ is on $I_1 J_1$. In the right triangle $II_3 A_1$ let $K_1$ be the common midpoint of $II_1'$ and $I_2 I_3$; then $IK_1 \cdot IA_1 = (II_3)^2 = r^2$. This, with above equations, gives $I_1 \overline{H} \cdot I_1 J_1 = 2r^2 = I_1 I \cdot I_1 Q_1$, where $I_1 Q_1$ is a diameter of $(I)$. Hence $I\overline{H}$ and $J_1 Q_1$ are antiparallel with respect to angle $J_1 I_1 Q_1$; and, since angles $I_1 \phi Q_1$ and $I_1 \phi J_1$ are right angles, $\phi$ is the projection of $I_1$ on $J_1 Q_1$. From this and the antiparallelism above it follows that $I_1 \phi$ and the perpendicular $I_1 \bar{I}_1$ to $I\overline{H}$ are isogonal conjugates with respect to angle $II_1 \overline{H}$. Since $I$ and $\overline{H}$ are isogonal conjugates with respect to triangle $I_1 I_2 I_3$, it follows that $I_1 \bar{I}_1$ and $I_1 \phi$ are isogonal conjugates also with respect to angle $I_3 I_1 I_2$. Since $\phi$ is on $(I)$ it is the focus of a parabola tangent to the sides of $I_1 I_2 I_3$, and the directrix passes through $\overline{H}$. We now use the theorem that the tangents from any point to a conic are isogonal conjugates with respect to the angle formed by the straight lines from the point to the foci. For the parabola the above results show that $I\overline{H}$ must be the directrix, and we have the proof of the focal property of $\phi$.

Since $I_1 \overline{H}$ and $II_1'$ are equal and parallel, $I\overline{H}$ and $I_1 I_1'$ have $\overline{N}$ as a common midpoint, where $\overline{N}$ is the nine-point circle center for $I_1 I_2 I_3$. Thus $I_1 I_1'$, $I_2 I_2'$,

$I_3I_3'$ have the common midpoint $\overline{N}$. This gives another proof that the circles $(I_1I_2I_3)$, $(I_1I_2'I_3')$, $(I_2I_3'I_1')$, $(I_3I_1'I_2')$ meet in a point; see *Note* following solutions of 3797 [1938, 487]. The hexagon $I_1I_3'I_2I_1'I_3I_2'$ has its opposite sides equal and parallel, and hence it is inscribed in a conic $\{\overline{N}\}$ with center at $\overline{N}$. It was shown in the above-mentioned *Note* that this conic passes through $\phi$. Also $\psi$, the symmetric of $\phi$ with respect to $\overline{N}$, lies on $\{\overline{N}\}$. It is easily shown that $\overline{H}$ is the circumcenter of the triangle $B_1B_2B_3$ whose vertices are the exmedian points of $I_1I_2I_3$, that is the two triangles have a common centroid as center of similitude with the ratio $2:-1$. Also $\overline{H}B_i$ has $I_i'$ for its midpoint. It then follows from the above reference that $\phi$ is the center of the rectangular hyperbola through the four points $\overline{H}$, $B_1$, $B_2$, $B_3$. The asymptotes are easily constructed by methods given in the reference.

Another proof that the parabola has $I\overline{H}$ for directrix will be given. The feet of the perpendiculars from $\phi$ to the sides of $I_1I_2I_3$ lie on the tangent $t$ to the parabola at its vertex. The circle on $I_1\phi$ as diameter passes through the feet on $I_1I_2$ and $I_1I_3$, and $J_1Q_1$ is tangent at $\phi$. It follows that $t$ is antiparallel to $J_1Q_1$ with respect to angle $I_2I_1I_3$, and then also with respect to angle $J_1I_1Q_1$. It was shown that $J_1Q_1$ is antiparallel to $I\overline{H}$ with respect to the last angle. Therefore $I\overline{H}$ is parallel to $t$, and it must be the directrix since the directrix passes through $\overline{H}$.

3839 [1937, 395]. *Proposed by V. Thébault, Le Mans, France.*

A transversal $\Delta$ cuts the sides $BC$, $CA$, $AB$ of a triangle in $\alpha$, $\beta$, $\gamma$. Parallels, with arbitrary direction, through the vertices of the triangle cut $\Delta$ in $\alpha'$, $\beta'$, $\gamma'$. Prove that the parallels to $BC$, $CA$, $AB$ through $\alpha'$, $\beta'$, $\gamma'$ divide in the same ratio the straight line segments $O_aH_a$, $O_bH_b$, $O_cH_c$ which join the circumcenters $O_a$, $O_b$, $O_c$ to the orthocenters $H_a$, $H_b$, $H_c$ of triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$. See 3818 [1937, 111].

*Editorial Note.* Three straight lines through the vertex $A$ cut from the transversal $\Delta$ two segments $\alpha_1'\alpha_2'$, $\alpha_2'\alpha_3'$, and parallels through $B$ and $C$ cut from $\Delta$ corresponding pairs of segments having the same ratio, say $r$. We shall suppose that $\Delta$ is not parallel to a side of $ABC$. Parallels to $BC$ through $\alpha_1'$, $\alpha_2'$, $\alpha_3'$ cut from any transveral corresponding segments in the same ratio $r$; and, similarly, for parallels to $CA$ and $AB$ through the corresponding points on $\Delta$. It suffices then to show that, if parallels to $BC$, $CA$, $AB$ through $O_a$, $O_b$, $O_c$, respectively, cut $\Delta$ in $\alpha_0'$, $\beta_0'$, $\gamma_0'$, and similar parallels through $H_a$, $H_b$, $H_c$ cut $\Delta$ in $\alpha_h'$, $\beta_h'$, $\gamma_h'$, then $A\alpha_0'$, $B\beta_0'$, $C\gamma_0'$ form one parallel set and $A\alpha_h'$, $B\beta_h'$, $C\gamma_h'$ form another parallel set. The following analytic proof of this is tedious, and a more elegant proof is desirable. The triangle and its transversal form a complete quadrilateral, $ABC$, $\Delta$, whose sides are tangent to a parabola $y^2=4ax$; and the equations of these sides may be taken as $m_iy-x-am_i^2=0$, $i=1, 2, 3, 4$. We shall denote by $\sigma_j$ the four elementary symmetric functions of the four $m_i$'s, and by $\sigma_j^k$ the three elementary symmetric functions of the three $m_i$'s obtained by omitting $m_k$. We then have the following results in coördinates and equations:

$$
\begin{array}{ll}
A: & am_2m_3, \qquad a(m_2+m_3), \\
O_a: & \dfrac{a}{2}(1+\sigma_2^1), \qquad \dfrac{a}{2}(\sigma_1^1-\sigma_3^1), \\
H_a: & -a, \qquad a(\sigma_1^1+\sigma_3^1), \\
O_a\alpha_0': & 2(m_1y-x)-a(m_1\sigma_1^1-\sigma_4-\sigma_2^1-1)=0, \\
\Delta: & m_4y-x-am_4^2=0, \\
A\alpha_0': & y-\lambda_0x-a(m_2+m_3-\lambda_0m_2m_3)=0, \\
H_a\alpha_h': & m_1y-x-a(m_1\sigma_1^1+\sigma_4+1)=0, \\
A\alpha_h': & y-\lambda_hx-a(m_2+m_3-\lambda_hm_2m_3)=0.
\end{array}
$$

The slope $\lambda_0$ of $A\alpha_0'$ will be determined in the following manner: equate identically the sum of the left member of the equation for $O_a\alpha_0'$ and $p$ times that of $\Delta$ to $q$ times that of $A\alpha_0'$. The three equations resulting from the identity give easily

$$(\lambda_0m_4-1)p=-2(\lambda_0m_1-1), \qquad (\lambda_0m_4-1)q=2(m_4-m_1);$$

and we then find that

$$\lambda_0[m_4(\sigma_2^4-1)-m_4^2(\sigma_3^4+\sigma_1^4)-2\sigma_3^4]+[2m_4^2+1+m_4(\sigma_3^4-\sigma_1^4)+\sigma_2^4]=0.$$

This shows that this value of $\lambda_0$ is also the slope of $B\beta_0'$ and of $C\gamma_0'$. Using the same method for $\lambda_h$ we find that the multipliers $p$ and $q$ are one-half of those above; and it then follows that

$$\lambda_h[m_4(\sigma_2^4+1)+(m_4^2-1)\sigma_3^4]+[m_4^2-m_4(\sigma_3^4+\sigma_1^4)-1]=0.$$

Hence this value of $\lambda_h$ is the common slope of $A\alpha_h'$, $B\beta_h'$, $C\gamma_h'$; and this completes the proof.

The theorem of this problem is related to that of 3818 [1939, 178]. In the latter theorem ($ABC$, $\Delta$) denotes the straight line joining the midpoints of the diagonals of the indicated complete quadrilateral, and $\Delta'$ denotes the straight line through $H$, $H_a$, $H_b$, $H_c$, the directrix of the parabola. It will be shown that the theorem of 3818 may be stated in the more general form.

(a) The sides of the triangles symmetrically equal to $ABC$, $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ with respect to $P$, the intersection of ($ABC$, $\Delta$) and ($ABC$, $\Delta'$), pass respectively through the orthocenters ($H_a$, $H_b$, $H_c$), ($H$, $H_b$, $H_c$), ($H$, $H_c$, $H_a$), ($H$, $H_a$, $H_b$).

(b) The points $H$ and $P$ are on the same side of $\Delta$, and the distance of $P$ from $\Delta$ is one-half of that of $H$ from $\Delta$.

(c) The straight lines ($A\beta\gamma$, $\Delta'$), ($B\gamma\alpha$, $\Delta'$), ($C\alpha\beta$, $\Delta'$) pass also through $P$.

The lines $H_a\alpha_h'$, $H_b\beta_h'$, $H_c\gamma_h'$ form a triangle $A'B'C'$ similar to $ABC$. The equation of $B'C'$ is the one given for $H_a\alpha_h'$; and, if we replace in it $m_1$ by $m_2$ and the superscript 1 by 2, we have the equation for $H_b\beta_h'$, or $C'A'$. The coördinates of $C'$ are then found to be

$$C': \qquad -a(m_1m_2+\sigma_4+1), \qquad a(m_3+m_4).$$

The midpoint of $CC'$ is

$$P: \qquad -\frac{a}{2}(\sigma_4+1), \qquad \frac{a}{2}\sigma_1;$$

and it is clear that $P$ is also the midpoint of $AA'$ and of $BB'$. Hence with respect to $P$ the triangles $ABC$ and $A'B'C'$ are symmetrically equal. The form of the coördinates of $P$ shows that we obtain the same point $P$ for the remaining three sets $(H, H_b, H_c)$, $(H, H_c, H_a)$, $(H, H_a, H_b)$. It is obvious that $P$ lies on $(ABC, \Delta)$ which has the equation $y = a\sigma_1/2$. The equation of $(ABC, \Delta')$ is

$$2(y\sigma_3^4 + x) - a(\sigma_1^4\sigma_3^4 - 1) = 0;$$

and the equation of $(A\beta\gamma, \Delta')$ is obtained from the above by replacing the superscript 4 by 1. It is easily verified that these two lines intersect in $P$. It is clear that any pair of lines in (c) gives the same result. If we insert the coördinates of $H$ in the left member of the equation for $\Delta$, we get $a(m_4\sigma_1+\sigma_4+1-2m_4^2)$. If we insert the coördinates of $P$ in the left member of $\Delta$, we get one-half of this expression. This proves (b). Moreover, we have similar results for $H_a$ and $P$ with respect to $BC$, $H_b$ and $P$ with respect to $CA$, $H_c$ and $P$ with respect to $AB$, using the proper subscripts for $m$.

It is easily shown synthetically that $O$, $O_a$, $O_b$, $O_c$ lie on a circle through the focus $F$.

3841 [1937, 543]. *Proposed by V. Thébault, Le Mans, France.*

Consider the orthocentric tetrahedron $ABCD$, any point $P$, and the inverse points $A'$, $B'$, $C'$, $D'$ of $A$, $B$, $C$, $D$ in an inversion with the pole $P$ and with the arbitrary modulus $k$. The planes perpendicular to the lines $PA$, $PB$, $PC$, $PD$ passing through $A'$, $B'$, $C'$, $D'$ determine a tetrahedron $A_1B_1C_1D_1$. Let $P'$ be the isogonal conjugate of $P$ with respect to the tetrahedron $A_1B_1C_1D_1$. (a) Show that the pedal tetrahedron $A_1'B_1'C_1'D_1'$ of the point $P'$ with respect to $A_1B_1C_1D_1$ is orthocentric. (b) Show that the planes through $P$ parallel to the faces of $A_1'B_1'C_1'D_1'$ cut the planes of the faces of $A_1B_1C_1D_1$ in four straight lines lying in the same plane perpendicular to the line joining $P'$ to the orthocenter of $A_1'B_1'C_1'D_1'$.

*Editorial Note.* The proposer gave a solution using inversion. The use of the polar theory for the greater part appears to be better suited to the nature of the problem. Let $(O)$ and $(O')$ be the circumspheres with centers $O$ and $O'$ and radii $r$ and $r'$ of the inverse pair of tetrahedrons $T \equiv ABCD$ and $T' \equiv A'B'C'D'$. The center $P$ of inversion is arbitrary, but not on $(O)$. Since $(O)$ and $(O')$ are inverses in this inversion, they have $P$ as a center of similitude with the ratio $\pm r/r'$. The ray $PA$ cuts $(O')$ in $A'$ and in another point $A''$, and thus the tetrahedron $T'' \equiv A''B''C''D''$ is similar to $T$, and it is therefore orthocentric. We may now discard $T$ and consider $T''$ in its place. For $T'$ and $T''$ have corresponding vertices as inverse pairs in an inversion for which $(O')$ is self-corre-

sponding with respect to the sphere $(P)$ with center $P$ and radius whose square is $(PO')^2-(r')^2$. With respect to this new sphere the polar reciprocal of $T''$ is $T_1 \equiv A_1B_1C_1D_1$; and with respect to this last tetrahedron $T'$ is the pedal tetrahedron of $P$. If we produce $PO'$ to $P'$ so that $O'P'=PO'$, then $P'$ and $P$ are isogonal conjugates with respect to $T_1$, and the projections of $P'$ on the faces of $T_1$ lie also on $(O')$, giving the pedal tetrahedron $T_1' \equiv A_1'B_1'C_1'D_1'$ of $P'$. The two tetrahedrons $T_1'$ and $T''$ are symmetric with respect to $O'$, and hence the first is also orthocentric. Let $H_1'$ and $H''$ be their respective orthocenters. The plane $\alpha$ through $P$ parallel to the face $B_1'C_1'D_1'$ is also parallel to the face $B''C''D''$, and $\alpha$ cuts the face $B_1C_1D_1$ in a straight line $a$. The altitude $A''H''$ is perpendicular to the plane $\alpha$, and hence the pole of $\alpha$ is the point at infinity on $A''H''$; while the pole of face $B_1C_1D_1$ is $A''$. Hence the polar plane $h''$ of $H''$ passes through $a$; and in the same way we see that $h''$ passes through the remaining three straight lines $b$, $c$, $d$, which are defined in the same way as $a$. The plane $h''$ is necessarily perpendicular to $PH''$ and to its parallel $P'H_1'$. This completes the proof. A similar theorem for two dimensions is given in 3640 [1935, 397]; and the extension to $n$ dimensions requires merely a slight change in wording of the above solution. See the solution of 3821 [1939, 241] for a related theorem and extension to $n$ dimensions.

3842 [1937, 543]. *Proposed by V. Thébault, Le Mans, France.*

A given plane $(P)$ cuts the edges $A_2A_3$, $A_3A_1$, $A_1A_2$, $A_4A_1$, $A_4A_2$, $A_4A_3$, of a tetrahedron $A_1A_2A_3A_4$ in $a_1$, $a_2$, $a_3$, $a_1'$, $a_2'$, $a_3'$, respectively. Parallel lines, with arbitrary direction, through $A_1$, $A_2$, $A_3$, $A_4$ cut respectively the spheres circumscribing the tetrahedrons $A_1a_2a_3a_1'$, $A_2a_3a_1a_2'$, $A_3a_1a_2a_3'$, $A_4a_1'a_2'a_3'$ in the points $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$. Show that these points lie in a plane $(Q)$ which passes through a fixed point in the plane $(P)$ when the direction of the parallels varies.

*Editorial Note.* The proposer stated, as an indication of a solution, that this problem is an application of a previous generalization of Roberts' theorem, and that it may be proved directly that the four spheres of the problem intersect in a point $S$ of the plane $(P)$; and then it may be shown directly that the points $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ lie in a plane $(Q)$ passing through $S$. No other details were given.

We shall supply details of what is probably meant by the above two direct proofs. It will be assumed that the plane $(P)$ cuts the faces of the given tetrahedron in four straight lines forming an actual complete quadrilateral with the vertices $a_1$, $a_2$, $a_3$, $a_1'$, $a_2'$, $a_3'$. The four spheres

$$(1)\qquad (A_1, a_1', a_2, a_3), \quad (A_2, a_2', a_3, a_1), \quad (A_3, a_3', a_1, a_2), \quad (A_4, a_1', a_2', a_3')$$

cut the plane $(P)$ in four circles

$$(2)\qquad (a_1', a_2, a_3), \quad (a_2', a_3, a_1), \quad (a_3', a_1, a_2), \quad (a_1', a_2', a_3'),$$

which circumscribe the triangles, with the vertices in the parentheses, of the complete quadrilateral. It is known that these four circles meet in a point $S$. Hence the four spheres (1) meet in this point $S$ on the plane $(P)$. It is easily

seen that $S$ cannot lie also on a face of the given tetrahedron; for, if it did, the complete quadrilateral would degenerate, and this is excluded.

We now apply the proposer's theorem in 3819 [1939, 239] using for the point $Q$ *in that theorem* the point at infinity in the direction of the four parallels $A_1\alpha_1$, *etc.* The sphere of that theorem is now a plane, denoted by $(Q)$ in the present problem, passing through $S$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, where $S$ is shown above to be fixed in the plane of $(P)$.

3843 [1937, 543]. *Proposed by V. Thébault, Le Mans, France.*

Given a tetrahedron $ABCD$, with an orthocenter $H$, inscribed in a sphere with center $O$. (a) Show that the tetrahedron with vertices $O_a$, $O_b$, $O_c$, $O_d$ at the centers of the spheres $HBCD$, $HCDA$, $HDAB$, $HABC$ is orthocentric. (b) The center of the sphere $O_aO_bO_cO_d$ is the symmetric of the centroid with respect to, $H$. (c) The points $O$ and $H$ are isogonal conjugates with respect to $O_aO_bO_cO_d$.

*Solution by the Proposer.*

The first part results from the fact that the tetrahedrons $HBCD$, $HCDA$ $HDAB$, $HABC$ are also orthocentric. Moreover, the tetrahedrons $O_aO_bO_cO_d$ and $ABCD$ are similar in the ratio $-3:2$, and the center of similitude $P$ divides $OH$ in the same ratio. From this (b) easily follows.

(c) The spheres $(HBCD)$, $(HCDA)$, $(HDAB)$, $(HABC)$ intersect in $H$ and also sets of three intersect in $A$, $B$, $C$, $D$. From one of our theorems in *Annales de la Soc. Scientifique de Bruxelles*, 1921, p. 240, the points $O$ and $H$ are isogonal conjugates with respect to the tetrahedron $O_aO_bO_cO_d$. This may also be shown from the fact that $O$ is the orthocenter of $O_aO_bO_cO_d$ and $H$ is the orthocenter of the tetrahedron formed by the centroids of the faces of $O_aO_bO_cO_d$.

*Editorial Note.* These results are easily extended to space of $n$ dimensions. Let $S$ be an orthocentric simplex with the vertices $A_i$, $i=1, 2, \cdots, n+1$, and with the orthocenter $H$, which we suppose does not lie in any face of $S$. The vertices of $S$ and $H$ form an orthocentric set of $n+2$ points such that each point of the set is the orthocenter of the simplex formed by the remaining $n+1$ points of the set. Let $H$ be the origin of vectors $\mathbf{a}_i$ to the vertices $A_i$, then $\mathbf{a}_i\cdot\mathbf{a}_j=\mathbf{a}_j\cdot\mathbf{a}_k=\mathbf{a}_k\cdot\mathbf{a}_i$, where $i, j, k$ are distinct; and the above statement follows easily. Let $\mathbf{g}$ and $\mathbf{c}$ be the vectors of $G$ and $C$, the centroid and circumcenter of $S$. Then $\mathbf{c}$ is uniquely defined, for any non-degenerate $S$, by the set of equations

$$(1)\qquad \left(\mathbf{c}-\frac{\mathbf{a}_i+\mathbf{a}_j}{2}\right)\cdot(\mathbf{a}_i-\mathbf{a}_j) = 0.$$

Also $(n+1)\mathbf{g}\cdot(\mathbf{a}_i-\mathbf{a}_j)=(\mathbf{a}_i+\mathbf{a}_j)\cdot(\mathbf{a}_i-\mathbf{a}_j)$, since $\mathbf{a}_h\cdot(\mathbf{a}_i-\mathbf{a}_j)=0$, for $h$, $i$, $j$ distinct. Thus

$$(2)\qquad \left(\frac{n+1}{2}\mathbf{g}-\frac{\mathbf{a}_i+\mathbf{a}_j}{2}\right)\cdot(\mathbf{a}_i-\mathbf{a}_j) = 0;$$

and, since the solution of (1) is unique, we must have

$$\mathbf{c} = \frac{n+1}{2}\mathbf{g}, \qquad \text{or} \qquad HC/HG = (n+1)/2. \tag{3}$$

Let $S_i$ be the simplex obtained by omitting $A_i$ from the orthocentric set of $n+2$ points; and let its centroid $G_i$, its orthocenter $H_i$, and its circumcenter $C_i$ have the vectors $\mathbf{g}_i$, $\mathbf{h}_i$, $\mathbf{c}_i$. From the above we have $\mathbf{h}_i = \mathbf{a}_i$; and $(n+1)\mathbf{g}_i = (n+1)\mathbf{g} - \mathbf{a}_i$ or $\mathbf{g}_i = \mathbf{g} - \mathbf{a}_i/(n+1)$. From the relation $H_iC_i/H_iG_i = (n+1)/2$, we have the last of the equations below:

$$\mathbf{g}_i = \mathbf{g} - \mathbf{a}_i/(n+1), \qquad \mathbf{h}_i = \mathbf{a}_i, \qquad 2\mathbf{c}_i = (n+1)\mathbf{g} - n\mathbf{a}_i. \tag{4}$$

Consider now the simplex $S'$ with the vertices $C_i$. The last equation of (4) says that the end of the vector $(n+1)\mathbf{g}/(n+2)$ is the center of similitude for $S'$ and $S$ with the ratio $n: -2$. Hence $S'$ is also orthocentric, and we now find easily from the last equation of (4) that its centroid $G'$, its orthocenter $H'$, and its circumcenter $C'$ have the vectors

$$\mathbf{g}' = \mathbf{g}/2, \qquad \mathbf{h}' = (n+1)\mathbf{g}/2 = \mathbf{c}, \qquad \mathbf{c}' = -(n+1)(n-2)\mathbf{g}/4. \tag{5}$$

Hence the points $C'$, $H$, $C \equiv H'$, $G'$, $G$ lie in a straight line. We have now to determine $F'$, the isogonal conjugate of $H'$ with respect to $S'$. In the note on the solution of 3821 [1939, 243], it was shown that $H'F' = 2H'C'/n$ and, since $H'C' = -n(n+1)\mathbf{g}/4$, $H'F' = -(n+1)\mathbf{g}/2$. Therefore $F' \equiv H$, and $H$ and $C$ are isogonal conjugates with respect to $S'$. In the same way we find that $F$, the isogonal conjugate of $H$ with respect to $S$, has the vector $(n+1)\mathbf{g}/n$. The special case of $n=2$, where $S$ is a triangle, is worthy of notice, for it is only in this case that the orthocenter and the circumcenter are isogonal conjugates with respect to the simplex.

The last sentence of the proposer's solution states for $n=3$ the theorem: The isogonal conjugate $F$ of the orthocenter $H$ of a simplex $S$ with respect to the same simplex is the orthocenter of the simplex $B$ whose vertices $B_i$ are the centroids of the faces of $S$. With $H$ as origin, let $\mathbf{b}_i$ be the vector of $B_i$; then $n\mathbf{b}_i = (n+1)\mathbf{g} - \mathbf{a}_i$, or $n(\mathbf{b}_i - \mathbf{g}) = -(\mathbf{a}_i - \mathbf{g})$. Hence $B$ and $S$ have $G$ as center of similitude with the ratio $1: -n$; and it follows that $B$ is also orthocentric. It also follows that the vector of the orthocenter of $B$ is $(n+1)\mathbf{g}/n$, and as pointed out above this is the vector of $F$.

The other theorem used by the proposer in the proof of (c) can also be extended to $n$ dimensions. Given any simplex $S$ with the vertices $A_i$, and any arbitrarily chosen point $P$, let now $C_i$ be the center of the sphere $(C_i)$ passing through $P$ and all of the vertices of $S$ except $A_i$, and let $S'$ be the simplex having the vertices $C_i$. Thus $(C_i)$ may be denoted by $(P, A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_{n+1})$ and hence $(C_1)$, $(C_2)$, $\cdots$, $(C_{i-1})$, $(C_{i+1})$, $\cdots$, $(C_{n+1})$ have the common chord $PA_i$. Let $M_i$ be the midpoint of $PA_i$, then $C_jM_i$, $j \neq i$, is perpendicular to $PA_i$ at $M_i$. Denote by $\pi_i$ the plane of the face of $S'$ opposite to $C_i$; then from the above, $\pi_i$ is perpendicular to $PA_i$ at its midpoint $M_i$. The simplexes $M$ with the

vertices $M_i$ and $S$ with the vertices $A_i$ have $P$ as center of similitude with the ratio 1:2. Denote by $Q$ and $C$ the centers of the circumspheres $(M_1, M_2, \cdots, M_{n+1})$, $(A_1, A_2, \cdots, A_{n+1})$; then $PQ=QC$. Thus $(M_1, M_2, \cdots, M_{n+1})$ with the center $Q$ is the pedal sphere of $P$ with respect to $S'$, and hence $P$ and $C$ are isogonal conjugates with respect to $S'$.

## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

The annual Pi Mu Epsilon breakfast will be held on Thursday morning, December 28, at eight o'clock in the Pomerene Hall Refectory of Ohio State University, Columbus, Ohio.

Professor Emil Artin of the University of Indiana will give monthly lectures in group theory and the theory of numbers at the University of Notre Dame during 1939–1940.

Applications are being received for Benjamin Peirce Instructorships in Mathematics at Harvard University for the academic year 1940–41. Candidates should ordinarily have the doctorate or its equivalent. Applications should be sent to the Chairman of the Department of Mathematics, from whom further information may be received.

Professor G. D. Birkhoff of Harvard University was awarded an honorary doctorate by the University of Sofia on the occasion of the celebration of the fiftieth anniversary of the founding of that institution. On September 1, 1939, he retired as dean of the faculty of arts and sciences of Harvard University, and will spend the second semester of the current year as exchange professor in France.

Harvard University has conferred an honorary doctorate upon Dr. P. W. Bridgman, professor of mathematics and natural philosophy at Harvard University.

Professor Karl Menger of the University of Notre Dame is conducting a fortnightly seminar at the Armour Institute of Technology which will run throughout the year.

Associate Professor V. W. Adkisson of the University of Arkansas has been promoted to a professorship.

Dr. C. B. Allendoerfer of Haverford College has been promoted to an assistant professorship.

H. F. Archibald of Keuka College, Keuka Park, New York, has been promoted to an assistant professorship.

Assistant Professor J. L. Barnes of Tufts College has been promoted to an associate professorship.

Dr. W. Z. Birnbaum of New York University has been appointed to an assistant professorship at the University of Washington.

Assistant Professor R. V. Blair of Vanderbilt University has been promoted to an associate professorship.

Associate Professor H. E. Bray of Rice Institute has been promoted to a professorship.

At the North Carolina State College Assistant Professors R. C. Bullock and Jack Levine have been promoted to associate professorships.

Assistant Professor R. S. Burington of the Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor J. W. Calkin of the University of New Hampshire has been appointed an assistant professor at the Armour Institute of Technology.

O. C. Collins of the University of Nebraska has been promoted to an assistant professorship.

Dr. A. P. Cowgill of Syracuse University has been promoted to an assistant professorship of applied mathematics.

Assistant Professor D. C. Dearborn of Catawba College, Salisbury, North Carolina, has been promoted to an associate professorship.

At the University of Chicago Professor L. E. Dickson and Associate Professor E. R. Breslich have retired.

Dr. F. G. Dressel of Duke University has been granted a leave of absence for the year 1939–40. He will be at Brown University for this period.

Leaves of absence for the year 1939–1940 have been granted to Assistant Professors W. W. Flexner and J. B. Rosser of Cornell University. The former will spend the year at the Institute for Advanced Study at Princeton, and the latter will lecture at Princeton University.

Dr. B. E. Gatewood of the University of Texas has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Associate Professor A. E. Gault of Bradley Polytechnic Institute has been promoted to a professorship.

Associate Professor J. J. Gergen of Duke University has been promoted to a professorship.

Dr. Israel Halperin of Harvard University has been appointed to an assistant professorship at Queen's University, Kingston, Ontario.

Coleman Herpel of the Pennsylvania State College Undergraduate Center at Hazleton has been promoted to an assistant professorship.

Dr. R. A. Higdon of Iowa State College has been promoted to an assistant professorship.

Dr. I. M. Hostetter of Howard College has been promoted to a professorship.

Dr. H. S. Kaltenborn of the University of Texas has been appointed an assistant professor at Louisiana Polytechnic Institute.

Dr. L. A. Knowler of the University of Iowa has been promoted to an assistant professorship.

Dean G. H. Ling of the University of Saskatchewan retired in December 1938 and is now living in Toronto.

Associate Professor J. B. Linker of the University of North Carolina has been promoted to a professorship.

Assistant Professor M. H. Martin of the University of Maryland has been promoted to an associate professorship.

Associate Professor A. D. Michal of the California Institute of Technology has been promoted to a professorship.

Dr. W. A. Patterson of Fenn College has been promoted to an assistant professorship.

Dr. Sallie E. Pence of the University of Kentucky has been promoted to an assistant professorship.

Assistant Professor H. A. Rademacher of the University of Pennsylvania has been promoted to the rank of professor.

Mr. N. S. Risley has been appointed assistant professor of mathematics at Fenn College. In the September number of this MONTHLY it was erroneously announced that he was at the Case School of Applied Science.

Dr. A. E. Ross of St. Louis University has been promoted to an assistant professorship.

Dr. J. B. Rosser of Cornell University has been promoted to an assistant professorship.

Assistant Professor H. C. Schaub of Washington and Jefferson College has been promoted to an associate professorship.

Associate Professor H. M. Sheffer of Harvard University has been promoted to a professorship.

Associate Professor J. A. Shohat of the University of Pennsylvania has been

granted a leave of absence during the second term of 1939–40 for the purpose of writing the second part of a book on Orthogonal Polynomials.

Dr. A. H. Smith of Purdue University has been promoted to an assistant professorship.

Dr. S. Ulam has been appointed Lecturer in mathematics at Harvard University for the academic year 1939–40.

The following appointments to instructorships are announced:

University of Alabama: Dr. E. S. Kennedy.
Armour Institute of Technology: Dr. G. E. Hay, Dr. A. T. Lonseth
Brown University: Dr. D. W. Hall, L. B. Hedge
University of California (Berkeley): Dr. Joel Brenner
Cornell University: Theodore Hailperin, Dr. Mark Kac, Dr. G. E. Schweigert
University of Detroit: Dr. Alvin Sugar.
George Washington University: Dr. J. W. Wrench, Jr.
Georgia School of Technology: Dr. L. J. Green, Dr. Nelson Robinson
Haverford College: E. E. Betz
University of Illinois: Dr. Ivan Niven
University of Michigan: Dr. W. T. Scott
University of Minnesota: Fulton Koehler
University of Nevada: W. G. Palm, Dr. E. P. Vance.
New York University: Dr. H. E. Robbins
Ohio State University: M. Hendrickson, Dr. P. V. Reichelderfer, Dr. W. S. Snyder
Purdue University: Dr. C. E. Clark, J. J. Eachus
Rensselaer Polytechnic Institute: Dr. B. A. Lengyel
Stanford University: Dr. A. C. Schaeffer
Tulane University: Albert Neuhaus
Wayne University: Morris Friedman, Joseph Levin
College of William and Mary: Dr. Wilfred Kaplan.
University of Wyoming: T. C. Doyle

Dr. S. C. Harry of Baltimore, Maryland, died September 19, 1939. He was a charter member of the Association.

Associate Professor W. H. Lyons of Kansas State College died October 21, 1939. Professor Lyons had been connected with the College for fifteen years, and was a member of the Mathematical Association since 1921.

Dr. W. C. Risselman died at Inglewood, California, on June 26, 1939. He was head of the department of mathematics at Arizona State Teachers College, Flagstaff, from 1929 until 1937, when he was taken ill. He had been a member of the Association for twelve years.

Associate Professor A. A. Shaw of the University of Arizona died on October 11, 1939, in San Francisco, California.

## SPECIAL RATE FOR THE DUKE MATHEMATICAL JOURNAL

Duke University Press makes the following announcement about the Duke Mathematical Journal. Mathematical Association members may purchase back numbers to complete their files at half the regular price, if the order is placed before December 31, 1940; thereafter no reduction will be made on back volumes. Institutional members will henceforth be entitled to the same reductions as individuals. All orders at reduced prices must be marked "Member of M.A.A." and sent direct to the Duke University Press, Durham, N. C.

## ANNOUNCEMENT OF "MATHEMATICAL REVIEWS"

The American Mathematical Society is in the fortunate position of having the financial backing, through the munificence of two of the great Foundations, to found a new international mathematical abstracting journal to be known as *Mathematical Reviews.* During the past quarter-century while the United States and Canada have been gradually assuming a more prominent part in mathematical research, there has been sentiment expressed from time to time among mathematicians that there should be a review journal sponsored by American organizations. But doubts as to whether the scientific and financial resources could be spared caused the postponement of the undertaking. However, the rapid growth of American mathematical resources and the availability of funds have resolved these doubts, and it has been decided to proceed immediately.

The first number of *Mathematical Reviews* is to appear late in 1939 or early in 1940; the material to be reviewed begins with the latter half of 1939. It is proposed to review all fields of pure mathematics and also those of applied mathematics and mathematical physics which are of pronounced interest to mathematicians. The new journal, which will be issued approximately once a month, will contain several thousand reviews annually and will run to approximately eight hundred large double-column pages. Professors J. D. Tamarkin and Otto Neugebauer will be the first editors. A strong group of collaborators for the initial period is assured.

The Carnegie Corporation has appropriated $60,000 as a backlog for the new journal. The Rockefeller Foundation has made a gift of $12,000 to cover some of the initial costs. Brown University is housing the project and aiding in the editorial work. The American Mathematical Society and the Mathematical Association of America are each starting off with a subsidy of $1,000 for the first year. Annual subsidies are being sought from other organizations, and plans for the permanent financing of the project are being considered. On account of the generous subventions, the subscription price will be set drastically below actual cost, $13.00 per volume to general subscribers, and $6.50 to members of the American Mathematical Society or of the Mathematical Association of America.

Partly with a view to aiding indirectly in the support of this journal, the Rockefeller Foundation has made a handsome gift to Brown University for an experiment in the dissemination of mathematical publications through the distribution of microfilm. This money is to be used to augment the mathematical library at that university, a collection which is already internationally known as outstanding. Out-of-print journals will be put on film and made available to mathematicians; rare books of general use will be filmed; on request from a subscriber to the new journal, any article reviewed will be sent on film or as film-print. This service will be extended to all parts of the world at a price not exceeding cost. It should be of greatest value to mathematicians located in the smaller universities and colleges and should be a factor in encouraging young men to continue with their investigations. This interesting experiment in the promotion of a new aid to learning should prove to be an asset not only to *Mathematical Reviews*, but also to American mathematics in general.

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For a list of all officers and trustees for preceding years, see pages 71–72

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American Association for the Advancement of Science—C. S. ATCHISON, Washington and Jefferson College, and F. D. MURNAGHAN, Johns Hopkins University, for the year 1939.

American Documentation Institute—J. H. TAYLOR, George Washington University, for term ending January 1940.

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*Closed for printing October 16, 1939*

NOTE. This list gives data for the members of the Association as taken from the Register issued in November 1937, unless changes in position, mailing address, etc., have been reported to the Secretary. Records may be inexact or incomplete through the failure of members to return fresh information to the Secretary, but great effort has been put forth to be full and correct. When members are known to be occupied in other lines than mathematics, this fact is indicated.

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ADAMS, O. S., D.Sc. (Kenyon) Senior Mathematician, U. S. Coast and Geodetic Survey, Washington, D.C. *727 Massachusetts Ave. N.E.*

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ALEXANDER, Prof. J. W., Ph.D. (Princeton) Inst. for Advanced Study, Princeton, N.J. *29 Cleveland Lane*

ALFIERI, F. A. Audit clerk, Metropolitan Life Ins. Co., New York, N.Y.; Student, Coll. of City of New York

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ZORN, Asso. Prof. M. A., Dr. res mat. (Hamburg) Univ. of California at Los Angeles, 405 N. Hilgard Ave., Los Angeles, Calif.

## RECAPITULATION OF MEMBERSHIP

| | |
|---|---|
| Individual members October 16, 1939 | 2,125 |
| Institutional members October 16, 1939 | 100 |
| Total membership October 16, 1939 | 2,225 |

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University of California at Los Angeles, Los Angeles
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Laval University, Quebec

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Knox College, Galesburg
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Purdue University, LaFayette
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St. Mary-of-the-Woods College, St. Mary-of-the-Woods

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FREEPORT. Mensenkamp.
GALESBURG. Heren, Sellew, Smyth, Stephens
HIGHLAND PARK. MacMartin.
JACKSONVILLE. Miller.
JOLIET. F. C. Smith.
KENILWORTH. Hart.
LAKE FOREST. Curtis.
LANCASTER. Oursler.
LA SALLE. Carus.
LEBANON. Stowell.
LINCOLN. Balof.
MACOMB. Ayre, Ginnings, Schreiber.
MAYWOOD. Hildebrandt, Olson.
MONMOUTH. Beveridge. Winbigler.
MURPHYSBORO. Bock.
NORMAL. Atkin, Flagg, Larsen, Mills.
PEORIA. Comstock, Gault.
RIVER FOREST. Dobbin.
ROCKFORD. McGavock.
ROCK ISLAND. Cederberg.
SALEM. Gonzalez.
SANDWICH. Rumney.
SPRINGFIELD. Wells.
TAYLORVILLE. Dappert.
URBANA. Armstrong, Bailey, Bower, Carmichael, Coble, Crathorne, Hartley, Hazlett, Ketchum, Levy, Miles, Miller, Moore, Pepper, Peters, Strobel, Vaughan.

WHEATON. Boyce, Brandt, Taylor.
WINNETKA. Gadske.

INDIANA. (53)

BLOOMINGTON. Artin, Hennel, Rothrock, Wells, Williams, Wolfe.
COLLEGEVILLE. Zanolar.
CRAWFORDSVILLE. Carscallen.
EARLHAM. Long, Pyle.
FORT WAYNE. Hutchison, Paxton.
GARY. Copp.
GOSHEN. Hartzler, Lehman.
GREENCASTLE. Arnold, Edington, Greenleaf.
HANOVER. Meyer.
HOLY CROSS. Callahan.
INDIANAPOLIS. Beal, Getchell, McColgin, Suter, Welchons.
MARION. Porter.
MUNCIE. Edwards, Shively.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparó, Maurus, Menger.
REYNOLDS. Erwin.
TERRE HAUTE. Kennedy, Shriner, Sousley.
UPLAND. Draper.
WEST LAFAYETTE. Black, Doan, Downing, Graves, Hadley, Hardman, Hazard, Hodge, Keller, Lanczos, Little, Miller, Robbins, Shreve, Stone, Webster.

IOWA. (46)

AMES. Brandner, Daniells, Fleming, Gouwens, Herr, J. V. McKelvey, M. M. McKelvey, Robertson, E. R. Smith, Snedecor.
CEDAR FALLS. Kearney, Van Engen, Wester.
CEDAR RAPIDS. Coffin, Yothers.
DECORAH. Ellingson.
DES MOINES. Neff, Westemeier.
DUBUQUE. Beach, Mary Resignata, Theobald.
FAYETTE. Deming.
GRINNELL. McClenon, Rusk.
HOPKINTON. Earhart.
INDIANOLA. Emmons.
IOWA CITY. Chittenden, Conkwright, Craig, Knowler, Lane, McLaughlin, Oberg, Reilly, Rietz, Ward, Woods, Wylie.
IOWA FALLS. Kreider.
KEOKUK. West.
MOUNT VERNON. McGaw, Moots.
ORANGE CITY. Hattan.
SIOUX CITY. Graber, Rochford.
WAVERLY. Chellevold.

KANSAS. (45)

ATCHISON. Pretz, Sullivan.
BALDWIN. Garrett.
EMPORIA. Peterson, Tucker.
HAYS. Colyer, Grabbe.
HESSTON. Driver.
HIGHLAND. Culbertson.
INDEPENDENCE, Bell.
KANSAS CITY. Dougherty, Thornton.
LAWRENCE. Babcock, Black, Jordan, Mitchell, Price, G. W. Smith, Stouffer, Wheeler.
LEAVENWORTH. Ann Elizabeth.
LINDSBORG. Marm.
MANHATTAN. Babcock, Daugherty, Hyde, Janes, Lewis, Lyons, Mossman, Remick, Sigley, Stratton, White.
NORTH NEWTON. Richert.
PITTSBURG. Shirk, R. G. Smith.
SALINA. Arnoldy, Jensen.
STERLING. Bell.
TOPEKA. Harshbarger, Householder.
WICHITA. Beito, Hoare, Longenecker, Read. Reagan.

KENTUCKY. (37)

ASHLAND. Taylor, Williams.
BEREA. Hutcherson, Pugsley.
BOWLING GREEN. Howard, Perry, Strayhorn, Yarbrough.
BURLINGTON. Barnett.
COVINGTON. Thuener.
DANVILLE. Fehn.
GEORGETOWN. Hatfield.
HOPKINSVILLE. Nowlan.
LEXINGTON. Boyd, Brown, Cohen, Downing, John, Latimer, LeStourgeon, Pence, South, Wright.
LOUISVILLE. Bloom, Bullitt, Moore, Morrison, Simester, Stevenson.
MAPLE MOUNT. Sheeran.
MAY'S LICK. Reckzeh.
MOREHEAD. Black, Fair.
MURRAY. Carman.
RICHMOND. Jenkins, Park.
WINCHESTER. Allison.

LOUISIANA. (42)

BAKER. Johnson.
BATON ROUGE. Freas, Nichols, O'Quinn, Parker, Sanders, H. L. Smith.
HAMMOND. Cordrey, Tucker.
LAFAYETTE. Buchanan.
NATCHITOCHES. Blair, Killen, Maddox.
NEW ORLEANS. Buchanan, Cramer, Duren, Fleddermann, Frankenbush, Hopkins, Humphreys, Many, Menuet, Monasterio, Nolan, Rayl, Spencer, Stevens, Thomson, Weiss.
PINEVILLE. Temple.
RUSTON. Gentry, Kaltenborn, Schroeder, P. K. Smith.
SCOTLANDVILLE. James.
SHREVEPORT. Banks
UNIVERSITY. Aucoin, Karnes, Rickey, Scott, White, Yates.

MAINE. (12)

BRUNSWICK. Hammond, Holmes, Korgen, Moody.
HOULTON. Morse.
LEWISTON. Ramsdell, Wilkins.
LISBON FALLS. Schultz.
ORONO. Bryan, Kimball.
WATERVILLE. Ashcraft, Schoenberg.

MARYLAND. (42)

ABERDEEN. Dederick.

ANNAPOLIS. Ayres, Ball, Bingley, Bramble, Buchanan, Church, Clements, Currier, Dillingham, Echols, Kells, Lamb, Leiper, Littauer, Lyle, Moore, Rawlins, Root, Scarborough, Vedova, Wilson.
BALTIMORE. Bacon, Cohen, Lewis, Mary Cordia, Morrill, Murnaghan, Reed, Roman, H. R. Smith, Torrey, Williamson, Zariski.
CHELTENHAM. Hartnell.
COLLEGE PARK. Gilbert, Richeson, Taliaferro.
EMMITSBURG. Burke.
FREDERICK. Brown.
PORT DEPOSIT. Haviland.
SILVER SPRING. Majella.

MASSACHUSETTS. (88)

ACTON. Phalen.
AMESBURY. Dame.
AMHERST. Boutelle, Esty, Miller, Moore.
BOSTON. Brown, Bruce, Gould, Hemenway, Hubbard, Laurentine, Mode, Skofield, Spear, Weaver, Wilson.
BROOKLINE. Miller.
CAMBRIDGE. Beatley, Birkhoff, Cameron, Clifford, Coolidge, Crum, Douglass, Franklin, Gere, Graustein, Harvey, Huntington, MacGregor, MacLane, Moon, Pettis, Rabinow, Rule, Rulon, Rutledge, Stone, Walsh, Whitman, Widder, Woods, Zeldin.
CHESTNUT HILL. Marcou, O'Donnell.
DORCHESTER. Davis.
GROTON. Holt, Nash.
MEDFORD. Lapidus.
NORTHAMPTON. Benedict, McCoy, Montgomery, Munroe, Rambo.
NORTON. Garabedian, Watt.
PETERSHAM. Moriarty.
PITTSFIELD. Washburne.
SCITUATE. Gillespie.
SOUTHBOROUGH. Gottschalk.
SOUTHBRIDGE. Boeder.
SOUTH HADLEY. Baker, Doak, Litzinger.
SWAMPSCOTT. Evans.
TUFTS COLLEGE. Mergendahl, Ransom.
TYNGSBORO. Richmond.
WELLESLEY. Copeland, Merrill, Russell, C. E. Smith, Stark, Young.
WESTON. Burke.
WILLIAMSTOWN. Agard, Hardy, Wells.
WINCHESTER. Harter.
WOLLASTON. Dennison.
WORCESTER. Brown, Gay, Melville, Morley, O'Callahan, Rice, Wheeler.

MICHIGAN. (77)

ALBION. Ingalls, Sleight.
ANN ARBOR. Anning, Ayres, Bradshaw, Churchill, Coe, Copeland, Craig, Dwyer, Field, Ford, Gaskell, Goldstine, Greville, Hildebrandt, Hopkins, Johnson, Karpinski, Kazarinoff, Love, Nyswander, Rainich, Rainville, Rouse, Rufus, Running, Schorling, Scott, Wilder.
BAY CITY. Shellenbarger.
DETROIT. Baldwin, Borgman, Butler, Coral, Darnell, Denton, Fischer, Folley, Goldman, Johnson, Johnston, McCarthy, Mary Paula, Morrow, Nelson, Pixley, Shires, Sugar.
EAST LANSING. Barbour, Baten, Grove, Hill, Plant, Powell, Speeker, Welmers.
FLINT. Swanson.
GRAND RAPIDS. Warren.
HART. Burdick.
HIGHLAND PARK. Peterson.
HILLSDALE. Beeler.
HOLLAND. Lampen.
HOWELL. Olson.
IRONWOOD. Eittreim, Field.
JACKSON. Richards.
KALAMAZOO. Ackley, Blair, Everett, Walton.
MARQUETTE. Spooner.
MILLFORD. McNeal.
MOUNT PLEASANT. Richtmeyer.
MUSKEGON. Emeline.
YPSILANTI. Erikson, Lindquist.

MINNESOTA. (63)

COLERAINE. Tangjerd.
COLLEGEVILLE. Danzl, Winkelmann.
CROSBY. Novak.
DULUTH. Cothran, Herr, Strane.
EVELETH. Pollard.
GILBERT. Schey.
GRANITE FALLS. Ostrom.
MINNEAPOLIS. Bearman, Brink, Brooke, Bussey, Carlson, Dalaker, Daoust, Edwards, Eggers, Gibbens, Hart, Hartig, Jackson, Kirchner, Koehler, McEwen, Martin, Ness, Poole, Priester, Quaid, Saunders, Scammon, Scherberg, Shawhan, Shuman, Shumway, Teeter, Thorp, Tucker, Turrittin, Underhill, Wilder.
MOORHEAD. Andersen, Brennum, Mundhjeld.
NORTHFIELD. Carlson, Gingrich, Shover.
ROCHESTER. Hickman.
ST. JOSEPH. Claudette.
ST. PAUL. Blackall, Bush, Camp, Polansky, Rysgaard, Taylor, Thielman, Wegner, Wilson.
ST. PETER. Rundstrom.
VIRGINIA. Hancock.
WINONA. De La Salle.

MISSISSIPPI. (14)

CLEVELAND. Sanders.
CLINTON. Hitt.
JACKSON. Babbitt, McCoy, Mitchell.
RAYMOND. MacDonald.
STATE COLLEGE. Cox, Murray, Ollivier, C. D. Smith.
UNIVERSITY. Bickerstaff, Hume, Quarles.
WESSON. Felder.

MISSOURI. (44)

CANTON. Ingold.
CLAYTON. Haertter, Rosskopf.
COLUMBIA. Blumenthal, Callaway, Ewing, Haynes, Wahlin, Westfall.

FAYETTE. Fleet.
FULTON. Butchart, Sweazey.
JEFFERSON CITY. Jason, Talbot.
KANSAS CITY. Cutting, Pierson.
KIRKSVILLE. Cosby, Jamison.
KIRKWOOD. Harris.
ROLLA. Hinsch, Miles.
ST. CHARLES. Karr.
ST. LOUIS. Callaghan, Case, Dunkel, Gove, King, Middlemiss, Nagle, Osborn, Pennell, Quinn, Rider, Roever, Siroky, E. Stephens, J. Y. Stephens.
SPRINGFIELD. Finkel, Graves, H'Doubler.
SULLIVAN. Beasley.
WARRENSBURG. Cole, Urban.
WEBSTER GROVES. Clarke.

MONTANA. (6)

BOZEMAN. Frick, Hurst.
HELENA. Canning, Topel.
MISSOULA. Carey, Merrill.

NEBRASKA. (27)

CHADRON. Berry.
CRETE. Hawkes.
GILEAD. Erwin.
HASTINGS. McDill.
KEARNEY. Hanthorn.
LINCOLN. Basoco, Brenke, Camp, Candy, Congdon, Cox, Daum, Dribin, Gaba, Howie, Ogden, Pierce, Runge.
OMAHA. Bettinger, Earl, Fitzpatrick, Marrin.
PERU. Hill.
WAYNE. Boyce, Hove.
YORK. Bell, Feemster.

NEVADA.

RENO. Wood.

NEW HAMPSHIRE. (16)

CONCORD. Conwell.
DURHAM. Slobin.
EXETER. Adkins, Funkhouser, Pennell, Sweet.
HANOVER. Brown, Forsyth, Mathewson, Morgan, Perkins, Robinson, Silverman, Wilder.
MANCHESTER. O'Leary.
PLYMOUTH. G. M. Smith.

NEW JERSEY. (54)

BELLEPLAIN. Durell.
COLLINGSWOOD. High.
EAST ORANGE. Nordgaard.
ENGLEWOOD. Echols.
HIGHTSTOWN. Harrison, Litterick.
HOBOKEN. Hazeltine, Murray.
LAWRENCEVILLE. Kimball, Mikesh.
LEONIA. Karapetoff.
MONTCLAIR. Clifford, Davis, Fehr, Mallory, Turner.
NEWARK. Conkling, Klein, Mac Donald, Mosesson, Strock.
NEW BRUNSWICK. Bunyan, Galbraith, Grant, Meder, Morris, Nelson, Starke, Walter, Wilson.
PRINCETON. Adams, Alexander, Begle, Eisenhart, Flood, Gillespie, Lefschetz, Lewis, Morse, Mosteller, Olmsted, Stephens, Tompkins, Tucker, Tukey, Veblen, von Neumann, Wedderburn.
SOUTH ORANGE. Stanwick.
TEANECK. Hallett.
TRENTON. Shuster.
UPPER MONTCLAIR. Campbell, Hildebrandt.
WEST ORANGE. Edison.

NEW MEXICO. (16)

ALBUQUERQUE. Anderman, Barker, Barnhart, Bauer, Byram, Haskins, Larsen, Newsom.
LAS VEGAS. Roberts, Rodgers.
PORTALES. MacKay.
ROSWELL. Harp.
SILVER CITY. Mickelson.
SOCORRO. Reece.
STATE COLLEGE. Branson, Swingle.

NEW YORK. (252)

ALBANY. Beaver, Birchenough, DoBell, Frankel, Lester, Stokes.
ALFRED. Lowenstein, Polan, Seidlin, Titsworth, Whitford.
AURORA. Hollcroft, Rusk.
BATAVIA. Rood.
BROOKFIELD. Whitford.
BROOKLYN. Antonina, Berry, Borofsky, Bowden, Boyer, Charosh, Cowles, Fleisher, Forman, Griffin, Harkin, Hertzler, R. A. Johnson, Kaplan, Kennison, Koch, Kubis, A. W. Landers, Lavoie, Levy, Lieber, Locke, MacNeish, Milkman, Miller, Moore, Penn, Ruderman, Rush, Simpson, F. E. Smith, Singer, Tabatchnik, Thompson, Welkowitz, Whitford, Woodbridge, Zaslavsky.
BUFFALO. Gehman, Harrington, Montague, Ott, Podmele, Pound, Smokowski.
CLINTON. Brown, Carruth, Ferry, Fitch, Patterson.
CROWN POINT. Henderson.
ELMIRA. Suffa, Wright.
FLUSHING. Archibald, A. B. Brown, Cairns, Cope, Raudenbush, Sard.
GENEVA. Durfee, Hubbs.
GROTON. Rhodes.
HAMILTON. Aude, Munshower, A. W. Smith, Wardwell.
HOUGHTON. Davison.
ITHACA. Agnew, Carver, Curtiss, Flexner, Hurwitz, B. W. Jones, Karapetoff, Murray, Snyder, Trevor, Walker, Wray.
KENMORE. Brockett.
KEW GARDENS. Walker.
LOUDONVILLE. Nickol.
NEW YORK. Alfieri, Allen, Allison, Anderson, Arorian, Bakst, Berger, Bergstresser, Berkeley, Bernard, Bernstein, Berry, Bleick, Boehm, Bowden, Bradley, Burgess, Mrs. J. H. Bushey, J. H. Bushey, G. A. Campbell, Cooley, Cooper, Courant, Darkow, D'Atri, Dix, Doermann, Eisele,

Fagerstrom, Farnum, Feld, Fiske, Fite, Flanders, Foster, Fry, Gentzler, Gilder, Gill, A. M. Ginsburg, J. Ginsburg, Graham, Gray, Greenberg, Grove, Hamilton Hawkes, Henderson, Hill, Hlavaty, Hubert, Hurwitz, Hussey, Jablonower, Joffe, John, P. C. Jones, Karnow, Kasner, Katsh, Kirby, Kutman, M. K. Landers, Landin, Lantz, Larkin, Lawton, Lazar, Lehmann, Linehan, MacColl, MacEwen, McKenna, Maria, Mead, Miller, Mirick, Molina, Moore, Mullins, Nehrbas, Oehler, Payne, Penney, Peters, Post, Putnam, Quilty, Reddick, Rees, Reeve, Ritt, Robinson, Roos, Roth, Schelkunoff, Schlauch, Shaw, Sheridan, Shewhart, Siceloff, Simons, Skelding, D. E. Smith, Snoke, Swenson, Tanzola, Tilley, Turner, Upton, Wahlert, Walker, Wayne, Weaver, Wehausen, Weisner, Whelan, Whitford, Wirth, Wood, Wright, Yanosik.

NIAGARA FALLS. O'Connor.
ONEONTA. Newton, Sanford.
PELHAM. Milos.
POTSDAM. Waltz.
POUGHKEEPSIE. Fiske, Hopper, Wells.
ROCHESTER. Atkins, Betz, Chesna, Eastham, Gale, Harding, Long, Seidel, Watkeys.
ST. BONAVENTURE. Scheier, Wheeler.
SCHENECTADY. Fox, Morse, Poritsky, Snyder.
SYRACUSE. Campbell, Carroll, Decker, Harwood, Taylor.
TROY. Allen, McGiffert, Nash, Street.
WEST POINT. Echols, Jones.

NORTH CAROLINA. (39)

BOONE. Wright.
CHAPEL HILL. Blackwell, Browne, Cameron, Henderson, Hill, Kattsoff, Lasley, Linker, Mackie, R. E. Smith.
CHARLOTTE. O. M. Jones, Woodson.
DAVIDSON. McGavock, Mebane.
DURHAM. Boas, Dressel, Elliott, Gergen, Greenwood, Hickson, Lee, Patterson, Rankin, Thomas.
GREENSBORO. Barton, Pegram, Strong.
GREENVILLE. Graham, ReBarker.
MARS HILL. Robinson.
RALEIGH. Bullock, Cell, Eason, Levine, Winton.
RED SPRINGS. Prince.
SALISBURY. Dearborn.
WINGATE. Hendricks.

NORTH DAKOTA. (7)

DICKINSON. Muggli.
FARGO. Householder, I. W. Smith.
GRAND FORKS. Mason, Staley.
JAMESTOWN. Jackson.
VALLEY CITY. Harrell.

OHIO. (116)

ADA. Whitted.
AKRON. Bender, Selby.
ALLIANCE. Hildner.
ATHENS. Marquis, Miller, Reed, Starcher.
BEREA. Baur, Dustheimer.
BLUFFTON. Hirschler.
BOWLING GREEN. Mathias, Overman.
CANAL WINCHESTER. Bareis.
CHILLICOTHE. Mathias.
CINCINNATI. Barnett, Brand, Hancock, Justice, Kennedy, Kersten, Lubin, Merriman, Moore, Muehlman, Reilly, E. S. Smith, Szász, Yowell.
CLEVELAND. Boyce, O. E. Brown, Burington, Focke, Johnson, Jonah, Justin, Morris, Musselman, Nassau, Patterson, Rinehart, Risley, Sauté, Simon, Thomas, Tolar, Torrance.
CLEVELAND HEIGHTS. Joliat.
COLUMBUS. Albert, Bamforth, Beatty, Blumberg, Eason, M. E. Jones, Kuhn, LaPaz, Manson, Morris, Radó, Rasor, Rickard, Singer, Toops, Weaver, Wildermuth, Wylie.
DEFIANCE. MacCullough.
DELAWARE. Crane, Rowland.
FINDLAY. Roots.
GAMBIER. Bumer, MacNeille.
GRANVILLE. Ladner, Wiley.
HIRAM. Clarke.
KENT. Brooks, Harshbarger, Manchester, Rogers, Stelson.
MARIETTA. Bennett, Sandt.
MOUNT ST. JOSEPH. Corona.
NEW LEXINGTON. Hoops.
NORTH CANTON. Schug.
NORWOOD. Wishard.
OBERLIN. Cairns, Carr, Johnson, Sinclair, Smyth, Yeaton.
OXFORD. Anderson, Christofferson, Pollard, Spenceley, Tappan, Wolfe.
PAINESVILLE. Lewis, Peters.
SEVEN MILE. Baird.
SOUTH EUCLID. Garvin.
SPRINGFIELD. Tripp.
TIFFIN. Pierce.
TOLEDO. Brandeberry, Dancer, Koley, Mercedes, Welker, Winslow, Yeager.
WESTERVILLE. Glover.
WILMINGTON. Spinks.
WOOSTER. Knight, Williamson, Yanney.
YELLOW SPRINGS. Astrachan, Burr.
YOUNGSTOWN. Foard.

OKLAHOMA. (31)

ADA. Heimann, Winn.
ALVA. Gifford, Hall.
DURANT. Dragoo.
EDMOND. Johnson.
HOLDENVILLE. Wedel.
LANGSTON. Tinner.
NORMAN. Brixey, Court, Duval, Hassler, LaFon, McFarland, Randels, Reaves, Springer.
OKLAHOMA CITY. Whitney.
SHAWNEE. Doerfler, Short.
SHIDLER. Gassett.
STILLWATER. Allen, Barnett, Diamond, Flanders, Garretson, Hamilton, H. W. Smith, Zant.

TULSA. Veatch.
WEATHERFORD. McCormick.

OREGON. (17)

CORVALLIS. Beaty, Kirkham, Milne, Williams.
EUGENE. Aitchison. DeCou, Moursund, Peterson.
FOREST GROVE. Price.
GRANT'S PASS. Feinler.
MCMINNVILLE. Ramsey.
PORTLAND. Griffin, Hadley, Johnson, Keeler, Merriss.
SALEM. Luther.

PANAMA.

PANAMA CITY. Linares.

PENNSYLVANIA. (153)

ALLENTOWN. Deck, Kunkel.
ANNVILLE. Black.
BEAVER FALLS. Cleland.
BETHLEHEM. Ashbaugh, Cutler, Fort, Latshaw, Lehmer, Pitcher, Rau, Raynor, Reynolds, Shook, Smail, Van Arnam.
BOALSBURG. Graves.
BRYN ATHYN. Allen.
BRYN MAWR. Atkinson, Lehr, Wheeler, Williams.
CALIFORNIA. Salisbury.
CARLISLE. Ayres, Landis.
COLLEGEVILLE. Clawson, Dennis, Manning.
DENVER. Marburger.
DUBOIS. Shanks.
DUNCANNON. Wilson.
EASTON. Benner, Cawley, Kennedy, Hatch, W. M. Smith.
ELLWOOD CITY. Johnston.
ERIE. Benedicta, Kraus, Oergel, Sullivan, Wells.
FREEDOM, Stright.
GEORGE SCHOOL. Bates.
GETTYSBURG. Clutz.
GREENSBURG. McNeil.
GROVE CITY. Carpenter, Renwick.
HARRISBURG. McKee, Whited.
HAVERFORD. Oakley, Wilson.
HAZELTON. Herpel.
HUNTINGDON. Hess, Stayer.
JENKINTOWN. Durand.
JOHNSTOWN. Breiland.
KUTZTOWN. Knedler.
LANCASTER. Charles, Ikenberry, Long, Marburger, Murray, Worthington.
LATROBE. Seubert.
LEWISBURG. Gold, Miller, Richardson.
LOCKHAVEN. S. J. Smith.
MEADVILLE. Beisel.
NEW KENSINGTON. Sturm.
NEW WILMINGTON. Black.
PHILADELPHIA. Blake, R. D. Brown, Campbell, Caris, Constable, Davis, Eggert, Emmons, Evans, Fudge, Helms, Huff, Kline, Latshaw, Love, McDonough, Mitchell, Moore, Robertson, Safford, Shohat, Spencer, Tartler.
PITTSBURGH. Baird, Briant, Bryson, Buker, Calkins, Cowley, Dines, Foraker, Hicks, Hoover, Johnson, Karpov, Leifer, Moskovitz, Neelley, Olds, Riggs, Rosenbach, Saibel, S. R. Smith, Starr, Taber, Taylor, Wagner, Whitman.
READING. Speicher.
SCRANTON. Bertrand, Mary Daniel.
SHARON. Manning.
SHIPPENSBURG. Kieffer.
SLIPPERY ROCK. Lady.
STATE COLLEGE. Cohen, Curry, Dunlap, Frink, Gordon, Gravatt, Graves, Hagen, Moody, F. W. Owens, H. B. Owens, Rupp, Sheffer, West.
SWARTHMORE. Brinkmann, Dresden, Marriott.
SWISSVALE. Zimmerman.
WARREN. Lafferty.
WASHINGTON. Atchison, Bert, Dorwart, Shaub, Thomas.
WAYNESBURG. Moston.
WILLIAMSPORT. Johnson.
WYNNEWOOD. Stabler.
YORK. Baker.

PHILIPPINE ISLANDS

MANILA. Hizon.

RHODE ISLAND. (18)

NEWPORT. Chase.
PROVIDENCE. Adams. Archibald, Bennett, Carlen, Currier, Dressel, Frame, Gilman, Hall, Lorell, Manning, McKenney, Précourt, Richardson, Smiley, Tamarkin, Watt.

SOUTH CAROLINA. (18)

CHARLESTON. Doyle, Dye, Hair, Holt, Myers, Reves, Saunders.
CLINTON. Spencer.
COLUMBIA. Coker, Coleman, Jackson, Peele, Williams.
DUE WEST. Leslie.
HARTSVILLE. Reaves.
NEWBERRY. Gaver.
ROCK HILL. Stokes.
SPARTANBURG. Peck.

SOUTH DAKOTA. (10)

BROOKINGS. MacDougal, Walder, Wente.
HURON. Titt.
MITCHELL. Knox.
RAPID CITY. Davis, Petrie.
SPEARFISH. Hesseltine.
SPRINGFIELD. Hoopes.
VERMILLION. Ekman.

TENNESSEE. (26)

CHATTANOOGA. Massey.
CLEVELAND. Hutto.
COOKEVILLE. Hutchinson, Ward.
JEFFERSON CITY. Sloan.
JOHNSON CITY. Carson.
KNOXVILLE. Blincoe, Cooley, Eaves, Gillis, Lee, Purviance.

LEBANON. Donnell.
MARYVILLE. Keller, Knapp, Sisk.
MEMPHIS. Locke.
NASHVILLE. Blair, Hyden, McPherson, N. P. Miser, W. L. Miser, Moorman, Morrel, Van Horn, Wren.

TEXAS. (74)

ABILENE. Burnam, Mullings, Tate.
ALPINE. Gilley.
ARLINGTON. Howard, Lynch.
AUSTIN. Batchelder, Coleman, Craig, Decherd, Dodd, Ettlinger, Lubben, Moore, Vandiver.
BROWNSVILLE. De la Garza.
CANYON. Murray.
COLLEGE STATION. Binney, Chaney, Edmonson, Luther, McCulley, Ross.
CORPUS CHRISTI. Woodard.
DALLAS. Huff, Mouzon, Rees, Thomas.
DENTON. Brown, Hanson, White.
EDINBURGH. Searcy.
EL PASO. Leech.
FORT WORTH. Sherer.
GEORGETOWN. Wapple.
HEREFORD. Rice.
HOUSTON, Beckenbach, Blau, Bray, Dean, Lovett, Reade, W. A. Rees, Slotnick, Ulrich, Underwood.
HUNTSVILLE. Querry.
KINGSVILLE. Kennedy.
LUBBOCK. Cross, Hazlewood, Heineman, May, Michie, Miller, Ollmann, Sparks, Thompson Underwood, Wakerling Woodward.
NACOGDOCHES. Ferguson.
PRAIRIE VIEW. Randall.
SAN ANTONIO. Hurry, McNelly, Mary of Mercy, Schnepp, Tulloch, Wunder.
STEPHENVILLE. McSweeny, Redden.
TEAGUE. Notley.
WACO. Baker.
WAXAHACHIE. Newton.
WICHITA FALLS. Adams.

UTAH. (8)

LOGAN. Bird.
ST. GEORGE. Everett.
SALT LAKE CITY. Gibson, Hayes, Horsfall, McConnell, Pehrson, Stafford.

VERMONT. (10)

BURLINGTON. Bullard, Butterfield, Millington, Swift.
MIDDLEBURY. Bowker, Hazeltine, Perkins, Wiley.
NORTHFIELD. Dix.
RANDOLPH. Alliot.

VIRGINIA. (47)

ASHLAND. Simpson.
AYLOR. Aylor.
BLACKSBURG. Hatcher, O'Shaughnessy, Rasche, Williams.
BLUEFIELD. Wright.
CHARLOTTESVILLE. Hancock, Hedlund, McShane, Wallace.
EMORY. Miller.
FARMVILLE. Taliaferro.
HAMPTON. Perkins.
HOLLINS. Allen.
LANGLEY FIELD. Pinkerton.
LEXINGTON. Byrne, Knox, Paxton, Purdie, L. W. Smith.
LYNCHBURG. Larew, Wiggin.
MIDDLEBURG. Keppler.
MILLER SCHOOL. Watson.
MONTEREY. Colaw.
NORFOLK. A. L. Smith.
PORTSMOUTH. Downing.
RICHMOND. Drew, Gaines, Harris, Wheeler.
SALEM. Carpenter.
SOUTH BOSTON. Patten.
STAUNTON. Taylor.
SWEET BRIAR. Cole, Morenus.
UNIVERSITY. Kelley, Linfield, Oglesby, Sparrow, Whyburn.
WILLIAMSBURG. Calkins, Gregory, Kaplan, Russell, Stetson.

WASHINGTON. (19)

COLLEGE PLACE. Godfrey.
PULLMAN. Butler, Hacker, Knebelman.
SEATTLE. Ballantine, Beegle, Cramlet, Haller, Jerbert, McFarlan, Moritz, Mullemeister, Neikirk, Winger.
SPOKANE. Carlson.
TACOMA. Martin.
WALLA WALLA. Bratton, Stewart.
YAKIMA. Whitney.

WEST VIRGINIA. (13)

HUNTINGTON. Hackney.
KINGSTON. Neely.
MONTGOMERY. Milo, W. F. Smith.
MORGANTOWN. Colwell, Davis, Eiesland, Reynolds, Turner, Vehse, Vest.
WEST LIBERTY. Kiplinger.
WHEELING. Bagby.

WISCONSIN. (48)

BELOIT. Bigelow, Conwell, Huffer.
GREEN BAY. Kalcik.
LACROSSE. Adkins.
MADISON. Allen, Cook, Evans, Ingraham, Langer, MacDuffee, March, Sokolnikoff, Stuckey, Trump, Ullsvik, Van Vleck.
MILWAUKEE. Bardell, Beckwith, Ericson, Fitzpatrick, Harner, Jautz, Knight, Luteyn, Marden, Mary Felice, Mary Gertrude, Nordhaus, Norris, Parkinson, Pettit, Roth, Turner, Vass, Wilczewski.
OSHKOSH. Beenken, Price.
RIVER FALLS. Eide.
SHEBOYGAN. Battig.
SUPERIOR. Flogstad, C. W. Smith.
WAUKESHA. Batha, Dancey, Hopkins.
WEST ALLIS. Wolf.
WEST DE PERE. De Cleene.
WISCONSIN RAPIDS. McMillan.

WYOMING. (4)

LARAMIE. Barr, Bellamy, Neubauer, Rechard.

FOREIGN MEMBERS

(Other than Canada.)

ARGENTINA. (2)

BUENOS AIRES. Baidaff, Barral-Souto.

BELGIUM

UCCLE. Errera.

BRITISH HONDURAS.

COROZAL. Zimmerman.

CEYLON

VADDUKODDAI. Lockwood.

CHILE

SANTIAGO. Salas-Edwards.

CHINA (4)

An-CHING. Chang.
CANTON. MacDonald, Woo.
PEKING. Shen-Fu.

EIRE

DUBLIN. Rowe.

FINLAND

HELSINGFORS. Ahlfors.

FRANCE. (2)

LE MANS. Thébault.
PARIS. Fréchet.

GREAT BRITAIN. (7)

BELFAST. McCrea.
CAMBRIDGE. Hardy.
CHIPPING NORTON. O'Hara.
LONDON. Dalal, Todd.
NOTTINGHAM. Piaggio.
OXFORD. Frecheville.

INDIA (6)

ALLAHABAD CITY. Mitra.
BANGALORE. Iyengar.
DACCA. Vijayaraghavan.
MADRAS. Durairajan.
POONA. Banerji.
SURAT. Shah.

ITALY. (4)

BOLOGNA. Bortolotti.
NAPLES. Crudeli.
ROME. Enriques, Laboccetta.

JAPAN.

TOKYO. Kobayashi.

NEW SOUTH WALES

SYDNEY. Hallett.

NEW ZEALAND

DUNDEDIN. Martyn.

PERU

LIMA. de Losada y Puga

POLAND

WARSAW. Dickstein.

PORTUGAL

LISBON. da Cunha.

ROUMANIA

BUCHAREST. Claudian.

SIAM. (2)

BANGKOK. Hadlock, Vadhana.

SOUTH AUSTRALIA.

ADELAIDE. Wilton.

STRAITS SETTLEMENTS.

SINGAPORE. Oppenheim.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHÂTEL. DuPasquier.

SYRIA.

BEIRUT. Jurdak.

UKRAINE.

KIEFF. Kryloff.

UNION OF SOUTH AFRICA. (2)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.

## BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

### Article I—Name, Purpose and Corporate Seal

1. This organization shall be known as

The Mathematical Association of America (Incorporated)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

### Article II—Membership

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

### Article III—Board of Trustees and Officers

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for reëlection. The Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief and two other members, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Board shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees, and the supervision and safekeeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

### Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

### Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

### Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

### Article VII—Dues

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro-rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to the American Annuitants' Table (Male) based upon three and one-half ($3\frac{1}{2}$) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the American Annuitants' Table (Male), three and one-half ($3\frac{1}{2}$) per cent; and the reserve thus computed shall be held as a liability.

### Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

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## PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION

### Presidents

| | | | |
|---|---|---|---|
| E. R. Hedrick | 1916 | J. L. Coolidge | 1925 |
| Florian Cajori | 1917 | Dunham Jackson | 1926 |
| E. V. Huntington | 1918 | W. B. Ford | 1927–1928 |
| H. E. Slaught | 1919 | J. W. Young | 1929–1930 |
| D. E. Smith | 1920 | E. T. Bell | 1931–1932 |
| G. A. Miller | 1921 | Arnold Dresden | 1933–1934 |
| R. C. Archibald | 1922 | D. R. Curtiss | 1935–1936 |
| R. D. Carmichael | 1923 | A. J. Kempner | 1937–1938 |
| H. L. Rietz | 1924 | W. B. Carver | 1939– |

### Vice-Presidents

- E. V. Huntington .......... 1916
- G. A. Miller .......... 1916
- D. N. Lehmer .......... 1917, 1918
- Oswald Veblen .......... 1917
- J. W. Young .......... 1918, 1926
- R. G. D. Richardson .......... 1919
- H. L. Rietz .......... 1919
- Helen A. Merrill .......... 1920
- E. J. Wilczynski .......... 1920
- R. C. Archibald .......... 1921
- R. D. Carmichael .......... 1921, 1922
- B. F. Finkel .......... 1922
- A. B. Chace .......... 1923
- L. P. Eisenhart .......... 1923
- J. L. Coolidge .......... 1924
- Dunham Jackson .......... 1924, 1925
- A. A. Bennett .......... 1925, 1933, 1934
- W. B. Ford .......... 1926
- A. J. Kempner .......... 1927, 1928, 1935
- Clara E. Smith .......... 1927
- F. D. Murnaghan .......... 1928
- E. T. Bell .......... 1929, 1930
- W. C. Graustein .......... 1929, 1930
- Arnold Dresden .......... 1931
- C. N. Moore .......... 1931
- W. H. Bussey .......... 1932
- G. C. Evans .......... 1932
- E. B. Stouffer .......... 1933
- E. P. Lane .......... 1934
- L. L. Dines .......... 1935
- N. A. Court .......... 1936
- T. C. Fry .......... 1936
- T. H. Hildebrandt .......... 1937
- E. J. Moulton .......... 1937, 1938
- H. E. Buchanan .......... 1938
- W. L. Hart .......... 1939
- F. D. Murnaghan .......... 1939

### Secretary-Treasurer

Appointed by the Trustees after 1918)

(W. D. Cairns .......... 1916–)

### Committee on Official Journal

(Appointed by the Trustees)

- H. E. Slaught .......... 1916–1937
- R. D. Carmichael .......... 1916–1918
- W. H. Bussey .......... 1916–1918, 1926–1931
- R. C. Archibald .......... 1919–1921
- W. A. Hurwitz .......... 1919–1921
- A. A. Bennett .......... 1922
- H. P. Manning .......... 1921–1922
- W. B. Ford .......... 1923–1925
- J. L. Coolidge .......... 1923
- A. J. Kempner .......... 1924–
- W. B. Carver .......... 1932–1936, 1937–
- E. J. Moulton .......... 1937–

### Elected Members of the Board

- D. N. Lehmer .......... 1916–1918, 1922–1924, 1930–1932
- R. E. Moritz .......... 1916–1918
- K. D. Swartzel .......... 1916
- Oswald Veblen .......... 1916, 1920–1922, 1926–1931
- R. C. Archibald .......... 1916–1917, 1923–1930
- Florian Cajori .......... 1916, 1918–1923, 1926–1930
- M. B. Porter .......... 1916–1917
- J. W. Young .......... 1916–1917, 1920–1922
- B. F. Finkel .......... 1916–1921, 1930–1935
- E. H. Moore .......... 1916–1921, 1923–1928
- Alexander Ziwet .......... 1916–1918
- E. R. Hedrick .......... 1917–1922, 1924–1929, 1932–1937
- J. N. Van der Vries .......... 1916–1918
- Helen A. Merrill .......... 1917–1919
- D. E. Smith .. 1917–1919, 1921–1926, 1937–
- Elizabeth B. Cowley .......... 1918–1920
- G. A. Miller .......... 1918–1920, 1922–1924
- E. J. Wilczynski .......... 1918–1919, 1922–1926
- L. P. Eisenhart .......... 1919–1922, 1925–1930
- E. V. Huntington .......... 1917, 1919–1927, 1933–1935
- E. L. Dodd .......... 1920
- R. D. Carmichael .......... 1920, 1924–1929, 1939–
- A. A. Bennett 1921, 1930–1932, 1939–
- H. L. Rietz .......... 1921–1923, 1925–1930, 1934–1936
- C. F. Gummer .......... 1921–1925
- Dunham Jackson .......... 1923–1929
- Clara E. Smith .......... 1923–1925
- A. B. Chace .......... 1924–1925
- J. L. Coolidge .......... 1926–1931
- E. T. Bell .......... 1927–1928
- E. P. Lane .......... 1928–1933
- W. B. Ford .......... 1929–1934
- E. R. Smith .......... 1929
- W. L. Hart .......... 1930–1935
- Lao G. Simons .......... 1930–1931
- L. L. Dines .......... 1931–1933
- T. C. Fry .......... 1931–1933
- J. W. Glover .......... 1931–1933
- H. E. Buchanan .......... 1932–1937
- W. R. Longley .......... 1932–1934, 1936–1938
- E. J. Moulton .......... 1933–1936
- R. W. Brink .......... 1934–
- D. R. Curtiss .......... 1934, 1937–
- J. L. Walsh .......... 1934–1936
- Arnold Dresden .......... 1935–
- J. O. Hassler .......... 1935–1936
- F. D. Murnaghan .......... 1935–1937
- G. C. Evans .......... 1936–
- Mary Emily Sinclair .......... 1936–1938
- J. M. Thomas .......... 1937–
- Marie J. Weiss .......... 1937–1938
- William Betz .......... 1938–
- A. B. Coble .......... 1938–
- J. H. Weaver .......... 1938–
- A. J. Kempner .......... 1939–

# The Carus Mathematical Monographs

The CARUS MONOGRAPHS are fulfilling their mission as intended by the generous donor, MRS. MARY HEGELER CARUS, and her son, DR. EDWARD H. CARUS.

Somewhat more than one-half the members of the ASSOCIATION have taken advantage of the distribution at cost of these Monographs. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. Each member is entitled to one copy of each Monograph at the special price of $1.25.

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It is believed that the ASSOCIATION is rendering a great service to mathematics by this enterprise, and a liberal support from the membership constitutes an appropriate vote of confidence in the undertaking.

## MONOGRAPHS THUS FAR PUBLISHED

No. 1. *Calculus of Variations,* by PROFESSOR G. A. BLISS. (First Impression, 1925; Second Impression, 1927; Third Impression, 1935.)

No. 2. *Analytic Functions of a Complex Variable,* by PROFESSOR D. R. CURTISS. (First Impression, 1926; Second Impression, 1930.)

No. 3. *Mathematical Statistics,* by PROFESSOR H. L. RIETZ. (First Impression, 1927; Second Impression, 1929; Third Impression, 1936.)

No. 4. *Projective Geometry,* by PROFESSOR J. W. YOUNG. (First Impression, 1930; Second Impression, 1938.)

No. 5. *History of Mathematics in America before 1900,* by PROFESSORS DAVID EUGENE SMITH and JEKUTHIEL GINSBURG. (First Impression, 1934.)

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By R. G. SANGER, The University of Chicago

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Numbers in black-face type refer to problems, those in light face to pages.

## NEWS AND NOTICES

Edited by R. G. SANGER, University of Chicago



## PERSONAL MENTION

This section contains the names of persons taking any active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

## NECROLOGY



## POEMS

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

VOLUME 46 NOVEMBER 1939 PART I NUMBER 9


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PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND EVANSTON, ILL.

# The AMERICAN MATHEMATICAL MONTHLY

*EDITED BY*

ELTON JAMES MOULTON, Editor-in-Chief

WALTER BUCKINGHAM CARVER

AUBREY JOHN KEMPNER

*WITH THE CO-OPERATION OF*





EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, E. J. MOULTON, Northwestern University, Evanston, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio.

NOTICE OF CHANGE OF ADDRESS by members of the Association should be sent promptly to the SECRETARY-TREASURER, W. D. CAIRNS, 97 Elm Street, Oberlin, Ohio, to reach him before the tenth of the month in which the change becomes effective.


## MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Annual Meeting, Columbus, Ohio, December 26–30, 1939.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1939 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Greenville, Pa., May 13; California, Pa., October 7.

ILLINOIS, Galesburg, May 12–13.

INDIANA, Muncie, April 28–29.

IOWA, Ames, April 21–22.

KANSAS, Topeka, April 1.

KENTUCKY, Murray, April 28–29.

LOUISIANA-MISSISSIPPI, Baton Rouge, La., March 3–4.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Aberdeen Proving Ground, Md., May 13; WASHINGTON, D. C., December.

MICHIGAN, Ann Arbor, March 18; Kalamazoo, November 18.

MINNESOTA, Northfield, May 13.

MISSOURI, Springfield, April 28.

NEBRASKA, Lincoln, May 5.

NORTHERN CALIFORNIA, San Francisco, January 28.

OHIO, Columbus, April 8.

OKLAHOMA, Tulsa, February 10.

PHILADELPHIA, Bethlehem, Pa., December 2.

ROCKY MOUNTAIN, Laramie, Wyo., April 28–29.

SOUTHEASTERN, Charleston, S.C., March 24–25.

SOUTHERN CALIFORNIA, Whittier, March 4.

SOUTHWESTERN, Alpine, Texas, May 2–3.

TEXAS, Abilene, March 31–April 1.

WISCONSIN, Milwaukee, May 6.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.


Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 412, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, 45 cents a Single Copy.
To Others, $5 a Year, 60 cents a Single Copy.

## MAY MEETING OF THE ILLINOIS SECTION

The twentieth regular meeting of the Illinois Section was held at Knox College, Galesburg, Illinois, on Friday and Saturday, May 12–13, 1939. Professor J. R. Mayor, chairman of the Section, presided at all of the sessions.

The attendance was forty-one, including the following thirty-three members of the Association: Edith I. Atkin, H. G. Ayre, H. R. Beveridge, S. F. Bibb, W. E. Cederberg, C. E. Comstock, D. R. Curtiss, H. T. Davis, J. E. Davis, W. M. Davis, A. E. Gault, R. M. Ginnings, G. D. Gore, Mabel M. Heren, Mildred Hunt, P. W. Ketchum, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Sister Mary Esther, J. R. Mayor, E. B. Miller, C. N. Mills, G. E. Moore, E. J. Moulton, Mary W. Newson, J. W. Peters, E. W. Ploenges, E. W. Schreiber, H. A. Simmons, Zens L. Smith, R. C. Stephens, M. E. Wescott.

At the annual business meeting the following officers of the Section were elected: Chairman, G. D. Gore, Central Y. M. C. A. College, Chicago; Vice-Chairman, Mildred Hunt, Illinois Wesleyan University; Secretary-Treasurer, C. N. Mills, Illinois State Normal University. The next meeting of the Section will be held at Illinois Wesleyan University, Bloomington, on May 3–4, 1940. On Friday evening Professor E. J. Moulton, Northwestern University, gave an illustrated lecture entitled "Genesis I, 1–16," in which he discussed the origin of the solar system.

The following ten papers were read:

1. "A configuration associated with a tetrahedron" by Dr. J. W. Peters, University of Illinois.
2. "Certain properties of statistical elements" by Professor C. N. Mills, Illinois State Normal University.
3. "A new brachistochrone problem" by J. F. Paydon, Northwestern University, introduced by Professor Simmons.
4. "A limit on the validity of subjective grading" by Professor R. M. Ginnings, Western State Teachers College.
5. "Elementary mathematical instruments from the historical point of view" (illustrated) by Professor E. W. Schreiber, Western State Teachers College.
6. "Some new applications of the geometry of the circle and triangle in constructing a parabola" by Dr. G. E. Moore, University of Illinois.
7. "Numbers that divide their reverses" by Professor P. W. Ketchum, University of Illinois.
8. "Curriculum problems in mathematics in the secondary schools" by Velma F. White, Galesburg High School, introduced by Professor Heren.
9. "Random numbers" by Professor H. T. Davis, Northwestern University.
10. "Mathematical difficulties encountered by university freshmen in the general course in the physical sciences" by Professor Zens L. Smith, University of Chicago.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Peters discussed a configuration associated with a tetrahedron that is analogous to the Brocard-Beltrami configuration associated with a triangle. The paper has appeared in the *National Mathematics Magazine.*

2. In educational literature many examples are found where the coefficients of correlation are averaged. This procedure is open to criticism. Using raw scores, Professor Mills derived a formula by which the coefficient of correlation for a total group of paired items can be expressed in terms of correlation of sub-groups of paired items. The formula is

$$\begin{aligned} r_t N_t \sigma_X \sigma_Y = {} & r_1 N_1 \sigma_{x_1} \sigma_{y_1} + r_2 N_2 \sigma_{x_2} \sigma_{y_2} + \cdots \\ & + N_1 M_{x_1} M_{y_1} + N_2 M_{x_2} M_{y_2} + \cdots - N_t M_X M_Y . \end{aligned}$$

The mean of the total group in terms of the means of the sub-groups is expressed by $N_t M_t = \sum_{i=1}^{i=n} N_i M_i$. The standard deviation of the total group in terms of the standard deviations of the sub-groups is not a new problem. However, a new formula is given, $N_t \sigma_t^2 = \sum_{i=1}^{i=n} N_i \sigma_i^2 + \sum_{i=1}^{i=n} N_i M_i^2 - N_t M_t^2$.

3. Given a straight line $L$ and a point $P$ not on $L$, Mr. Paydon gave a simple method for locating on $L$ the point $Q$ from which a particle will fall from $L$ to $P$ in the shortest time. Further, given a vertical circle $C$ (which contains a plumb line) and a point $P$ not on $C$ and not above its highest point; a convenient method of locating on $C$ the point $Q$ from which a particle will fall from $C$ to $P$ in the shortest time was given. The paper also included envelopes from which important tangents can be drawn, tangents which classify line segments of points $P$. By means of these tangents, all points $P$ under consideration become classified; if two points $P$ belong to the same class, then all points on a line segment containing them belong to this class.

4. College classes were tested as to ability in grading line segments, drawn on the blackboard, which were one-half the corresponding heights of college freshmen. Professor Ginnings concluded (1) since line segments are easier to grade subjectively than examination papers, it is probably a conservative estimate that half or more of the grades given subjectively in classes of twenty or twenty-five students are one or more ranks wrong; (2) picked classes representing only a portion of the range of the whole probability curve are very much more difficult to grade subjectively than the class with the range in achievement of the whole probability curve.

5. Professor Schreiber presented by means of slides some interesting examples of early mathematical instruments from the Mensing Collection in Chicago. Old measuring sticks, French and English sectors, drafting instruments of various kinds, instruments for measuring angles, three-legged compasses, callipers, and old brass rulers from Italy and Germany, were among the types of instruments thrown upon the screen. Perhaps the greatest news item was the showing of the oldest fountain pen (plume san fin) in America which was made in France about 1700. In his concluding remarks he made a plea for more instrumental work in mathematics classes, especially on the junior and senior high school level.

6. Dr. Moore gave a construction for a parabola, when given two tangents and their points of tangency, by first constructing two additional tangents by the method outlined by Bullard (this MONTHLY, vol. 42, 1935, p. 606). Use was then made of the following two theorems: (a) Given four lines in the plane, the circumcircles of the four triangles formed intersect in a common point, which is the focus of the parabola tangent to the four lines; (b) The Simson line of the focus with respect to any triangle of tangents to a parabola is the tangent at the vertex of the parabola. This construction was shown to be superior to those in current use in that it enables one to locate the focus, vertex, axis, and directrix of the parabola.

7. Professor Ketchum gave methods for determining numbers which divide their reverses. All such numbers can be obtained as suitable modifications of the two numbers 1089 and 2178. When bases other than 10 are used, the problem is more complicated. In particular, bases 8 and 11 present anomalies. Simple rules may be obtained which yield certain solutions to the problem for any base. For small bases it is possible to find all the solutions.

8. In discussing problems of content, Miss White pointed out that definite changes in the first nine grades are agreed upon and are taking place; also, that better training of teachers of mathematics, and coöperation between teachers from junior and senior high schools and from colleges would aid in making curriculum adjustments in the senior high school. She discussed the problems of individual differences and emotional attitudes in relation to the pupil, the teacher, and the curriculum content; and suggested that aid in solving problems could be obtained from the report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, urging that all should coöperate with that committee.

9. The theory of random numbers is designed to provide a statistical basis for the theory of time series in economics, similar to that furnished by the curve of error and its generalizations for the analysis of frequency distributions. The problem began with a paper by the British physicist J. H. Poynting in 1884, and its development includes the work of many statisticians, most prominent among these being G. U. Yule. Professor Davis sketched the history of the problem, established a basis for a rational definition of random numbers by means of serial correlations, and developed theorems about the structure of various series derived from random variables transformed by various linear operations.

10. Between 600 and 700 students at the University of Chicago take each year the required one-year general course in the physical sciences. One outstanding source of student difficulty in the course is lack of mathematical understanding. In the fall of 1938, at the opening meeting thirty questions on elementary arithmetic and algebra were given the group. The following three lecture hours were devoted to a discussion of elementary mathematical procedures viewed as *reasoning processes*. One week later, a test comparable to the first was given. The number taking both tests was 378. With 100 representing a perfect score, the arithmetic means were respectively 63 and 83, their stand-

ard deviations 21 and 14. Professor Smith indicated that college freshmen in a very short time could make decided improvement in their elementary mathematical ability provided they are encouraged to reason in mathematics rather than to blindly remember rules.

C. N. MILLS, *Secretary*

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## THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Aberdeen Proving Ground, Aberdeen, Maryland, on Saturday, May 13, 1939. The chairman, Mr. Michael Goldberg, presided over the morning session, while Dr. L. S. Dederick directed the activities of the afternoon session.

The meeting was attended by seventy persons, including the following twenty-four members of the Association: H. C. Ayres, N. H. Ball, Archie Blake, C. C. Bramble, Randolph Church, L. S. Dederick, Alexander Dillingham, E. J. Finan, Michael Goldberg, L. M. Kells, W. D. Lambert, A. E. Landry, C. L. Leiper, C. M. Lennahan, S. B. Littauer, Carol V. McCamman, T. W. Moore, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, Irwin Roman, R. E. Root, J. L. Stearn, Clement Winston.

At the business session the Section accepted an invitation from Catholic University to hold the Fall 1939 meeting there during the celebration of the fiftieth anniversary of the university's founding. Discussion was held on the possibility of incorporating symposia in the Section's programs and the membership was invited to send suggestions to the Executive Committee. The following officers were elected for 1939–40: Chairman, L. S. Dederick, Aberdeen Proving Ground; Secretary, S. B. Littauer, U. S. Naval Academy; Executive Committee, W. E. Byrne, Virginia Military Institute and R. C. Yates, University of Maryland. A rising vote of thanks was accorded to the Commanding Officer and members of the staff of Aberdeen Proving Ground for their generous hospitality.

Six papers were read at the morning session, and three papers descriptive of some of the scientific activities undertaken at the Proving Ground were read at the afternoon session. Following these papers, demonstrations of the processes described were performed by members of the Proving Ground staff. After an address of welcome by the Commanding Officer, the following papers were read:

1. "A basis for interpreting results of samples of indefinite size" by Captain L. E. Simon, U. S. Army, Aberdeen Proving Ground, Maryland, introduced by the Secretary.

2. "The application of normalization to fitting a hypersurface possessing observational errors following the normal multivariate law" by F. V. Reno, Aberdeen Proving Ground, Maryland, introduced by the Secretary.

3. "Affine geometry from the projective point of view" by Dr. G. E. Alrich, University of Maryland, introduced by the Secretary.

4. "Osculation under conformal transformation" by T. C. G. Wagner, University of Maryland, introduced by the Secretary.

5. "Solution of simultaneous equations by division" by Professor John Tyler, U. S. Naval Academy.

6. "Use of conics to smooth distributions" by G. B. Dantzig, Bureau of Labor Statistics, Washington, D. C., introduced by the Secretary.

7. "The differential analyzer" by Dr. L. S. Dederick, Aberdeen Proving Ground, Maryland.

8. "High pressure and high velocity measurements" by Dr. A. H. Hodge, Aberdeen Proving Ground, Maryland, introduced by Dr. Dederick.

9. "The miniature cannon" by R. H. Kent, Aberdeen Proving Ground, Maryland, introduced by Dr. Dederick.

Abstracts of these papers follow, the numbers corresponding to the numbers as listed above:

1. In the case of industrial products where the presence of defects is a criterion of quality, *e.g.*, a finished or plated surface, the sample size with respect to the product differs from the sample size with respect to the quality characteristic defects. This is so because the number of defects possible is indefinitely large even in the case of a small sample of the product. Captain Simon presented a method, applicable to samples of this type, for distinguishing between the variation in number of defects owing to mere sampling fluctuation and the variation resulting from a significant difference of quality.

2. Mr. Reno showed that the normal multivariate errors form can be reduced to a sum of squares by a rotation in which the rotational angles are functions of the original independent variables. He then employed the reduced error form to derive a system of normal equations whose coefficients are products of the transformed independent variables, and examined some properties of the transformation matrix.

3. Dr. Alrich discussed the general concepts of distance and angle, employing for this purpose the circles of non-euclidean geometry. The method of approach afforded an opportunity to exhibit the characteristics of the elliptic, hyperbolic, and parabolic lines. The geometric significance of the rotations of affine geometry was also indicated.

4. Mr. Wagner endeavored to investigate local properties of osculation under conformal transformations. The point $z$ and its transform $Z$ are plotted in the same plane; then the intersection of corresponding tangents and normals is a circle. It was shown that if the center of curvature at $z$ lies on a circle about $z$, the corresponding center will lie on a conic with $Z$ as a focus and a fixed directrix. If there is an inflection at $z$, the center of curvature at $Z$ will lie on this directrix. A cusp transforms into a cusp. For a fixed direction the correspondence between centers of curvature is one-dimensional projective, and on the basis of this a construction was given.

5. Professor Tyler demonstrated by a few examples a very compact method, which combines elimination with indirect determinant evaluation, for solving

simultaneous linear equations. He then indicated the theory of the "solution by division" and its advantages over solution by determinants.

6. Mr. Dantzig indicated by specific examples taken from actual samples in the field of labor statistics how the obtained distributions could be smoothed effectively by use of conics.

7. Dr. Dederick explained the theory and operation of the integraph used at the Proving Ground. Later the gathering watched the actual solution of a trajectory problem.

8. Dr. Hodge explained the theory of high pressure and high velocity measurements and demonstrated these processes in operation.

9. Mr. Kent sketched some points in the theory of projectile flight and then with his miniature cannon made visible the phenomena of drift, yaw, precession, and stability or instability.

S. B. LITTAUER, *Secretary*

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## THE FALL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The thirteenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at State Teachers College, California, Pennsylvania, on Saturday, October 7, 1939. Professor H. L. Black, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was thirty-one, including the following eighteen members of the Association: C. S. Atchison, L. C. Bagby, H. L. Black, P. N. Carpenter, H. A. Davis, L. L. Dines, H. L. Dorwart, H. C. Hicks, V. V. Johnston, David Moskovitz, L. T. Moston, J. H. Neelley, C. N. Reynolds, E. G. Salisbury, J. S. Taylor, R. W. Thomas, C. H. Vehse, M. L. Vest.

The following officers were elected for the coming year: Chairman, J. S. Taylor, University of Pittsburgh; Secretary, David Moskovitz, Carnegie Institute of Technology; Member of Executive Committee, H. L. Dorwart, Washington and Jefferson College. Professor L. T. Moston, Waynesburg College, continues in office for the second year of his term as the additional member of the Executive Committee.

After the opening address by Dr. Robert M. Steele, President of California State Teachers College, the following six papers were read:

1. "Constructed trisections of angles by means of several rational plane curves" by Professor J. H. Neelley, Carnegie Institute of Technology.

2. "Classification of cubic curves" by Professor L. T. Moston, Waynesburg College.

3. "Discussion of the four-color problem" by Professor C. N. Reynolds, University of West Virginia.

4. "An extension of the desmic configuration" by Professor J. K. Stewart, University of West Virginia, introduced by the Secretary.

5. "A birational transformation belonging to the complex of secants of the twisted cubic" by Professor H. A. Davis, University of West Virginia, and Pro-

fessor A. H. Black, Lebanon Valley College. The paper was read by Professor Davis.

6. "Some dynamical models of curved spaces" by Henry Lawton, Westinghouse Electric and Manufacturing Company, introduced by Professor H. L. Black.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Neelley showed how an arbitrary angle can be trisected by using the limaçon, the equilateral hyperbola, the nodal cubic, or the conchoid of Nicomedes. He also showed how the construction of the $(1/n)$th part of an angle could be performed by using the curve $r \sin(\theta/n) = \sin[(n+1)/n]\theta$.

2. Professor Moston gave a summary of the theorems leading up to the problem of classification of cubics, and then enumerated and described the genera, species, and varieties of cubics, following closely the procedure of Salmon. Corresponding to many of the varieties of cubics he had found algebraic equations and displayed their graphs.

3. Professor Reynolds considered the properties of those maps of regions of simply connected closed surfaces which simultaneously (a) require five colors in coloring them, (b) minimize the number of regions, and (c) maximize certain other topologically definable numerical characteristics of maps.

4. Professor Stewart considered two complete quadrangles, lying in different planes, but having a line of points in common. If opposite pairs of noncommon points be joined, three additional points are located, which constitute the $[12_4 16_3]$ configuration. An obvious extension of this method of generation is developed by considering two desmic sets lying in different linear three-spaces, having in common six points in the common plane. This leads to a $[20_4 40_3]$ which includes five desmic sets, and ten $10_3$ configurations. The three-space desmic sets conjugate to the five lead to a total of 65 points in $S_4$ which are found to be two overlapping $[40_3 30_4]$ configurations. By easy deletions five $[24_3 18_4]$ configurations are found in the set of 65 points, as pointed out by Schroeter for $S_3$. There are 80 ways that $[15_4 20_3]$ configurations may be picked from the set, these configurations projecting into the Cayleyan configuration in the plane. By central projection on a linear $S_3$ and by duality point-line systems $[20_6 40_3]$ and $[40_3 30_4]$ are established. Generalizing the method to $S_n$ a point-line configuration in all spaces from 2 to $n$ dimensions of the type $[(n-1)n_{2(n-1)}, \frac{2}{3}(n-1)n(n-1)_3]$ is established.

5. Professor Davis defined a transformation $T_{13}$ related to Caldarera's $T_{15}$ by means of two pencils of quadrics through the twisted cubic $r$, and a regulus of bisecants of $r$, all projectively related. The study of this $T_{13}$ discloses, among other things of interest, a regulus of parasitic lines and six isolated parasitic lines. Several cases of contact along $r$ occur in the intersection table.

6. Mr. Lawton's paper was based on a paper by J. L. Synge entitled *The geometry of dynamics* appearing in the Philosophical Transactions of the Royal Society of London. In this paper, Professor Synge applies Riemann geometry to

the general problems of classical dynamics using geometrical concepts and reasoning to more clearly picture the results of analytical dynamical reasoning. Mr. Lawton proposes to use dynamical systems as a means of illustrating some of the properties of curved spaces. For example, the problem of two discs rotating freely in space gives the geometrical metric $ds^2 = I_1 d\theta_1^2 + I_2 d\theta_2^2$, where the moments of inertia $I_1$ and $I_2$ are constant. This gives a space locally flat and having the properties of a torus in the large, *i.e.* the well known flat torus. Two additional problems from classical mechanics were used to illustrate Riemann spaces with interesting properties.

DAVID MOSKOVITZ, *Secretary*

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## ADJOINTS OF LINEAR DIFFERENTIAL OPERATORS

E. D. RAINVILLE, University of Michigan

**1. Introduction.** The classical treatment of the theory of linear ordinary differential equations contains much use of the adjoint of a linear differential equation. Particularly in recent years much stress has been laid on the concept of the adjoint problem of a boundary value problem in one or more linear differential equations. It is, however, of some interest to define and study an operator which, applied to a linear differential operator, yields the adjoint of that linear differential operator. Such a study will naturally disassociate the concept of the adjoint of an operator from any consideration of boundary conditions.

Most advanced books on ordinary differential equations treat the adjoint concept in more or less detail but, we believe, not from the standpoint which is used in the present exposition.

**2. Simple properties of the adjoint operator.** DEFINITION 1. Let $D \equiv d/dx$ be the usual symbol for differentiation with respect to $x$. We define a *linear differential operator* to be any linear combination of terms of the type $p_n(x)D^n$ $n = 0, 1, 2, \cdots$.

It is to be noted that since we are concerned only with a formal calculus of operators we may assume the various $p_n(x)$ to be unlimitedly differentiable.

DEFINITION 2. The *adjoint operator* $\alpha$ is a linear operator which acts upon linear differential operators as indicated in

$$\alpha[p(x)D^n] = (-1)^n D^n p(x); \qquad D^0 \equiv 1, \tag{1}$$

and

$$\alpha\left[\sum_s b_s p_s(x) D^s\right] = \sum_s b_s \alpha\left[p_s(x) D^s\right], \tag{2}$$

where the $b_s$ are any constants.

We wish to emphasize the fact that the symbol $D^s p(x)$ represents a linear differential operator which is not the $s$th derivative of $p(x)$. For instance, con-

sider $D^2x$. Let $F$ be an indeterminate function of $x$. Then $D^2x$ is the operational coefficient of $F$ in the second derivative of $(xF)$. Since,

$$\frac{d^2}{dx^2}(xF) = x\frac{d^2F}{dx^2} + 2\frac{dF}{dx},$$

we see that $D^2x$ is the operator $(xD^2+2D)$.

*Example* 1. The linear differential operator $y=xD^2-x^3D+3$ is transformed by $\alpha$ as follows:

$$\begin{aligned}\alpha y &= \alpha(xD^2) - \alpha(x^3D) + \alpha(3) = (-1)^2D^2x - (-1)^1Dx^3 + 3\\ &= xD^2 + 2D + x^3D + 3x^2 + 3\\ &= xD^2 + (x^3+2)D + 3x^2 + 3.\end{aligned}$$

We speak of $\alpha y$ as the adjoint of $y$. By $\alpha^2 y$ we mean $\alpha(\alpha y)$.

By means of a series of lemmas, we shall prove the following theorem:

THEOREM 1. *If $A$ and $B$ are linear differential operators, then* $\alpha(AB) = (\alpha B)(\alpha A)$.

We shall let $p^{(k)}$ denote the $k$th derivative of $p(x)$, $p^{(0)}\equiv p(x)$, and similarly for $q(x)$. And we shall use $\binom{n}{k}$ as a symbol for the binomial coefficient $n!/[k!(n-k)!]$.

LEMMA 1. *If $k$ is a non-negative integer, then*

$$\alpha(pqD^k) = (\alpha qD^k)(\alpha p).$$

*Proof*: We readily obtain the chain of relations

$$\alpha(pqD^k) = (-1)^kD^kpq = (-1)^kD^kqp = [(-1)^kD^kq]p = (\alpha qD^k)(\alpha p),$$

which proves the lemma.

LEMMA 2. *If $k$ is a non-negative integer, then*

$$\alpha(DqD^k) = (\alpha qD^k)(\alpha D).$$

*Proof*: We may verify the following steps:

$$\begin{aligned}\alpha(DqD^k) &= \alpha[qD^{k+1} + q^{(1)}D^k] = (-1)^{k+1}D^{k+1}q + (-1)^kD^kq^{(1)}\\ &= (-1)^{k+1}\sum_{s=0}^{k+1}\binom{k+1}{s}q^{(s)}D^{k+1-s} + (-1)^k\sum_{s=0}^{k}\binom{k}{s}q^{(s+1)}D^{k-s}\\ &= (-1)^{k+1}\sum_{s=0}^{k+1}\binom{k+1}{s}q^{(s)}D^{k+1-s} + (-1)^k\sum_{s=1}^{k+1}\binom{k}{s-1}q^{(s)}D^{k+1-s}\\ &= (-1)^{k+1}qD^{k+1} + (-1)^{k+1}\sum_{s=1}^{k+1}\left\{\binom{k+1}{s} - \binom{k}{s-1}\right\}q^{(s)}D^{k+1-s}.\end{aligned}$$

Now

$$\binom{k+1}{s} - \binom{k}{s-1} = \binom{k}{s}, \quad \text{for } s \leqq k,$$
$$= 0, \qquad \text{for } s = k+1.$$

Hence,

$$\begin{aligned} \alpha(DqD^k) &= (-1)^{k+1}qD^{k+1} + (-1)^{k+1}\sum_{s=1}^{k}\binom{k}{s}q^{(s)}D^{k+1-s} \\ &= (-1)^{k+1}\sum_{s=0}^{k}\binom{k}{s}q^{(s)}D^{k+1-s} = [(-1)^kD^kq](-D) \\ &= (\alpha qD^k)(\alpha D), \end{aligned}$$

and Lemma 2 is proved.

LEMMA 3. *If $n$, $k$ are non-negative integers, then*

$$\alpha(D^nqD^k) = (\alpha qD^k)(\alpha D^n).$$

*Proof*: By repeated application of Lemma 2 and use of the associative character of linear differential operators, we see that

$$\begin{aligned} \alpha(D^nqD^k) &= (\alpha D^{n-1}qD^k)(\alpha D) \\ &= (\alpha D^{n-2}qD^k)(\alpha D)(\alpha D) = (\alpha D^{n-2}qD^k)(\alpha D^2) \\ &\cdots\cdots\cdots\cdots \\ &= (\alpha DqD^k)(\alpha D^{n-1}) \\ &= (\alpha qD^k)(\alpha D^n), \end{aligned}$$

which was to be proved.

LEMMA 4. *If $n$, $k$ are non-negative integers, then*

$$\alpha(pD^nqD^k) = (\alpha qD^k)(\alpha pD^n).$$

*Proof*: By Lemma 1, we have

$$\alpha(pD^nqD^k) = (\alpha D^nqD^k)(\alpha p).$$

Then Lemma 3 shows that

$$\alpha(pD^nqD^k) = (\alpha qD^k)(\alpha D^n)(\alpha p).$$

Another application of Lemma 1 yields

$$\alpha(pD^nqD^k) = (\alpha qD^k)(\alpha pD^n),$$

which was to be proved.

Theorem 1 follows from Lemma 4 and the linearity of $\alpha$. The reasoning used in concluding Theorem 1 from Lemma 4 is exemplified in its application to the case of the simple operators $A_1 = p_1D^{n_1} + p_2D^{n_2}$ and $B_1 = g_1D^{k_1} + g_2D^{k_2}$. Since,

$$A_1B_1 = p_1D^{n_1}q_1D^{k_1} + p_1D^{n_1}q_2D^{k_2} + p_2D^{n_2}q_1D^{k_1} + p_2D^{n_2}q_2D^{k_2},$$

the application of Lemma 4 yields

$$\alpha(A_1B_1) = (\alpha q_1D^{k_1})(\alpha p_1D^{n_1}) + (\alpha q_2D^{k_2})(\alpha p_1D^{n_1}) + (\alpha q_1D^{k_1})(\alpha p_2D^{n_2}) + (\alpha q_2D^{k_2})(\alpha p_2D^{n_2}).$$

We may now group terms of the right member and find

$$\alpha(A_1B_1) = (\alpha B_1)(\alpha p_1D^{n_1}) + (\alpha B_1)(\alpha p_2D^{n_2}).$$

Thus

$$\alpha(A_1B_1) = (\alpha B_1)(\alpha A_1),$$

from which Theorem 1 follows.

THEOREM 2. *If $A$ is a linear differential operator, then $\alpha^2A = A$.*

*Proof*: Consider a representative term $pD^n$ of $A$. By Theorem 1,

$$\alpha pD^n = (\alpha D^n)(\alpha p).$$

Hence,

$$\alpha^2pD^n = (\alpha^2p)(\alpha^2D^n).$$

But,

$$\alpha^2p = \alpha p = p;$$
$$\alpha^2D^n = \alpha[(-1)^nD^n] = (-1)^{2n}D^n = D^n.$$

Then

$$\alpha^2pD^n = pD^n.$$

Theorem 2 now follows from the linearity of $\alpha$.

We are now able to state the following:

COROLLARY 1. *If $A$ is a linear differential operator, then $(A+\alpha A)$ is invariant under $\alpha$.*

Since $\alpha A+\alpha^2A=\alpha A+A$, Corollary 1 follows immediately from Theorem 2.

**3. An invariant related to second order linear differential operators.** Consider the operator $y=a_0D^2+a_1D+a_2$, where the $a_i$, $(i=0, 1, 2)$, are functions of $x$ and the notations $a_i^{(1)}\equiv d/dx(a_i)$, *etc.*, are used. Define

$$T(y) = 4a_0a_2 - 2a_0a_1^{(1)} + 2a_0^{(1)}a_1 - a_1^2,$$

a function of the coefficients of $y$. Now,

$$\alpha y = a_0D^2 + [2a_0^{(1)} - a_1]D + a_0^{(2)} - a_1^{(1)} + a_2,$$

and

$$T(\alpha y) = 4a_0[a_0^{(2)} - a_1^{(1)} + a_2] - 2a_0[2a_0^{(2)} - a_1^{(1)}] + 2a_0^{(1)}[2a_0^{(1)} - a_1] - [2a_0^{(1)} - a_1]^2.$$

Finally,

$$T(\alpha y) = -2a_0a_1^{(1)} + 4a_0a_2 + [2a_0^{(1)} - a_1]a_1 = T(y).$$

Thus we have proved the following:

THEOREM 3. *The function*

$$T \equiv 4a_0a_2 - 2a_0\frac{da_1}{dx} + 2a_1\frac{da_0}{dx} - a_1^2$$

*of the coefficients of the linear differential operator*

$$a_0D^2 + a_1D + a_2$$

*remains invariant when the operator is transformed by* $\alpha$.

In studying the equation

$$\frac{d^2F}{dx^2} + p(x)\frac{dF}{dx} + q(x)F = 0,$$

we frequently transform it to a normal form,

$$\frac{d^2v}{dx^2} + Iv = 0.$$

If we put $a_0 = 1$, $a_1 = p(x)$, $a_2 = q(x)$, then the $T$ of Theorem 3 and the $I$ of this normal form are connected by the simple relation $T = 4I$.

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# AN ANALOG OF THE NINE-POINT CIRCLE IN THE KASNER PLANE

JOHN DE CICCO, Brooklyn College

**1. Introduction.** We shall begin by giving some fundamental definitions. A *simple horn-set* consists of all the curves (differential elements of the third order) which possess a common point and a common direction. Let $x$ denote the curvature and $y$ the rate of variation of the curvature $x$ with respect to the arc length $s$ (that is $y = dx/ds$) of any curve $C$ of a simple horn-set at the common point. Then any curve $C$ of a simple horn-set is defined by an ordered pair of numbers $(x, y)$. Therefore a simple horn-set is a two-dimensional space called the *Kasner plane* $K_2$ where any point of $K_2$ is a curve $(x, y)$ of the simple horn-set. Our object is to study the triangle geometry induced in the Kasner plane $K_2$ by the group of conformal transformations. The Kasner plane is an abstract two-dimensional space of a special Finsler type. However, for a picturization of our theorems, we shall use the cartesian plane with $x$ as abscissa and $y$ as ordinate.

## MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

*This department of the* MONTHLY *affords a place for the discussions of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

## ALGEBRA FOR THE UNDERGRADUATE*

MARIE J. WEISS, Newcomb College, Tulane University

The subject-matter of the undergraduate curriculum in mathematics seems to be, upon inspection of college catalogues, both a matter of taste and tradition. It is not my purpose to consider the undergraduate mathematical curriculum as a whole, but rather that part of it which deals with what is called algebra, and more particularly, algebra for the junior or senior in the liberal arts college. Usually an undergraduate in a liberal arts college, even if a major in mathematics, can devote no more than a year's course to the general field of algebra. Naturally, in such limited time, choice of subject-matter must be made. I shall try to justify both the choice to which my taste has led, and my neglect of some traditional subject-matter.

A first consideration, obviously, in preparing any course is the student. A department of mathematics in a liberal arts college not only must care for the occasional mathematical specialist who is to go to the graduate school, but also must consider the needs of the prospective mathematics teacher of the primary and secondary schools, must provide the necessary mathematical tools for the physical scientist, and must interest the general student, who always wishes to know what mathematics is all about. For these, a course in algebra must be prepared.

In recent years algebra has been dominated by the axiomatic point of view, and perhaps now we may say that the algebraist knows what algebra is all about. Some years ago Ore† gave as a definition of algebra: "Algebra deals with the formal combination of symbols according to prescribed rules." This is perhaps a too comprehensive description, for from the axiomatic point of view it might be taken as the definition of all mathematics. Weyl‡ in describing the algebraist's approach to the number system gives a similar, but more specific, point of view. He says: "The pure algebraist can do nothing with his numbers except perform upon them the four species, addition, subtraction, multiplication, and division." Recently MacLane§ describes the field of algebra as follows:

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* An address delivered before the Mathematical Association of America at Madison, Wisconsin, September 4, 1939, at the invitation of the program committee.

† Oystein Ore, Some recent developments in abstract algebra, Bulletin of the American Mathematical Society, vol. 37, 1931, p. 537.

‡ Hermann Weyl, Emmy Noether, Scripta Mathematica, vol. 3, 1935, pp. 1–20.

§ Saunders Mac Lane, Some recent advances in algebra, this MONTHLY, vol. 46, 1939, pp. 3–19.

"Algebra tends to the study of the explicit structure of postulationally defined systems closed with respect to one or more rational operations." This point of view, then, I believe we may take as our guide for an introduction to algebra; for although, as is well known, algebra makes use of ideas far afield from those embodied in the closure of a set under rational operations, the beginnings of the subject lie in this concept. The closed systems which have brought order into and thrown light upon many diverse algebraic topics are those of group, ring, integral domain, and field, with their accompanying theories of homomorphism and isomorphism. These are some of what Weyl calls the "right general notions," and I believe an introduction to them should form the basis of an undergraduate course in algebra.

These concepts, if well understood, will give the prospective teacher and the general student an introduction to the domain of algebra and an understanding of a mathematical system, and, surprisingly enough, the scientist some indispensable tools for advanced work. Further, the student will have some understanding of those concepts which are of interest to the working mathematician of today; and any contact, however slight, with the mathematics that is alive in the student's generation, has always seemed to me to be an important consideration in the problem of choosing mathematical material for the undergraduate. In 1928 Weyl wrote in his introduction to his *Gruppentheorie und Quantenmechanik*:* "Es ist ein wenig langweilig dass man die lineare Algebra, deren Grundbegriffe überall in der Mathematik und Physik auftauchen und deren Kenntnis darum eigentlich dieselbe Verbreitung haben sollte wie die Elemente der Differential- und Integralrechnung, heute noch immer von neuem auseinander setzen muss." Is it necessary to say the same thing eleven years later or is eleven years hoary enough with tradition to remedy the situation? Recently, I saw the notes of a graduate course in physics given at one of the large universities. It was interesting to see how great a part of the course was devoted to an elementary exposition of the calculus of matrices and to the elementary properties of groups. This material could easily have been included in a course in algebra for juniors or seniors.

Although I propose that an introduction to the abstract concepts just mentioned should form the motif of the course, I do not believe that a course for an undergraduate can begin with their satisfying axiomatic development, for the usual undergraduate has not sufficient background to make these concepts real to himself. Before these abstract concepts can be given to the student, material must be at hand so that he can see the need for such generalization and abstraction. To demand commutivity of an operation seems meaningless if the student has not had experience with non-commutative operations; to prohibit

* Hermann Weyl, Gruppentheorie und Quantenmechanik, second edition, 1931, p. 3. In the edition translated by H. P. Robertson this statement reads: "It is somewhat distressing that the theory of linear algebras must again and again be developed from the beginning, for the fundamental concepts of this branch of mathematics crop up everywhere in mathematics and physics and a knowledge of them should be as widely disseminated as the elements of the differential calculus."

the presence of zero divisors in a set seems extraneous if the student has not had examples of sets in which zero divisors occur naturally. However, this point of view may have a more traditional than pedagogical basis. Now what material can be given so the student has sufficient mathematical experience to absorb these abstract concepts? All students need to know something of polynomials, determinants, systems of simultaneous linear equations, linear dependence and independence, matrices, and transformations. If this material is given with the stamp of modern algebra, the later discussion of abstract concepts seems a unifying and natural development.

Students usually have fairly vague ideas of the rational aspects of the number system. Hence, it seems necessary to begin with a discussion of the number system, and since it is a course in algebra, only the algebraic aspect and not the continuous aspect of the number system need be discussed. Such points as closure under operations, the definition of rational numbers as ordered pairs of integers, and the complex numbers as ordered pairs of real numbers may be emphasized. These definitions of the rational numbers and the complex numbers, perhaps new to the student, give him the opportunity to see the necessity for proving the ordinary laws governing addition and multiplication for these numbers. Again, an interesting point to make with respect to the rational numbers is to indicate how the definition of the equality of two rational numbers is not the logical identity, but an equivalence relation, by means of which, for example, the ordered integer pair $1/2$ may be replaced by any ordered pair equivalent to it, such as $a/2a$, $a$ any integer not zero. In this way a discussion of sets and equivalence relations can be given, showing how an equivalence relation separates the elements of a set into mutually exclusive classes, such that any two members of a class are equivalent.

Number rings and number fields may be introduced naturally when closure is discussed. The elementary properties of the factorization and division of polynomials should, I believe, be taught with reference to the number ring or number field of coefficients. Moreover, I have yet to meet the student who has heard of the Euclidean Algorithm. The elementary theory of the algebraic equation in the rational number field, usually known to the student either from high school or from freshman college work, may now be reviewed from the point of view of the new concepts, ring and field. Weisner does this admirably in the first chapters of his *Introduction to the Theory of Equations*.* The derivative of the polynomial may be defined without continuity considerations, and thus the theory of multiple roots given from a purely algebraic point of view. May I comment that the theory of the algebraic equation in the field of real numbers, especially that dealing with the computation and isolation of real roots, is a topic that might as justifiably be part of a calculus course as part of a course in algebra. The theory of elimination and resultants may be postponed until a study of determinants has been made.

In teaching the theory of determinants, another useful tool may be intro-

* Louis Weisner, Introduction to the Theory of Equations, New York, 1938.

duced to the student; namely, the permutation as a mapping of a finite set of elements upon itself. By using the elementary theory of permutations, the properties of determinants may be logically and simply developed. J. M. Thomas does so in his *Theory of Equations*;* Hasse in his small book, *Höhere Algebra*,† gives an extended discussion from this point of view, and of course it is the method of classic Weber.‡ The theory of permutations finds application not only in mathematics, but also in the other sciences. However, from the point of view of the general concepts to be developed, their importance lies for the student in the circumstance that he now has a first illustration of a law of combination different from those involved in the number system, and more important, an example of a non-commutative operation. Further, examples of inverses and an identity are at hand, all very useful when the theory of groups is studied.

In the development of the subject of determinants, I believe that the definition of a matrix with elements in a number field should first be given and the determinant as a valuation of a square matrix. The subject of determinants thus leads naturally to the study of the rational operations of matrices in a number field. The associative law of multiplication and the distributive laws may be proved, and their proofs give good drill to the student in double subscript and summation notation. Again the failure of the commutative law may be discussed, the nature of division emphasized, zero divisors and the inverse illustrated.

Naturally, the subject of matrices is an extended one and the question of limitation of subject-matter a difficult one to answer. However, the theory of equivalence of matrices in a number field, which of course gives the theorems on the rank of matrices as corollaries, is a useful and a simple one. Albert in his recent book,§ *Modern Higher Algebra*, has given an excellent exposition of the subject, first studying equivalence in an integral domain and then in a field, of course giving the subject in abstract integral domains and fields. For the undergraduate at this stage, this means equivalence in the ring of integers and in a number field. If, following the exposition given by Albert, equivalence of matrices is defined by means of the elementary transformations, (already familiar to the student from the study of determinants), and if their matric equivalents are given, proofs are easy and easily remembered. Again the student's attention should be drawn to the essential nature of the study of an equivalence relation and of a reduction to a canonical form; namely, that all $m\times n$ matrices in a number field are separated into classes according to their rank, and that in each class there is a matrix of simple form to which all matrices in the class are equivalent. Further, the equivalence of bilinear forms in a number field is a corollary of the theory of equivalence of matrices in a number

* J. M. Thomas, Theory of Equations, New York, 1938.

† Helmut Hasse, Höhere Algebra, vol. 1, second edition, Berlin, 1933.

‡ Heinrich Weber, Lehrbuch der Algebra, vol. 1, second edition, 1898.

§ A. A. Albert, Modern Higher Algebra, Chicago, 1937, chap. 3.

field. Equivalence in the integral domain of all polynomials in the indeterminate $\lambda$ with coefficients in a field might be a subject of further study if time allows.

The theory of simultaneous linear equations in a number field is given great simplicity by the introduction of the matrix notation; for example, a system of $m$ simultaneous linear equations in $n$ unknowns $y$,

$$\sum_{j=1}^{n} a_{ij}y_j = x_i, \qquad (i = 1, \cdots, m),$$

may be denoted simply by $X=AY$, where $X$ and $Y$ are the one-columned matrices

$$X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_m \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{pmatrix},$$

and $A$ is the matrix of coefficients. To illustrate the simplicity of the theory in the case of $n$ linear equations in $n$ unknowns when $A$ is non-singular, we find that the solution is given immediately by $Y=A^{-1}X$, and since the inverse is unique, the solution is unique; and further, the solution obviously satisfies the system of equations, for $X=AY=A(A^{-1}X)=X$. Cramer's rule follows immediately, of course, from the form of the inverse and the rule for the expansion of determinants by minors. The cases $m\neq n$, $A$ singular, and the homogeneous case are slightly more complicated. May I add that the theory of $m$ simultaneous linear equations in $n$ unknowns should be given in detail and in all generality, for it is precisely the general theory of solution that is needed in the applications. And the general theory can be given effectively only by means of the rank of matrices and the accompanying concept of linear dependence and independence.

After these preliminaries there is illustrative material enough at hand to begin the definitions of abstract groups and rings and to develop some of the elementary properties of their structure. The group is one of the simplest mathematical systems. A first question in discussing the group is of course how redundant to make the set of postulates. For the sake of pedagogy, I have always given the postulates in what might be called super-redundant form. After demanding the closure of the set and the associative law for the law of combination, I have demanded the existence of a commutative identity and of a commutative inverse for each element of the set. If training in postulate work is wanted, these postulates may of course be weakened and the student left to prove these fundamental properties of the identity and inverse. It should be pointed out that the postulates give unique solutions for the equations, $ax=b$ and $ya=b$, and the postulational equivalence should be shown of the demand for a solution of these equations with the demand for an identity and inverse. To make the concept of a group real it is necessary to give numerous and diverse examples of sets, some of which are, and some of which are not groups with

respect to the given law of combination. The symmetry mappings of regular polygons and polyhedra interest the student, and can of course be made to correspond to permutation groups. Examples of infinite groups are available from the number system and matrices. The student is always amused by the additive and multiplicative groups of the residue class rings in the ring of integers. At first startled by such laws of combination as $2+2=1$, the class usually has at least one student who sees how such apparently absurd laws of combination are constructed. Non-commutative groups may be illustrated by permutation groups and matric groups. I have found it feasible to make a thorough study of the cyclic group, both finite and infinite. The $n$ $n$th roots of unity combined with respect to multiplication make good illustrations of the cyclic group, and the group significance of the primitive $n$th roots of unity make the concept of primitive $n$th roots of unity vital to the student. The symmetric and alternating groups should also be familiar to the student.

The problem again is how to limit the material to be given in the theory of groups. Perhaps for all applications of group theory and for the theory itself some knowledge of the theory of homomorphism, isomorphism, and automorphism of groups is necessary. As preparation for its study, the separation with respect to a subgroup of the elements of a group into classes, called cosets, of mutually exclusive elements and the theory of the composition of these classes is of importance. When invariant subgroups and factor or quotient groups have been studied, the student is ready for the homomorphism theory of groups. Van der Waerden* gives this theory very simply and his terminology seems a good one. According to Van der Waerden, a group $\overline{G}$ is the homomorphic image of a group $G$ if to every element $a$ of $G$ there corresponds an element $\bar{a}$ of $\overline{G}$, such that if $a \rightarrow \bar{a}$ and $b \rightarrow \bar{b}$, then $ab \rightarrow \bar{a}\bar{b}$, and further if all the elements of $\overline{G}$ are exhausted in this mapping. It is even not necessary to assume that $\overline{G}$ is a group, for the homomorphic mapping makes $\overline{G}$ a group. The correspondence itself is called a homomorphism. If the correspondence is one to one, the homomorphism is called an isomorphism. If $\overline{G}$ is $G$ itself, the correspondence is called an automorphism. A homomorphic mapping of $G$ upon $\overline{G}$ separates the elements of $G$ into classes, for all those elements which have the same image in $\overline{G}$ are put into the same class. It is a simple step to prove that those elements whose image is the identity in $\overline{G}$ form an invariant subgroup of $G$, and that the remaining classes are cosets with respect to this invariant subgroup. Hence it is easily seen that $\overline{G}$ is isomorphic to a factor group of $G$, and the so-called homomorphism theorem of groups is proved. It is thus seen that the invariant subgroup is the instrument in forming new groups from the given one, and the key to the possible mapping of one group upon another. Using the theory of cosets and factor groups, the student may prove as exercises many of the elementary theorems on finite groups.

It may be said that only a beginning has been made in the study of groups, but I believe it must be remembered at this point that the student is to be given

* B. L. Van der Waerden, Moderne Algebra, vol. 1, second edition, Berlin, 1937.

an introduction to some of the unifying concepts in algebra and not a thorough study of any one theory. If more time is available for the study of groups, I suggest that composition series would form a subject of study having wide application.

The next simplest mathematical system to study is the ring, a set closed under two laws of combination, usually called addition and multiplication. With a knowledge of groups at hand, a simple definition may be given; namely, a commutative group with respect to addition, such that the associative law of multiplication and the distributive laws hold. The student may be given many examples of rings; the number rings and the polynomial rings, with which he is already familiar, the residue class rings in the ring of integers, which are finite rings, rings of matrices, the ring of quaternions, *etc.* Thus examples may be given to show rings with or without zero divisors, with or without unity element, and the student soon has diverse enough examples to realize that further specialization in kinds of rings is necessary to create order among such diverse entities. The integral domain, a commutative ring with unity element and without zero divisors; the division ring or quasi field, a ring all of whose elements except the zero element form a group with respect to multiplication; and the field, a ring in which the multiplicative group is commutative, are the most useful and most elementary classifications of rings. The term, division ring, emphasizes the fact that such a ring is one in which division is possible, whereas the terminology quasi emphasizes the fact that multiplication is non-commutative. Unfortunately, there seems to be no uniform terminology, the terminology varying with the author.

In connection with the integral domain it is interesting to point out to the student that every integral domain can be embedded in a field, called the quotient field, just as the ring of integers can be embedded in the rational number field. Students also find it interesting to prove the properties of the minus sign, which are evidently taken on faith in the elementary mathematical training of most students.

Since the ideas of homomorphism and isomorphism are prevalent in all mathematics, and since the student is now presumably familiar with the homomorphism theory of groups, it seems natural to give the homomorphism theory of rings. Hence, of course, the concept of ideal must be defined, and it is most simply defined as an additive subgroup of the additive group of the ring, such that if $r$ is in the ring and $a$ is in the additive subgroup, $ra$, $ar$, or both products are in the additive subgroup, depending upon whether a left-sided, right-sided, or two-sided ideal is to be defined. Of course in commutative rings all these different ideal concepts resolve into one. For the student at this stage the chief importance of the ideal lies in the rôle it plays in the homomorphism theory of rings. Since an ideal is an additive subgroup of the additive group of the ring, it defines a separation of the elements of the ring into classes, for which a sum and product can be defined. Thus an ideal defines a new ring, called a residue class ring or a difference ring, just as an invariant subgroup defines a

factor group in the theory of groups. If one ring is mapped homomorphically upon another ring, a homomorphic mapping now demanding the preservation of two laws of combination, and if again those elements which have the same image are put into the same class, the result is that those elements whose image is the zero element, naturally forming a group with respect to addition from the homomorphism theory of groups, now form a two-sided ideal, and the remaining classes are the residue classes with respect to this ideal. Examples of various residue class rings may be found in the ring of integers and the ring of polynomials. Moreover, the elementary theory of congruences takes on a new significance when the residue class rings are studied.

By means of the theory of residue class rings, it is easily proved that every field either contains as a subfield a field isomorphic to the field of rational numbers or a field isomorphic to the residue class ring of integers modulo the prime ideal $(p)$, $p$ a prime. A field of the first type is called a field of characteristic zero since the ideal used in its construction is the zero ideal, and a field of the second type a field of characteristic $p$, the ideal used in its construction being the principal ideal $(p)$. If time permits, a short study of the finite commutative fields could be made.

Anyone familiar with Van der Waerden's *Moderne Algebra* will recognize the outline given of the theory of groups and rings as practically the material of the first three chapters, which the author claims are fundamental to any further work in algebra. For the undergraduate, of course, this material must often be given in more elementary and not such general form, and it must be illustrated by many problems and examples. However, the spirit and point of view can be maintained.

For obvious reasons, courses must vary from college to college; and surely the same objective, namely, to introduce the undergraduate to some purely algebraic methods and theorems, can be attained in diverse ways. However, it is evident that the subject of algebra has been neglected in the undergraduate curriculum. One need point only to the Putnam Competitive Examination to see how meager is the algebraic material on which the examiners feel that they may legitimately set questions. Perhaps this situation is due to the fact that algebra has come into great prominence only in comparatively recent years. Moreover, there is the additional handicap of having some of the material written in a foreign language,* or given from a too advanced point of view for the undergraduate. However, its adaptation should be a stimulating challenge to the teacher. My only plea is that some of the elementary algebraic concepts and methods be taught the undergraduate; for surely they are not so sophisticated as the limiting process, the foundation of the calculus, and surely the finite and the discrete may be understood as easily as the infinite and the continuous.

* Saunders Mac Lane, Notes on Higher Algebra, Mimeographed notes, 1939, is an exposition written in English which has recently been called to my attention.

# QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

*The department of Questions, Discussions, and Notes in the* MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

## ON DIFFERENCES OF PAIRS OF PRIMES

G. F. CRAMER, Tulane University

Let $p_i$ denote the $i$th prime number. For any particular integer $n>3$, form all of the differences $d_{ij}=p_i-p_j$ for which $n\geqq i>j>1$. Then $d_{ij}$ is even, since $p_j>p_1=2$. There are $n-2$ of these differences which have $p_n$ as first term, $n-3$ which have $p_{n-1}$ as first term, $n-m-2$ which have $p_{n-m}$ as first term, and, finally, one which has $p_3$ as first term. Let $T_n$ be the total number of the differences considered above. Then $T_n=(n-2)+(n-3)+\cdots+2+1=(n-2)(n-1)/2$.

Let $E_n$ be the number of positive even integers $2k<p_n$, and let the $T_n$ differences $d_{ij}$ be separated into $E_n$ classes, so that any two differences which are equal to the same $2k$ are placed in the same class. Let $N(n, k)$ be the number of $d_{ij}$'s in the class determined by $k$, and let $B_n=\text{Max } N(n, k)$ be the largest of the $N(n, k)$'s. Then $B_n\geqq T_n/E_n$, and so $B_n+1>(n-1)(n-2)/2E_n$. But it is known that $E_n=[p_n/2]<p_n/2<\alpha_{10}(n \log n)/2$* where $\alpha_{10}$ is a fixed number $>0$. Hence $B_n+1>(n^2-3n+2)/(\alpha_{10}n \log n)$.

Consider the $\text{Lim}_{x\to\infty}(x^2-3x+2)/(\alpha_{10}x \log x)$. This is equal to

$$\text{Lim}_{x\to\infty}(2x-3)/\alpha_{10}(1+\log x)=\text{Lim}_{x\to\infty}(2/\alpha_{10}(1/x))=\infty.$$

Let $M>0$ be an arbitrarily large number. Then there is an $n$ large enough so that

$$B_n+1>(n^2-3n+2)/(\alpha_{10}n \log n)>M+1.$$

This completes the proof of the following theorem:

THEOREM. *For arbitrarily large $M>0$, there are integers $n$ and $k$ such that $N(n, k)>M$.*

## DUAL USAGE OF THE TERM "RATE OF CHANGE"

C. B. READ, University of Wichita

The student who has studied calculus is very likely to become confused if he attempts to read in many elementary texts in statistics the material regarding semi-logarithmic charts. He has learned that a constant rate of change means equal increments in the dependent variable corresponding to equal increments in the independent variable. In a large number of the statistics books, however, he will discover that an equal *percentage* increase in one variable over equal intervals of the other variable, usually time, is called a constant rate of change. Having learned from calculus that a function with a constant rate of change is a straight line in the ordinary coördinate system, he now learns, to

* Landau, Vorlesungen über Zahlentheorie, vol. 1, p. 68.

quote from one text, that on a semi-logarithmic chart "a constant rate of change appears as a straight line." Similar quotations indicate a rather loose usage of the term, for example:

> "If we see a curve ascending, and nearly straight, we know that the statistical magnitude it represents is increasing at a nearly uniform rate."
>
> "A series having a constant rate of change . . . is termed a geometric progression."
>
> "In logarithmic charts, if a curve has a steeper slope in one part than in another, it means that the steeper part is changing at a higher rate."

Some texts carefully avoid this term by using such expressions as *ratio of change*. This would seem to avoid any ambiguity, if it were not for such loose statements as:

> "It implies a comparison of the rate of change or ratios between each two successive items in a series . . . "

At least one text uses the terms *rate of change* and *ratio of change* as synonymous.

The ideal situation would be complete elimination of the dual usage of the expression; until that point is reached, the teacher might bear in mind the possible confusion to the beginning student.

## THE DERIVATIVE OF A POLYNOMIAL ON FURTHER ARCS OF THE COMPLEX DOMAIN

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In this note we extend to the limaçon and lemniscate results already established* for other curves and arcs of the complex domain. The method is the same as that used in previous papers.

The equation of the limaçon, which we shall denote by $C$, in polar coördinates is $r=a+b\cos\theta$; but on this curve we have

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = (a + b\cos\theta)(\cos\theta + i\sin\theta)$$

or

$$z = a\cos\theta + b\cos^2\theta + i(a\sin\theta + b\sin\theta\cos\theta). \tag{1}$$

Hence $P_n(z)$, a polynomial of degree $n$ in $z$, by the above relation is a trigonometric sum $Q_{2n}(\theta)$ of order $2n$, and $|P_n(z)| \leqq M$, $z$ on $C$, implies $|Q_{2n}(\theta)| \leqq M$ for all $\theta$. But $|Q_{2n}(\theta)| \leqq M$ implies† that the first derivative $Q_{2n}'(\theta)$ satisfies the inequality $|Q_{2n}'(\theta)| \leqq M(2n)$. Since

$$P_n'(z) = \frac{dP_n(z)}{dz} = \frac{dQ_{2n}(\theta)}{d\theta}\,\frac{d\theta}{dz},$$

* W. E. Sewell, On the polynomial derivative constant for an ellipse, this MONTHLY, vol. 44, 1937, pp. 577–578; The derivative of a polynomial on various arcs of the complex domain, National Mathematics Magazine, vol. 12, 1938, pp. 167–170.

† See, *e.g.*, S. Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions d'une variable réelle, Paris, 1926, p. 39.

we have to examine $dz/d\theta$; from (1)

$$\frac{dz}{d\theta} = -a\sin\theta - 2b\sin\theta\cos\theta + i(a\cos\theta + b\cos 2\theta),$$

$$\left|\frac{dz}{d\theta}\right| = \left\{\begin{matrix} a^2\sin^2\theta + 2ab\sin\theta\sin 2\theta + b^2\sin^2 2\theta \\ + a^2\cos^2\theta + 2ab\cos\theta\cos 2\theta + b^2\cos^2 2\theta\end{matrix}\right\}^{1/2}$$

$$= [a^2 + b^2 + 2ab\cos\theta]^{1/2}.$$

Hence

$$\left|\frac{dz}{d\theta}\right| \geqq \Big|\,|a| - |b|\,\Big|.$$

Thus we have

$$|P_n'(z)| \leqq \frac{M(2n)}{\Big|\,|a| - |b|\,\Big|}, \qquad z \text{ on } C, \qquad |a| \neq |b|.$$

The case $|a| = |b|$, the cardioid, is treated elsewhere.*

For the lemniscate $C$ we have in polar coördinates

$$r^2 = a^2\cos 2\theta,$$

and hence

$$z = a\sqrt{\cos 2\theta}\,(\cos\theta + i\sin\theta), \quad \text{or} \quad z^2 = a^2\cos 2\theta(\cos\theta + i\sin\theta)^2.$$

From this we see that a polynomial $P_n(z^2)$ of degree $n$ in $z^2$ is a trigonometric sum $Q_{4n}(\theta)$ of order $4n$. Hence by the above reasoning $|P_n(z^2)| \leqq M$ implies $|Q_{4n}'(\theta)| \leqq M(4n)$. Also

$$\frac{dz}{d\theta} = \frac{a(-2\sin 2\theta)}{2\sqrt{\cos 2\theta}}(\cos\theta + i\sin\theta) + a\sqrt{\cos 2\theta}\,(-\sin\theta + i\cos\theta)$$

$$= \frac{a}{\sqrt{\cos 2\theta}}\left\{-\sin 2\theta\cos\theta - \cos 2\theta\sin\theta + i(-\sin 2\theta\sin\theta + \cos 2\theta\cos\theta)\right\};$$

$$\left|\frac{dz}{d\theta}\right| = \frac{a}{\sqrt{\cos 2\theta}}\left\{\begin{matrix}\sin^2 2\theta\cos^2\theta + 2\sin 2\theta\cos 2\theta\cos\theta\sin\theta + \cos^2 2\theta\sin^2\theta \\ + \sin^2 2\theta\sin^2\theta - 2\sin 2\theta\cos 2\theta\sin\theta\cos\theta + \cos^2 2\theta\cos^2\theta\end{matrix}\right\}^{1/2}$$

$$= \frac{a}{\sqrt{\cos 2\theta}}.$$

Hence

$$|P_n'(z^2)| \leqq \frac{M(4n)\sqrt{\cos 2\theta}}{a}, \qquad z \text{ on } C.$$

By the same method the result can be extended to curves given by polar equations in which $2\theta$ is replaced by $m\theta$, where $m$ is an arbitrary positive integer.

* W. E. Sewell, National Mathematics Magazine, *loc. cit.*

## RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

*All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.*

### NEW BOOKS RECEIVED

*Principles of the Mathematical Theory of Correlation.* By A. A. Tschuprow. Translated by M. Kantorowitsch. London, Edinburgh, and Glasgow, William Hodge and Company, Limited, 1939. 10+194 pages. 12s/6 net.

*The Nature of Proof.* By H. P. Fawcett. New York, Bureau of Publications, Teachers College, Columbia University, 1938. 11+146 pages. $1.75.

*Introduction to Geometry.* By A. W. Siddons and K. S. Snell. Cambridge, The University Press, 1939. 7+167 pages. $0.80.

*Descriptive Geometry.* By James T. Larkins, Jr. New York, Prentice-Hall, Inc., 1939. 8+317 pages. $2.50.

*Projektive Geometrie.* By Heinz Prüfer. Leipzig, Akademische Verlags-Gesellschaft, 1939. 8+314 pages. RM 12.00.

*Selections Illustrating the History of Greek Mathematics.* By Ivor Thomas (translator). Vol. I: *From Thales to Euclid.* Cambridge, Mass., Harvard University Press; London, William Heinemann, Ltd., 1939. 15+505 pages.

*Piccola Storia della Matematica da Pittagora a Hilbert.* By Egmont Colerus. Translated by Spartaco Casavecchia. Torino, Guilio Einaudi, 1939. 359 pages.

### REVIEWS

*Fabre and Mathematics and Other Essays.* By Lao G. Simons. New York, Scripta Mathematica, 1939. 5+101 pages. $1.00.

Three of the essays in this volume have appeared previously, and there is one which makes its first appearance. The latter, entitled *The Interest of Alexander von Humboldt in Mathematics*, presents some of the evidence of (1) Humboldt's friendly relations with Gauss, Laplace, Dirichlet, Jacobi, and Eisenstein, (2) his special interest in the numeration and notation of primitive tribes and early peoples, and (3) his admiration for the methods of mathematics.

The leading article from which this volume takes its name consists almost entirely of quotations from Fabre's works, giving the story of his interest in mathematics and his insight into the various mathematical concepts and laws which are revealed in diverse phenomena in the insect world. This article is especially delightful and reading it should help increase one's "appreciation of mathematics as a part of the educational discipline for all students through the secondary school grade."

In the essay, *The Influence of French Mathematicians at the End of the Eighteenth Century upon the Teaching of Mathematics in American Colleges*, the author recounts many of the works of Lagrange, Laplace, and Legendre which,

in different translations, were used in various colleges and schools in the United States. "The École Polytechnic furnished a model for the United States Military Academy at West Point, a model followed afar off but, nevertheless, an inspiration."

The final essay, *Short Stories in Colonial Geometry*, gives a record from each of several colonial colleges showing "the ideals of the men connected with each one and the extent to which these ideals may have been reached."

MARIAN E. DANIELLS

*Differential and Integral Calculus.* Second edition. By J. H. Neelley and J. I. Tracey. New York, The Macmillan Co., 1939. 9+495 pages. $3.25.

The first edition of this book appeared in 1932; it was reviewed in this MONTHLY, vol. 40, 1933, pp. 235–236, by Samuel Borofsky.

The essential features of the earlier edition are retained, including review chapters on plane and solid analytic geometry and an introduction to ordinary differential equations. Parts of the book have been rewritten, to advantage, notably the sections on curvilinear motion and on simple harmonic motion. There has been added to chapter 8 a paragraph on space curves and to chapter 9 a discussion of the tangent planes to a surface, and a section on maxima and minima of a function of two variables.

The most notable and commendable change from the first to the second edition is the revision and increase in number of the problems. Choosing at random, after section 99 there are nineteen problems in the first edition and thirty-three in the present edition.

The second edition is more attractive in appearance due to the use of a larger page. However, the reviewer wishes the authors and publishers had taken advantage of this larger page to increase the size of some of the figures. These are profuse and generally excellent; but the reviewer found it quite an eye-strain to study Figure 139, for example.

The book is an excellent text for those who prefer to make the differential calculus and the integral calculus successive subjects.

L. L. LOWENSTEIN

*College Algebra.* Revised edition. By J. B. Rosenbach and E. A. Whitman. Boston, Ginn and Co., 1939. 10+431 pp. $2.25.

Thoroughly, yet not drastically rewritten, this well known text, first published in 1933, has been definitely improved by the revision.

This edition presents an entirely new set of exercises, and for all chapters the number has been generously increased. The various tables have also undergone some revision. Illustrative examples appear in greater numbers, and, as in the case of the first edition, have been carefully chosen. The general plan of this text, however, has not been changed. The numerous historical notes have been retained, as well as the "Warnings"—admonitions concerning the various algebraic "boners." This new edition should be a very satisfactory text.

MALCOLM FOSTER

*The Study of the History of Science.* By George Sarton. Cambridge, Mass., Harvard University Press, 1936. 6+75 pages.

*The Study of the History of Mathematics.* By George Sarton. Cambridge, Mass., 1936. 113 pages.

These two little publications of the Harvard University Press contain the substance of inaugural lectures delivered at Harvard University, during the academic years 1935–36, in semester courses in the History of Science and in the History of Mathematics. The lectures probably corresponded more particularly to pages 3–52 and 3–28 respectively. A bibliography then follows in each case (pp. 53–70, and pp. 39–65); each has an index (pp. 71–74, and pp. 105–113); but the second volume has also a "Note" and an "Appendix" (pp. 29–103), to be referred to later.

Both lectures are remarkably stimulating and thought-provoking and constitute extraordinarily appropriate introductions to study in the history of science and in the history of mathematics. Many things in the first lecture apply equally well to the history of mathematics, and since there is no repetition, the second lecture is the briefer.

As fundamental in his discussion, Dr. Sarton enunciates (a) a definition; (b) a theorem; (c) a corollary. (a) "Science is systematized positive knowledge or what has been taken as such at different ages and in different places." (b) "The acquisition and systematization of positive knowledge are the only human activities which are truly cumulative and progressive." (c) "The history of science is the only history which can illustrate the progress of mankind. In fact, progress has no definite and unquestionable meaning in other fields than the field of science."

The history of science is a relatively new discipline, not yet represented by an array of noble books constituting a definite kind of scholarship different from all others. One thinks of Whewell's *History of the Inductive Sciences* (1837) as a pioneer work, but Dr. Sarton feels that here the "conception of the history of science was still primitive and somewhat narrow, and his realization of it imperfect." "The first scholar to conceive that subject as an independent discipline and to realize its importance was the French philosopher Auguste Comte (1798–1857), and the scholar who deserves more than any other to be called the father of your studies."

The lecture develops a large number of themes such as: scientific and historical preparation for study, natural vs. historical laws; the fundamental nature of accuracy, and needless precision; pitfalls in establishing dates; measuring the past, with illustrations, including suggestions of the relative amounts of space to be devoted to different periods in a well-balanced complete history; the means of obtaining relative mastery of the whole field of science by starting with cross sections such as the development of one branch of science, the development of science and learning within a special period (such as a particular century), the development of science and learning within a certain country; historical criticism; scientific tradition; scientific education; the psychology of scientific dis-

covery; the human side of science; the need of thoroughness; thoroughness and altruism; the danger of pedantry; and, finally, the constant need of selection.

The bibliography lists works on historical methods, scientific methods, the chief reference books, journals and serials, treatises, handbooks, societies and congresses,—in the history of science. The work as a whole must be most helpful to all who are interested in taking up the teaching of the study of the history of science.

The second lecture gives a general outline on the history and development of mathematics (pp. 3–28) but does not take up the actual methods of study of the subject. This is followed by "Note on the study of modern mathematics" (pp. 29–38) which furnishes valuable guides and emphasizes an important point as follows: "the history of modern mathematics should be taught by mathematical teachers in the course of their ordinary teaching, while the history of older mathematics can be properly taught only by a specialist, who must be as much of an historian as of a mathematician, if not more."

As in the case of the history of science, the bibliography is arranged under a number of headings: general treatises, and other works completing them, with regard to the history of mathematics in Egypt, Mesopotamia, Greece, India, the Far East, and in medieval times; handbooks; treatises devoted to special branches of mathematics; mathematics in the nineteenth and twentieth centuries; philosophy and methodology; *etc.* The annotated bibliography deals only with material with which Dr. Sarton was familiar and there are frequent references to *Isis* for fuller information about works or people mentioned. The omission of any reference to *Zentralblatt für Mathematik* is serious and the comment about *Revue Sêmestrielle des Publications Mathêmatiques* is quite misleading. It is not made clear that the very valuable *A Source Book in Mathematics* was only edited by D. E. Smith; many authors collaborated. Then follows this note: "Unfortunately the selection begins only with the end of the fifteenth century; this is not due to the author's [?] fault, but is due to the stupid programme of the collection in which it is included. On that account, if on no other, Wieleitner's source-book is preferable." The reviewer finds this comment far from fair, and unscientific. A substitute statement telling what was in the volume would have been an illuminating guide.

The useful Appendix on "Biographies of modern mathematicians" (pp. 67–103) quotes for each of 118 mathematicians the best available biographies, editions of their collected works, and editions of their correspondence. In such a list, to leave out Salmon and to put in his translator Fiedler seems strange. No one but a bibliographic expert would know that of the two references under Sylow, the first is to an article in *Norsk Matematisk Tidsskrift*, and the second to Norsk Matematisk Forening, *Skrifter*, s. 2. On the whole, however, the references are exceedingly accurate, as one would expect when coming from the pen of a great scholar, and the leading living authority on the general history of science.

This book is a valuable one which should be widely known.

RAYMOND CLARE ARCHIBALD

### III. Advanced Theory of Projections of the Sphere and Spheroid

**19.** Deetz and Adams, Elements of Map Projections, Part II. "It is the purpose in Part II to give a comprehensive description of the nature, properties, and construction of the better systems of map projections in use at the present day." References are given to other sources for those projections for which the mathematical development is not given.

**20.** Hinks, A. R., Map Projections, Cambridge University Press, 1912. Theory of important projections of the sphere.

**21.** Brown, B. H., Conformal and equiareal world maps, this Monthly, vol. 42, pages 212–221. A summary of four important investigations of the types of conformal and equal-area projections in which the meridians and parallels are straight lines or conics.

**22.** Hughes, William, Treatise on the Construction of Maps, Longmans, Green, 1864.

**23.** Craig, T. A., Treatise on Projections, U. S. Coast and Geodetic Survey, 1882.

**24.** Germain, A., Traité des Projections des Cartes Géographiques, A. Bertrand, Paris, 1866.

**25.** Herz, Norbert, Lehrbuch der Landkarten Projektionen, Teubner, Leipzig, 1885.

**26.** Adams, O. S., General Theory of Polyconic Projections, U. S. Coast and Geodetic Survey, Special Publication No. 57, 1934.

**27.** Adams, O. S., Elliptic Functions Applied to Conformal Maps, U. S. Coast and Geodetic Survey, Special Publication No. 97, 1925.

*Note.* Several of the references appearing above were sent in by P. M. Whitman, Harvard University.—E. H. C. H.

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# PROBLEMS AND SOLUTIONS

Edited by Otto Dunkel, H. L. Olson, and H. S. M. Coxeter

## ELEMENTARY PROBLEMS

*Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.*

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

### PROBLEMS FOR SOLUTION

E 396. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Given a triangle $ABC$, construct a point $X$ such that the three lines drawn through $X$, each parallel to a side of the triangle and limited by the other two sides, are equal.

E 397. *Proposed by H. T. R. Aude, Colgate University.*

The symbols $a$, $b$, $c$, $d$, $e$, $f$ are distinct digits in the denary scale. Find the three-figured number $abc$ such that $(abc+1)^2=acdef$, and $(a+b+c)^{1/3}+1=(a+c+d+e+f)^{1/3}$.

E 398. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Given a triangle $ABC$, let $O$ be the circumcenter, $A'$ the projection of $A$ on $BC$, $M$ any other point of $BC$, and $B_1$, $C_1$ the respective projections of $B$, $C$ on $AM$. Let lines through $M$, parallel to $A'C_1$ and $A'B_1$, meet $AC$ and $AB$ in points $P$ and $Q$, respectively. Prove that the lines $PQ$ and $OM$ are perpendicular.

E 399. *Proposed by V. Thébault, Le Mans, France.*

Prove that the product of the first $n$ positive integers $(1 \cdot 2 \cdots n)$ is divisible by their sum $(1+2+\cdots+n)$ if and only if $n+1$ is not an odd prime.

E 400. *Proposed by H. S. M. Coxeter, University of Toronto.*

Show how to dissect a regular hexagon by straight cuts into the smallest possible number of pieces which can be reassembled to form an equilateral triangle (of the same area).

**SOLUTIONS**

E 358 [1939, 48]. *Proposed by L. S. Johnston, University of Detroit.*

Mr. Brown asked Mr. Smith to perform the following operations in the order named, without Mr. Brown's being able to see Mr. Smith's work:

(i) Write an integer, preferably of two digits, to save labor on the part of Mr. Smith.

(ii) Multiply this number by the next higher integer.

(iii) Multiply the result of (ii) by 225.

(iv) Add 56 to the result of (iii).

(v) Tell Mr. Brown all the result of (iv) except the two right-hand digits.

Mr. Smith gave 4064 in response to the request of (v), whereupon Mr. Brown, after a moment's computation, informed Mr. Smith that his result after step (iv) was 406406, and that the number he originally chose was 42. Mr. Smith confirmed these statements.

(A) How did Mr. Brown reach his conclusion?

(B) Generalize the problem.

I. *Solution by Hazel E. Schoonmaker, Annapolis, Maryland.*

(A) The largest number possible at the end of step (iv) is 406499. Subtract 56 from this, giving 406443. Divide by 225, giving a quotient of 1806 and a remainder of 93. Since we are working with integers, there should be no remainder. Hence 406499 is too large by 93, and the number at the end of step (iv) was 406406. The largest square in 1806 is the square of 42. Hence 42 is the number originally chosen.

(B) Any integers may be used in place of 56 and 225, so long as the latter is greater than 100.

II. *Solution by E. P. Starke, Rutgers University.*

(A) Let $n$ be the integer of (i), let $a$ be the result (v) as told to Mr. Brown, and let $b$ be the two-digit number omitted from (iv) to get (v). Then we have

$$225n(n+1) + 56 = 100a + b. \tag{1}$$

Since $n(n+1)$ is always even, and thus $225\,n(n+1)$ is a multiple of 50, $b$ must be 56 or 06. With these values of $b$, (1) may be put in the form $225\,n(n+1)+56 = 100\,a+(31 \pm 25)$, and reduced to

$$9n(n+1) + 1 = 4a \pm 1. \tag{2}$$

Thus $b$ is 56 or 06 according as $a$ is or is not divisible by 3. Further, (2) may be written $n(n+1)=(4a\pm 1-1)/9$. Since $n(n+1)$ lies between $n^2$ and $(n+1)^2$, differing from the latter by more than 2/9, it follows that

$$n^2 < 4a/9 < (n+1)^2.$$

Thus $n$ is the greatest integer which does not exceed $2a^{1/2}/3$; in short, $n=[2a^{1/2}/3]$.

In the cited instance: since $a=4064$ is not a multiple of 3, $b$ is 06, making $100\ a+b=406406$. From $\sqrt{4064}=63.7$, we have $n=42$.

(B) The constants 56 and 225 may be replaced by any number less than 100 and any odd multiple of 25, say $25k$, with no essential change in method. The choice between two possible values of $b$ (differing by 50) is determined by considering the divisibility of $a$ by any factor of $k$ (or, if $k=1$, by seeing whether $2a$ is a triangular number). The answer is $n=[2(a/k)^{1/2}]$.

As a sample of what could be done with a cubic relation, consider the following operations:

(i) Write an integer $n$.
(ii) Multiply this by the next two higher integers.
(iii) Multiply the result of (ii) by 125.
(iv) Subtract 156 from the result of (iii).
(v) Tell the result of (iv) except the three right-hand digits.

As before, let $a$ be the result reported in (v); then we have

$$125n(n+1)(n+2)-156=1000a+b,$$

and $n=[2a^{1/3}]$ (except that $n=1$ when $a=0$). If $n$ is even, $b=844$; otherwise $b=594$, 344, 094, or 844 acording as $n\equiv 1$, 3, 5, or 7 (mod 8). The proof depends on simple algebra similar to the above.

Also solved by H. T. R. Aude, Wm. B. Campbell, Wm. Douglas, C. W. Trigg, and the proposer.

*Editorial Note.* Campbell remarks that the last *three* digits could have been omitted from Mr. Smith's report (v). For, an increase of unity in the value of $n$ makes an increase of $450\ (n+1)$ in the result of (iii) or (iv), and this increase is greater than 1000 (except when $n=1$).

E 359 [1939, 48]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

A quadrilateral, $ABCD$, is inscribed in a circle. A second circle is passed through $A$ and $D$, tangent to $CD$. A third is passed through $B$ and $C$, tangent to $CD$. A fourth is passed through $A$ and $D$, tangent to $AB$. A fifth is passed through $B$ and $C$, tangent to $AB$. Points $E$, $F$, $G$, and $H$ are the respective second intersections of the second circle with $AB$, the third circle with $AB$, the fourth circle with $CD$, and the fifth circle with $CD$. Show that quadrilaterals $ABGH$, $CDEF$, and $EFGH$ are each inscriptible, and that the centers of the three new circles thus determined form a parallelogram with the center of the first circle.

(A regrettable error in the original enunciation was noted by each solver.)

*Solution by Herman Levy, Brooklyn, New York.*

Let $AB$ meet $CD$ in $O$. Considering the five circles in turn, we have:

$$OA \cdot OB = OC \cdot OD; \qquad OA \cdot OE = OD^2; \qquad OB \cdot OF = OC^2;$$
$$OA^2 = OD \cdot OG \cdot \qquad OB^2 = OC \cdot OH.$$

Therefore $OE \cdot OF = OC \ OD = OA \cdot OB = OG \cdot OH$; and $EFHG$ is cyclic, as well as $ABHG$ and $CDEF$. The center of the circumcircle of a cyclic polygon lies on the perpendicular bisectors of the sides. The perpendicular bisectors of segments $AB$, $EF$, and likewise of $DC$, $GH$, are parallel. Hence the centers of the circles circumscribing $ABCD$, $ABHG$, $EFHG$, $CDEF$ are the vertices of a parallelogram.

Also solved by W. B. Clarke, and K. W. Crain.

E 360 [1939, 49]. *Proposed by J. E. Trevor, Cornell University.*

Let $S_{ij}$ be the set of $j$ consecutive positive integers, starting with the integer $i$. Form all possible combinations of these integers, taken $1, 2, 3, \cdots, j$, at a time. Add the integers in all these combinations into a single total, $T_{ij}$. Prove that $T_{ij} = j(2i+j-1)2^{j-2}$.

*Solution by H. G. Landau, Carnegie Institute of Technology.*

Let $s_k$ be the sum obtained by all possible combinations of the integers $S_{ij}$ taken $k$ at a time. Then $s_1 = j(2i+j-1)/2$. In the combinations which go to make $s_k$, each integer of the set $S_{ij}$ will occur exactly $\binom{j-1}{k-1}$ times; that is, each integer occurs in all the combinations which can be formed by using the remaining $j-1$ integers taken $k-1$ at a time. Hence $s_k = \binom{j-1}{k-1} s_1$, and

$$T_{ij} = s_1 \sum_{k=1}^{j} \binom{j-1}{k-1} = s_1(1+1)^{j-1} = j(2i+j-1)2^{j-2}.$$

Also solved by Wm. Forman, V. W. Graham, E. R. Heineman, P. M. Hummel, E. P. Starke, W. R. Talbot, C. W. Trigg, and the proposer.

## ADVANCED PROBLEMS

*Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.*

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

### PROBLEMS FOR SOLUTION

3933. *Proposed by T. E. Naish, Major, Royal Engineers (retired), Penticton, Lake Okanagan, B. C.*

If $f(x)=(x-1)(x-2)\cdots(x-p+1)$, where $p$ is a prime, show that (1) each coefficient of $f(x)$ is divisible by $p$ except the first and the last, and the sum of these two is also divisible by $p$; (2) the sum of the even coefficients is equal to the sum of the odd, and this sum is $p!/2$.

*Editorial Note.* Since the proof of part (1) is obvious from the algebra of the residues, mod $p$, the following extension is offered: Let $f_m(x)$ denote the polynomial, the coefficient of whose highest power of $x$ is unity and which vanishes for and only for the positive residues, mod $p^m$, where $m$ is a positive integer, and the residues are prime to $p$ and less than $p^m$. Then

$$\begin{aligned} f_m(x) &\equiv (x^{p-1}-1)^{p^{m-1}}, \text{ mod } p^m, \text{ where } p \text{ is an odd prime;} \\ &\equiv (x^2-1)^{2^{m-2}}, \text{ mod } 2^m. \end{aligned}$$

O. D.

3934. *Prosposed by V. Thébault, Le Mans, France.*

Let $G$ be the centroid of the tetrahedron $ABCD$. Through $A$, $B$, $C$, $D$ planes are drawn perpendicular to $GA$, $GB$, $GC$, $GD$, respectively, forming the antipedal tetrahedron of $G$, with respect to $ABCD$, of volume $V_g$. Similarly, $V_a$, $V_b$, $V_c$, $V_d$ are the volumes of the antipedal tetrahedrons of $A$, $B$, $C$, $D$, with respect to the tetrahedrons $GBCD$, $GCDA$, $GDAB$, $GABC$. Show that $V_g=4V_a=4V_b=4V_c=4V_d$.

3935. *Proposed by V. Thébault, Le Mans, France.*

In what system of numeration is it true that a number formed from four identical figures *aaaa* is a perfect square provided that $a$ is a square? The solution is unique. What happens if $a$ is not a square?

Consider the same questions for a number formed from five identical figures *aaaaa*.

3936. *Proposed by N. A. Court, University of Oklahoma.*

If of the four circles determined by four coplanar points taken three at a time two circles are orthogonal, the remaining two circles are orthogonal. (*Mathesis*, 1929, p. 130, Q. 2515).

If of the five spheres determined by five points in space taken four at a time three spheres are mutually orthogonal, the remaining two spheres are orthogonal to each other. Prove, or disprove.

## SOLUTIONS

3829 [1937, 332]. *Proposed by J. D. Hill, Michigan State College.*

Let $C$ be a simple closed rectifiable plane curve and $P$ an arbitrary point inside of $C$. (a) Show that there exist two points $A$ and $B$ on $C$ such that $P$ bisects the chord $AB$. (b) Does this property remain true if the curve is nonrectifiable?

*Solution by H. E. Vaughan, University of Michigan.*

The proof given here holds for any simple closed curve $C$, rectifiable or not, lying in a plane $S$. Denote by $D_1$ and $D_2$ the interior and exterior of $C$ and let $P$ be any point of $D_1$. Let $d$ be a fixed line through $P$ and denote by $C_0$ and $C_\pi$ the intersections of $C$ with the two rays into which $d$ is separated by $P$. Denote by $C'$, $C_0'$, and $C_\pi'$, the sets obtained from $C$, $C_0$, and $C_\pi$ by rotating $S$ about $P$ through an angle $\pi$.

We wish to show that $C$ and $C'$ have at least one point in common, since any such point will be admissible as the point $A$ of the theorem, while the point mapped into it by the above rotation may be used for $B$. To prove the existence of such a point we shall show that either (1) $C_0'$ and $C_\pi$ have a point in common or (2) $C'$ contains points of both $D_1$ and $D_2$. If (1) holds, the theorem will be proved, since $C_0'$ and $C_\pi$ are subsets of $C'$ and $C$ respectively, so that a point common to them will be common to the latter pair also. If (2) holds, the connected set $C'$ contains points of both domains complementary to $C$ and hence contains at least one point of $C$, so that the theorem also follows in this case. (We make use here of the theorem that a connected set which contains points interior and exterior to a set contains at least one boundary point of the set. We might also use the more sophisticated Jordan Curve Theorem.)

Let us suppose, then, that $C_0'$ and $C_\pi$ have no points in common. Then any point of $C_0'$ is in $D_1$ or $D_2$. Without loss of generality we may assume that some point of $C_0'$ is in $D_1$. If, in addition, there is a point of $C_0'$ in $D_2$, we have case (2) and the theorem is proved. Since $C_0'$ is a closed subset of the ray issuing from $P$ and containing $C_0'$ and $C_\pi$, it has a last point, $R'$. If no point of $C_\pi$ lies beyond this, $R'$ lies in $D_2$ and the theorem is proved. If there is a point $Q$ of $C_\pi$ beyond all points of $C_0'$ on the ray in question, the corresponding point $Q'$ of $C_\pi'$ lies, on the opposite ray, beyond all points of $C_0$ and is a point of $C'$ in $D_2$.

As a matter of fact, the following more general theorem has been proved: Let $C$ be any plane continuum, $D$ a bounded domain complementary to $C$, and $P$ an arbitrary point of $D$. Then there exist two points $A$ and $B$ of the boundary of $D$ such that $P$ bisects the segment $AB$. It is easily seen that the property need not hold in case $D$ is not bounded.

*Second Solution.* This solution is less elementary than the preceding but seems to be of interest in view of its purely topological character. Suppose that $C$ is given in rectangular coördinates in $S$ by the equations $x=f(t)$, $y=g(t)$, $0\leqq t\leqq 1$. The locus $M$ of midpoints of chords of $C$ is given by the equations

$$(1)\qquad x = \tfrac{1}{2}[f(t_1) + f(t_2)], \qquad y = \tfrac{1}{2}[g(t_1) + g(t_2)]$$

where $(t_1, t_2)$ runs over the unit square in an auxiliary $(t_1, t_2)$-plane, and it is sufficient for the proof of the theorem to show that $M$ covers the interior of $C$. Due to symmetry the range of $(t_1, t_2)$ may be restricted to the triangle $t_1\leqq t_2$, $0\leqq t_2\leqq 1$, and also, for each number $s$, $0\leqq s\leqq 1$, the points $(0, s)$ and $(s, 1)$ may be identified, since the functional values $x$ and $y$ are the same for both points. In this process the points $(0, 0)$ and $(1, 1)$ are also identified since each is identified with $(0, 1)$. This identification introduces a "twist" so that the resulting surface is a cross-cap (topologically equivalent to a Moebius band). Its boundary mod 2 is the line $t_2=t_1$ which, through the identification of its endpoints, has become a simple closed curve. Now, by the mapping (1), this curve is mapped into $C$ and the cross-cap as a whole goes into the set $M$. In the language of combinatorial topology we may say that the singular cycle determined by the mapping on $C$ bounds, mod 2, the singular 2-chain determined by the mapping on $M$. But this necessitates that $M$ cover the interior of $C$ because of a theorem whose intuitive content is that if $P$ is a point interior to $C$, $C$ does not bound in $S-P$.

Solved also by the proposer.

3844 [1937, 601]. *Proposed by Gertrude S. Ketchum, Urbana, Ill.*

Let

$$p_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a polynomial of degree $n$. Then

$$\sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s}p_n(s) = a_nn!.$$

If the degree is less than $n$, then the above sum is zero. Also if

$$\sum_{s=0}^{n} p_n(s)\psi(n, s) = a_nn!$$

for every polynomial of degree $n$, then

$$\psi(n, s) = (-1)^{n+s}\binom{n}{s}.$$

*Solution by Roy MacKay, Eastern New Mexico Junior College.*

Set

$$A_{n-k} = \sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s} s^{n-k}, \qquad 0 \leqq k < n,$$

$$= \sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s}, \qquad k = n.$$

Then

$$\sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s} p_n(s) = \sum_{k=0}^{n} A_{n-k}a_{n-k}.$$

Now $A_{n-k}$ is $(n-k)!$ times the coefficient of $x^{n-k}$ in the expansion of $(e^x-1)^n$; for

$$(e^x - 1)^n = \sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s} e^{xs} = \sum_{s=0}^{n} (-1)^{n+s}\binom{n}{s} \sum_{r=0}^{\infty} \frac{x^r s^r}{r!} = \sum_{r=0}^{\infty} \frac{A_r x^r}{r!}.$$

Since

$$(e^x - 1)^n = x^n\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)^n,$$

it is clear that $A_n = n!$ and $A_{n-k} = 0$, $0 < k \leqq n$. Hence

$$\sum_{k=0}^{n} A_{n-k}a_{n-k} = n.a_n.$$

If the given polynomial is of degree less than $n$, $a_n = 0$ and the required sum is zero.

On the other hand, if

$$\sum_{s=0}^{n} p_n(s)\psi(n, s) = a_n n!$$

for every set of numbers $a_n, a_{n-1}, \cdots, a_0$, $(a_n \neq 0)$, then

$$\sum_{s=0}^{n}\left[(-1)^{n+s}\binom{n}{s} - \psi(n, s)\right] p_n(s) \equiv 0.$$

Denote the expression in brackets by $B_s$, then

$$\sum_{s=0}^{n} B_s s^{n-k} = 0, \quad 0 \leqq k < n; \qquad \sum_{s=0}^{n} B_s = 0.$$

This system of $n+1$ linear homogeneous equations in $n+1$ unknowns $B_s$ has a

determinant of the Vandermonde type which is easily evaluated and found to be different from zero. Hence

$$\psi(n, s) = (-1)^{n+s}\binom{n}{s}.$$

Solved also by the proposer.

*Editorial Note.* The solution of this problem follows immediately after setting

$$\sum_{s=0}^{n}(-1)^{n+s}\binom{n}{s}p_n(s) = \Delta^n p_n(0),$$

and using the facts that $\Delta^n(0)^k=0$, $0\leqq k<n$, and $\Delta^n(0)^n=n!$, where here $(0)^0=1$.

Since this difference formula is seldom used, we give its simple derivation. Denote by $U$ the operator such that $U^i f(x)=f(x+i)$, and by $\Delta$ the operator such that $\Delta f(x)=(U-1)f(x)=f(x+1)-f(x)$. Then

$$\begin{aligned}\Delta^n f(0) = (U-1)^n f(0) &\doteq \sum_{s=0}^{n}(-1)^{n-s}\binom{n}{s}U^s f(0),\\ &= \sum_{s=0}^{n}(-1)^{n+s}\binom{n}{s}f(s).\end{aligned}$$

3845. [1937, 601]. *Proposed by R. E. Gaines, University of Richmond.*

A right triangle $ABC$ is inscribed in a conic with the side $AB$ and the hypotenuse $CA$ as normal chords. Prove that the conic must be an ellipse. Determine the position of $A$ for such a triangle.

*Solution by the Proposer.*

If $A$ is the origin and $AB$ is the axis of $x$, the equation of the conic is

$$ax^2 - 2hxy + by^2 - 2gx = 0. \tag{1}$$

The coördinates of $B$ and $C$ are $(2g/a,\ 0)$ and $(2g/a,\ 4gh/ab)$. Imposing the condition that $AC$ is perpendicular to the tangent at $C$ gives

$$ab - b^2 = 4h^2, \qquad \text{or} \qquad ab - h^2 = 3h^2 + b^2, \tag{2}$$

which shows that the conic must be an ellipse. The center of the ellipse is $[bg/(ab-h^2),\ hg/(ab-h^2)]$ which reduces by (2) to $[4g/(3a+b),\ 4hg/b(3a+b)]$. The distance of $A$ from the center is

$$\rho^2 = 4g^2(a+3b)/b(3a+b)^2. \tag{3}$$

If the lengths of major and minor semi-axes be denoted by $\alpha$ and $\beta$, the equation for determining them is found to be

$$r^4 - \frac{16(a+b)g^2}{b(3a+b)^2} r^2 + \frac{64g^4}{b(3a+b)^3} = 0, \tag{4}$$

whence

$$\alpha^2 + \beta^2 = \frac{16(a+b)g^2}{b(3a+b)^2}, \qquad \alpha^2\beta^2 = \frac{64g^4}{b(3a+b)^3} \cdot \tag{5}$$

Eliminating $g$ from (3) and (5) gives two equations homogeneous in $a$ and $b$, from which we obtain by eliminating $a$ and $b$

$$\rho^2 = \left[5(\alpha^2 + \beta^2) - \sqrt{9(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}\right]/8. \tag{6}$$

If now we take the axes of the ellipse as coördinate axes so that the equation becomes $x^2/a^2 + y^2/b^2 = 1$, the value of $\rho$ in (6) with $a$ and $b$ replacing $\alpha$ and $\beta$ is simply the distance of $A$ from the center. The coördinates $(x', y')$ of $A$ are given by

$$\begin{aligned} 8(a^2 - b^2)x'^2 &= a^2\left[5a^2 - 3b^2 - \sqrt{9(a^2 - b^2)^2 + 4a^2b^2}\right], \\ 8(a^2 - b^2)y'^2 &= b^2\left[3a^2 - 5b^2 + \sqrt{9(a^2 - b^2)^2 + 4a^2b^2}\right], \end{aligned} \tag{7}$$

where $a$ and $b$ are not the same as the letters in (1), but are the lengths of the semi-axes. There are, of course, four such points $A$, one in each quadrant. If $a^2 = 15$, $b^2 = 6$, the point $A$ in the first quadrant is $(\sqrt{5}, 2)$.

*Editorial Note.* A circle circumscribing the right triangle $ABC$ must be tangent to the conic at $C$; hence the bisector of angle $ACB$ must be parallel to a principal diameter of the conic. It will then be easily seen that the conic must be an ellipse. The determination of $A$ for a given ellipse requires a tedious computation.

3846 [1937, 601]. *Proposed by V. Thébault, Le Mans, France.*

The five points $A_1, A_2, A_3, A_4, A_5$ on a sphere with the center $O$ determine five tetrahedrons by omitting in turn $A_1, A_2, \cdots$. Let $M_i$, $i = 1, 2, 3, 4, 5$, be the points which divide in the same ratio $k$ the segments which join $O$ with the corresponding Monge points $\Omega_i$. Show that the five straight lines $A_iM_i$ pass through a single point, and determine its position. Generalize for $n$ points on a sphere.

*Solution by Roy MacKay, Eastern New Mexico Junior College.*

A point $M$ which divides the segment $O\Omega$ from circumcenter to Monge point of an $r$-simplex in the ratio $k$ divides the segment $OG$ from circumcenter to centroid in the ratio $(r+1)k/(r-1-2k)$ as readily follows from the well known relations between $O$, $G$, and $\Omega$ of an $r$-simplex (see, for example, the *Editorial Note* in solution of 3757 (1937, 601)). This suggests the following generalization of the proposer's theorem.

Let $G$ be the centroid of a set of $n+1$ points $A_0, A_1, \cdots, A_n$ on the $r$-sphere with center $O$; let $G_i$ be the centroid of the subset formed by deleting $A_i$; and suppose $M_i$ divides $OG_i$ in the ratio $k(r+1)/(r-1-2k)$. Choose $G$ as origin for the vectors $\mathbf{a}_i$ to $A_i$, and for the vectors to $G_i$

$$\mathbf{b}_i = \frac{1}{n} \sum_{j\neq i} \mathbf{a}_j = -\frac{\mathbf{a}_i}{n},$$

and for the vector $\mathbf{c}$ to $O$. Then the vector $\overrightarrow{GM_i}$ is

$$\mathbf{d}_i = \frac{r-2k-1}{(r-1)(k+1)}\mathbf{c} - \frac{k(r+1)}{(k+1)(r-1)n}\mathbf{a}_i$$

and a vector equation of the line $A_iM_i$ is $\mathbf{x}=\mathbf{a}_i+t(\mathbf{a}_i-\mathbf{d}_i)$. Substituting for $\mathbf{d}_i$ we easily find by inspection that the $n+1$ lines $A_iM_i$ concur at the tip of the vector $n(r-2k-1)\mathbf{c}/[(k+1)\ (r-1)n+k(r+1)]$, which is a point dividing the segment $OG$ in the ratio $k(r+1)(n+1)/n(r-2k-1)$.

Solved also by W. T. Short, and the proposer.

3847 [1937, 667]. *Proposed by N. A. Court, University of Oklahoma.*

Three vertices and the circumradius of a variable tetrahedron are fixed, and the Monge point of the tetrahedron lies on the circumsphere. Find the locus of the fourth vertex.

I. *Solution by L. M. Kelly, Northeastern University, Boston, Mass.*

Since the Monge point $M$ is the symmetric of the circumcenter $O$ with respect to the centroid $G$ of the tetrahedron, $G$ lies on a sphere $(O, R/2)$ with center $O$ and radius $R/2$. If $G'$ is the centroid of the face of the three given vertices $A$, $B$, $C$ and $D$ is the fourth vertex, then $D$ lies on a sphere $(O_1, 2R)$ such that $G'$ is the center of similitude of $(O, R/2)$ and $(O_1, 2R)$ with the ratio $1:4$. Thus the locus of $D$ is the circle of intersection of the circumsphere $(O, R)$ and $(O_1, 2R)$.

II. *Solution by Roy MacKay, Eastern New Mexico Junior College.*

Let $A_1, A_2, \cdots, A_n$ be $n$ fixed vertices of a variable $n$-simplex $AA_1A_2 \cdots A_n$ with circumradius $R$. Denote the centroid, circumcenter, and circumradius of the $(n-1)$-simplex determined by the fixed vertices by $G'$, $O'$, and $R'$, respectively. Since $R$ is given, the circumcenter $O$ of the variable $n$-simplex is uniquely etermined except for its reflection in the $(n-1)$- plane determined by the given fixed vertices. If the Monge point $M$ of the variable $n$-simplex is to be on the circumsphere, the variable centroid $G$ of $AA_1A_2 \cdots A_n$ which divides $OM=R$ in the ratio $(n-1):(n+1)$ must lie at a distance $(n-1)R/(n+1)$ from $O$. On the other hand $G'G=G'A/(n+1)$, where $A$ is the variable vertex. In the triangle $G'AO$ it is easy to find that

$$G'A = [(n+1)\overline{G'O}^2 - (n-3)R^2]^{1/2}.$$

From the right triangles $OO'G'$ and $OO'A$, the Pythagorean relation easily yields:

## NEWS AND NOTICES

*Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.*

### INTERNATIONAL MATHEMATICAL CONGRESS

The Emergency Executive Committee of the International Congress of Mathematicians has decided definitely to postpone until some more favorable date the Congress which was to have been held in Cambridge in September 1940. Notice to that effect is being sent to the invited speakers and to other parties interested.

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### SO WE COMPUTE

By Maud Oakes Volandri, San Francisco

The rhythm of counting may not be confined
To concrete images, but when we add,
We add REAL things. Long use has made us blind
To dreamings of perfection men have had—
By their assumptions, one and one make two;
"Add LIKE things only:" on such ground they reared
The edifice of number. Think it through:
How close has true identity been neared?
No apples ever were exactly like,
Nor any grains of sand. Yes, even I
Am not like to myself. I change: I strike
New bargains, lose old fears, lay habits by.
We can assume things like, so we compute.
In number lies the dreamer's Absolute.

### THE BOUNDARY

By Maud Oakes Volandri, San Francisco

"The limit is infinity." When man
Grows tired of counting, when his imagery
Will stretch no further, with one word he can
Sum and represent immensity.
Annex twelve zeros to the number one,
And read a trillion. Add a trillion more
Of naughts, and add again; but when it's done,
The number's just as finite as before.
And yet, the concept of infinitude—
Abstract, outside man's comprehension—brings
Him concrete answers, with exactitude,
To problems which defy close measurings.
With what precise effrontery man's mind
Can leap beyond his universe and find!

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